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Solutions of a Lagrangian system on $\mathbb{T}^2$

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ABSTRACT **A Lagrangian system on $\mathbb{T}^2$ that has been studied earlier under a geometrical condition and found to possess a pair of solutions, $H^{\pm}$, homoclinic to periodic solutions, $v^{\pm}$, of a given homotopy type, is considered further. It is shown with the aid of $H^{\pm}$ and variational arguments that, in fact, there is a much richer structure of homoclinics and heteroclinics to $v^{\pm}$. Indeed, the system admits chaotic solutions.**

This paper studies the Lagrangian system on $\mathbb{R}^2$:

$$\text{(LS)} \qquad \frac{d}{dt}L_{\dot{q}} - L_q = 0$$

where the Lagrangian L is given by

$$L(q) = \sum_{i,j=1}^{2} a_{ij}(q)\dot{q}_i\dot{q}_j - V(q).$$

Assume

(V_1) $V \in C^2(\mathbb{R}^2, \mathbb{R})$ and is 1-periodic in q_1, q_2,

(V_2) $V(0) = 0 > V(x)$, $x \in \mathbb{R}^2\backslash\mathbb{Z}^2$,

(A) $(a_{ij}(q))$ is positive definite for all $q \in \mathbb{R}^2$,

and a_{ij} also satisfies (V_1).

Because of the periodicity of (LS) in q_1, q_2, it can be viewed as a system in $\mathbb{R}^2$ or on $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. For $V \equiv 0$, (LS) was considered by Morse (1) and Hedlund (2). They established the existence of a pair of geodesics (for the Riemannean metric associated with L) lying between adjacent periodic geodesics in a given homotopy class on $\mathbb{T}^2$ and heteroclinic to these periodic geodesics. When the potential V is present, the situation becomes more complicated due to the equilibrium solutions at $\mathbb{Z}^2$ given via (V_2). Under further geometrical conditions, there has been some work on the existence of zero energy periodic, heteroclinic, and homoclinic solutions of (LS) in refs. 3–6. In particular in ref. 6, it was shown that a geometrical condition led to a pair of periodic solutions v^+, v^- of (LS), and to homoclinics to v^+, v^- lying in the region between v^+ and v^-. The goal of this paper is to show that, in the setting of ref. 6, there is a much richer set of homoclinic and heteroclinic solutions of (LS). Indeed there, is a full symbolic dynamics of these and other solutions. Thus, (LS) admits chaotic solutions. This will be made precise and carried out in the next section.

A Symbolic Dynamics of Solutions

To describe our results, the framework of ref. 6 must be recalled. For $k \in \mathbb{Z}^2\backslash\{0\}$, let

$$F_k = \{q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)|\ \text{there is a } T = T(q) > 0 \ \text{such that } q(t + T) = q(t) + k\}.$$

Viewed on $\mathbb{T}^2$, F_k is the class of $W^{1,2}$ curves of homotopy type k. Let

$$G_k = \{q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)|q(-\infty) = 0,\ q(\infty) = k\}$$

The elements of G_k are candidates for heteroclinic solutions of (LS) (or homoclinics to 0 of homotopy type k viewed on $\mathbb{T}^2$).

For $q \in G_k$ and F_k respectively, let

$$I(q) = \int_{-\infty}^{\infty} L(q)dt,\ I_k(q) = \int_0^{T(q)} L(q)dt$$

and define

$$\bar{c}_k = \inf_{q\in G_k} I(q);\ c_k = \inf_{q\in F_k} I(q)$$

It was shown in refs. 3 and 4 that, if

$$\bar{c}_k > c_k, \qquad \textbf{[1]}$$

there is a $v \in F_k$ such that $I_k(v) = c_k$ and v is a solution of (LS) (of period $T(v)$ on $\mathbb{T}^2$). Moreover, there is a $u \in G_k$ such that $I(u) = \bar{c}_k$ and u is a solution of (LS) heteroclinic to 0 and k. Let

$$P_k = \{q \in F_k|I_k(q) = c_k\}.$$

The elements of P_k are only determined up to a phase shift because, if $\theta \in \mathbb{R}$ and $\tau_\theta q(t) \equiv q(t - \theta)$, then $I_k(q) = I_k(\tau_\theta q)$ for all $\theta \in \mathbb{R}$. Moreover, if $p \in F_k$, so is $p + j$ for all $j \in \mathbb{Z}^2$. It was shown in ref. 4 that $0 \notin p(\mathbb{R})$ for any $p \in P_k$. Therefore, 0 belongs to some component of $\mathbb{R}^2\backslash\{p(\mathbb{R})|p \in F_k\}$. This component is bounded by a pair of functions $v^+, v^- \in P_k$ and will be denoted by $\mathcal{R}$.

The region $\mathcal{R}$ will be subdivided as follows. For $i \in \mathbb{N}$, set $u_i = u + (i - 1)k$ and, for $-i \in \mathbb{N}$, set $u_i = u + ik$. Then, $U = \overline{\cup_{i\in\mathbb{Z}\backslash\{0\}}u_i(\mathbb{R})}$ divides $\mathcal{R}$ into $\mathcal{R}^+$ and $\mathcal{R}^-$ with $v^{\pm}(\mathbb{R})$ forming a boundary component of $\mathcal{R}^{\pm}$. Minimizing $\int_0^\infty L(\varphi)dt$ over the class of $W^{1,2}_{\text{loc}}$ curves, φ, with $\varphi(0) \in v^+(\mathbb{R})$ and $\varphi(\infty) = 0$ yields a C^2 solution, z_0^+ of (LS) in this class, joining v^+ and U. Similarly, there is a C^2 solution, z_0^-, of (LS) joining v^- and U with $z_0^-(0) \in v^-(\mathbb{R})$ and $z_0^-(\infty) = 0$. For $k \in \mathbb{Z}$, set $z_k^{\pm} = z_0^{\pm} + ik$. The curves U, $v^{\pm}$, and $z_i^{\pm}$ divide $\mathcal{R}$ in a natural way

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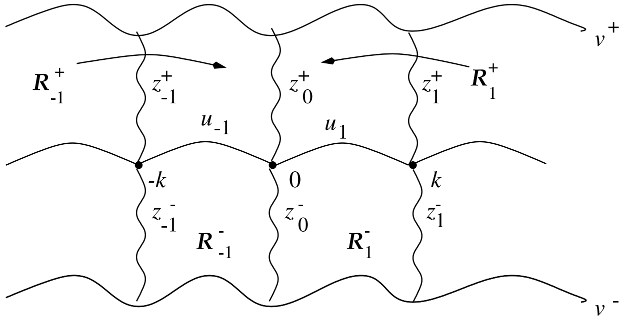

FIG. 1.

into "subrectangles," $\mathcal{R}_i^{\pm}$, $i \in \mathbb{Z}\backslash\{0\}$. See Fig. 1. Set $\mathcal{R}_i = \mathcal{R}_i^+ \cup \mathcal{R}_i^-$.

To continue, a stronger version of **[1]** is needed. Consider the class of $W^{1,2}$ curves joining $v^+([0, T(v^+))$ to $v^-([0, T(v^-))$. Minimizing $\int L(\cdot)dt$ over this class produces an infimum, b, of the functional. Suppose

$$\bar{c}_k - c_k > 2b, \qquad \textbf{[2]}$$

the strengthened geometrical condition. Then, there is a corresponding minimizer, ψ, of the functional that avoids $\mathbb{Z}^2$. By using **[2]**, it was shown in ref. 6 that (LS) possesses a pair of solutions, $H^{\pm}$ with $H^{\pm}$ homoclinic to $v^{\pm}$. Moreover, $H^{\pm}$ crosses $z_i^{\pm}$ for all $i \neq 0$ and also crosses $z_0^{\mp}$. In fact, $H^{\pm}(0) \in z_0^{\mp}(\mathbb{R})$, and the curves lie in $\mathcal{R}^{\pm}$ except for an interval in which they cross u_{-1} and $z_0^{\mp}$ and reenter $\mathcal{R}_1^{\mp}$ through u_1 (see Fig. 2). The functions $H^{\pm}$ are also minimal solutions of (LS) in the homotopy class of curves that cross the curves $z_i^{\pm}$ in the above fashion. "Minimal" means that, for all $x < y$, $H^{\pm}$ minimizes $\int L(w)dt$ over the class of $W^{1,2}$ curves w having the same endpoints and the same crossing (of $z_i^{\pm}$) properties as $H^{\pm}|_x^y$.

Observe that this minimality property implies that, for any $i \neq j \in \mathbb{Z}$, $\tau_i H^+(\mathbb{R}) \cap \tau_j H^-(\mathbb{R}) = \phi$.

With the aid of these preliminaries, $H^{\pm}$ will be used to help construct new homotopy classes of curves and a symbolic dynamics of solutions of (LS). Let

$$\sum = \{\sigma = (\sigma_i)_{i \in \mathbb{Z}} | \sigma_i \in \{+, -\}\}$$

A curve $q : \mathbb{R} \to \mathcal{R}$ will be said to have homotopy type $\sigma \in \Sigma$ if q crosses the curves $z_i^{\sigma_i}$, $i \in \mathbb{Z}$, in the order given by σ. Define $\sigma^{\pm} \in \Sigma$ by $\sigma_i^{\pm} = \pm$, $i \neq 0$, and $\sigma_0^{\pm} = \mp$. Then, $H^{\pm}$ the homotopy type $\sigma^{\pm}$.

Our main result is that, for each $\sigma \in \Sigma$, (LS) has a minimal solution of homotopy type σ. To be more precise, let $\sigma \in \Sigma$ and $i \in \mathbb{Z}$. Consider $\tau_i H\sigma_i$. It divides $\mathcal{R}$ into two subregions. Excise the region between $\tau_i H\sigma_i$ and $v\sigma_i$ from $\mathcal{R}$, calling the resulting region $\mathcal{R}(\tau_i H\sigma_i)$. Associate with σ the region $\cap_{i \in \mathbb{Z}} \mathcal{R}(\tau_i H\sigma_i) \equiv X_\sigma$. See Fig. 3, where $\sigma_i = -$, $i \leq 0$; $= +$, $i > 0$ and X_σ is the shaded region.

Now we have

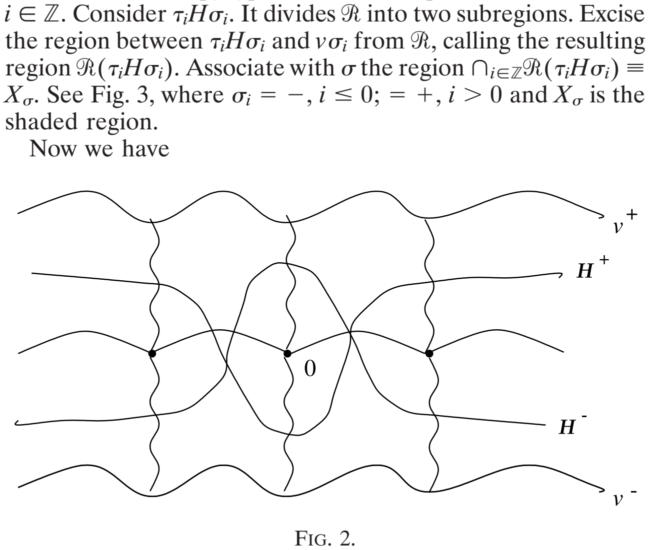

FIG. 2.

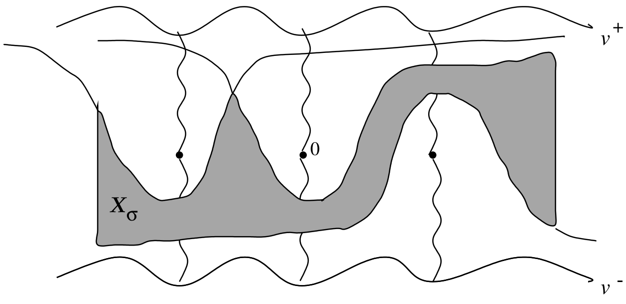

FIG. 3.

Theorem 3 *If* $(V_1) - (V_2)$, (A), *and [2] are satisfied, then, for each* $\sigma \in \Sigma$, *there exists a minimal solution* Q_σ *of (LS) of homotopy type* σ *lying in* X_σ.

Theorem 3 is a consequence of a related result for a subclass of Σ. For $p, r \in \{+, -\}$, let

$$\sum^{pr} = \left\{\sigma \in \sum \middle| \sigma_\ell = p \text{ for all large } \ell \in -\mathbb{N} \right.$$

$$\left. \text{and } \sigma_\ell = r \text{ for all large } \ell \in \mathbb{N}\right\}$$

Theorem 4 *If* $(V_1) - (V_2)$, (A), *and [2] are satisfied, then, for each* $\sigma \in \Sigma^{pr}$ *and* p, r $\in \{+, -\}$, *there is a minimal solution* Q_σ *of* (LS) *of homotopy type* σ *lying in* X_σ. *Moreover,* Q_σ *is heteroclinic from* v^p *to* v^r *if* p $\neq$ r *and is homoclinic to* v^p *if* p = r.

Theorem 4 will be proved first and then Theorem 3 follows from it by an approximation argument. As in refs. 4–6, the proof of Theorem 4 involves finding Q_σ as the minimizer of an appropriately renormalized functional over a class of curves lying in X_σ. Renormalization is necessary because the natural functional is infinite on the class of curves in X_σ. The first step in the proof is to introduce an appropriate class of curves. Let $\sigma \in \Sigma^{pr}$ and

$$\Gamma_\sigma = \{q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) | q \text{ satisfies } (\gamma_1) - (\gamma_5)\}$$

where

(γ_1) q lies in X_σ,

(γ_2) $q(0) \in u_1(\mathbb{R})$,

(γ_3) There is a monotone sequence $t_i = t_i(q)$, $i \in \mathbb{Z}$,

$$\text{such that } q(t_i(q)) \in z_i^{\sigma_i}(\mathbb{R}^+),$$

$$(\gamma_4) \qquad \begin{cases} q(t) \in \bar{\mathcal{R}}_{i+1} & \text{for } t \in [t_i, t_{i+1}],\ i \geq 0 \\ q(t) \in \mathcal{R}_i & \text{for } t \in [t_i, t_{i+1}],\ i \leq -1 \end{cases}$$

Because $\sigma \in \Sigma^{pr}$, there is a smallest $\ell^-, \ell^+ \in \mathbb{N}$ such that $\sigma_i = p$ for all $i \leq -\ell^-$ and $\sigma_i = r$ for all $i \geq \ell^+$. Define $s_i = s_i(q)$ via $q(t_i) = z_i^r(s_i(q))$, $i \geq \ell^+$ and $q(t_i) = z_i^p(s_i(q))$, $i \leq -\ell^-$. Then we require that

$$(\gamma_5) \qquad \begin{matrix} s_{i+1}(q) \leq s_i(q),\ i \geq \ell^+ \\ s_{i+1}(q) \leq s_i(q),\ i \leq -\ell^- \end{matrix}$$

Remark 5 *The sequence* $(t_i(q))$ *need not be unique. If* (t_i) *and* $(\tilde{t}_i)$ *are two such sequences, by* (γ_4), $q(t) \in z_i^{\pm}(\mathbb{R}^+)$ *for* $t \in [t_i, \tilde{t}_i]$.

The renormalized functional on Γ_σ will be defined as follows. Let $q \in \Gamma_\sigma$. Set

$$a_i(q) = \int_{t_{i-1}(q)}^{t_i(q)} L(q)dt - \alpha_i c_k$$

for $i \geq 1$ and

$$a_i(q) = \int_{t_i(q)}^{t_{i+1}(q)} L(q)dt - \alpha_i c_k$$

for $i \leq -1$ where $\alpha_i = 0$ if $-\ell^- \leq i \leq \ell^+$ and $\alpha_i = 1$ otherwise. Now define

$$J(q) = \sum_{i\in\mathbb{Z}\setminus\{0\}} a_i(q)$$

Because there may be more than one possible choice of $(t_i(q))$, it must be shown that $J(q)$ is independent of the choice of $(t_i(q))$. Thus, suppose that $J(q) < \infty$. Then $a_i(q) \to 0$ as $|i| \to \infty$, so

$$\int_{t_{i-1}(q)}^{t_i(q)} L(q)dt \leq c_k + 1 \quad \textbf{[6]}$$

for large $|i|$. By a simpler version of the proof of Proposition 3.12 of ref. 4,

$$\begin{cases} t_{i+1}(q) - t_i(q) \to T(v^p), \; i \to -\infty \\ t_{i+1}(q) - t_i(q) \to T(v^r), \; i \to \infty \end{cases} \quad \textbf{[7]}$$

and

$$\begin{cases} \|q - v^p\|_{L^\infty[t_i,t_{i+1}]} \to 0, \; i \to -\infty \\ \|q - v^r\|_{L^\infty[t_i,t_{i+1}]} \to 0, \; i \to \infty \end{cases} \quad \textbf{[8]}$$

Hence, $s_i(q) \to 0$ as $|i| \to \infty$. As in ref. 6, set

$$J_\ell(q) = \sum_{-\ell}^{\ell} a_i(q); \; \tilde{J}_\ell(q) = \sum_{-\ell}^{\ell} \tilde{a}_i(q),$$

where J corresponds to $t_i(q))$ and $\tilde{J}$ to $(\tilde{t}_i(q))$, with both $(t_i(q))$ and $(\tilde{t}_i(q))$ satisfying (γ_3). Then, **[8]** and $s_i(q) \to 0$ imply

$$|J_\ell(q) - \tilde{J}_\ell(q)| \leq \left| \int_{t_{-\ell}(q)}^{\tilde{t}_{-\ell}(q)} L(q)dt \right| + \left| \int_{t_\ell(q)}^{\tilde{t}_\ell(q)} L(q)dt \right|. \quad \textbf{[9]}$$

Because $q|_{t_\ell}^{\tilde{t}_\ell}$ lies on $z_\ell^r(\mathbb{R}^+)$, **[7]** and **[8]** show $|t_\ell(q) - \tilde{t}_\ell(q)| \to 0$ as $\ell \to \infty$ and similarly for $-\ell$. Hence, **[6]**, the right hand side of **[9]** $\to 0$ as $\ell \to \infty$. Consequently, $J(q) = \tilde{J}(q)$, and J is well defined.

Now define

$$c_\sigma = \inf_{q\in\Gamma_\sigma} J(q). \quad \textbf{[10]}$$

Theorem 4 will be proved by showing there is a $Q_\sigma \in \Gamma_\sigma$ such that $J(Q_\sigma) = c_\sigma$. Moreover, Q_σ is a minimal solution of (LS). Note that, by **[8]**, $Q_\sigma \in \Gamma_\sigma$ and $J(Q_\sigma) < \infty$ implies that Q_σ is asymptotic to v^p as $t \to -\infty$ and to v^r as $t \to \infty$. The minimization argument is related to that of ref. 6, and, therefore, ref. 6 will be referred to for details when appropriate.

If, for example, $i \geq \ell^+$, gluing $(z_i^r|_{s_i(q)}^{s_{i-1}(q)} - k)$ to $q|_{t_{i-1}(q)}^{t_i(q)}$ produces an element of F_k. Hence, by the definition of c_k,

$$a_i(q) \geq -\int_{s_i(q)}^{s_{i-1}(q)} L(z_0^r)dt. \quad \textbf{[11]}$$

Combining these estimates shows

$$J(q) \geq -\int_0^\infty (L(z_0^p) + L(z_0^r))dt \equiv -M_0, \quad \textbf{[12]}$$

that is, J is bounded from below on Γ_σ. An upper bound for c_σ is provided by gluing a curve in X_σ, joining $v^p(-\ell^- T(v^p))$ and $v^r(\ell^+ T(v^r))$ to $v^p|_{-\infty}^{-\ell^- T(v^p)}$ and $v^r|_{\ell^+ T(v^r)}^\infty$, yielding $H \in \Gamma_\sigma$ with $c_\sigma \leq J(H) < \infty$.

Let (q_m) be a minimizing sequence for **[10]**. Consider $\tau_{\ell^+} H^r(t)$. Now, $q_m(t_{\ell^+}(q_m))$ lies on $z_{\ell^+}^r(\mathbb{R}^+)$ between $\tau_{\ell^+-1} H^r(t_{\ell^+}(H^r))$ and $\tau_{\ell^+} H^{-r}(t_{\ell^+}(H^{-r}))$ and a fortiori between $\tau_{\ell^+}(H^r(t_{\ell^+}(H^r))$ and $\tau_{\ell^+-1} H^r(t_{\ell^+}(H^r))$. It can be assumed that $q_m|_{t_\ell^+(q_m)}^\infty$ lies between $\tau_{\ell^+-1} H^r|_{t_{\ell^+}}^\infty(H^r)$ and $t_{\ell^+} H^r|_{t_{\ell^+}}^\infty(H^r)$. Indeed, suppose $q_m((x_1, x_2))$ is outside of this region and $q_m(x_i) = \tau_{\ell^+} H^r(y_i)$, $i = 1, 2$. Replacing $q_m|_{x_1}^{x_2}$ by $\tau_{\ell^+} H^r|_{y_1}^{y_2}$ yields $\hat{q}_m \in \Gamma_\sigma$ with $J(\hat{q}_m) < J(q_m)$ via the minimality property of $H^\pm$. If $q_m|_{x_1}^\infty$ lies outside the region, replace $H^r|_{y_1}^\infty$ by $q_m|_{x_1}^\infty$, calling the resulting function $\hat{H}$. Because $\tau_{\ell^+} H^r$ is the minimizer of J in an associated class of curves (6) (containing $\hat{H}$), $J(\hat{H}) > J(\tau_{\ell^+} H^r)$, which implies

$$\int_{x_1}^{t_{\ell^++1}(q_m)} L(q_m)dt + \sum_{\ell^++1}^\infty a_i(q_m) > \int_{y_1}^{t_{\ell^++1}(\tau_{\ell^+} H^r)} L(\tau_{\ell^+} H^r)dt + \sum_{\ell^++1}^\infty a_i(\tau_{\ell^+} H^r). \quad \textbf{[13]}$$

Therefore, by **[13]** gluing $q_m|_{-\infty}^{x_1}$ to $\tau_{\ell^+} H^r|_{y_1}^\infty$ yields $\tilde{q}_m \in \Gamma_\sigma$ with $J(\tilde{q}_m) < J(q_m)$. Similar reasoning shows that $q_m|_{-\infty}^{-t_{\ell^-}}(q_m)$ lies between $\tau_{-\ell^-+1} H^p|_{-\infty}^{-t_{\ell^-}(H^p)}$ and $\tau_{-\ell^-} H^p|_{-\infty}^{-t_{\ell^-}(H^p)}$.

As in refs. 4–6, (q_m) is bounded in $W^{1,2}_{\text{loc}}$ and therefore, along a subsequence, converges weakly in $W^{1,2}_{\text{loc}}$ and strongly in L^∞_{loc} to $Q = Q_\sigma \in W^{1,2}_{\text{loc}}$, with Q satisfying $(\gamma_1) - (\gamma_2)$ as well as the constraints on (q_m) of the previous paragraph. As in ref. 6, there are numbers $A_i > 0$ such that

$$|t_i(q_m)| \leq A_i, \; i \in \mathbb{Z}. \quad \textbf{[14]}$$

By **[14]**, it can be assumed that $t_i(q_m) \to \tilde{t}_i$ for all $i \in \mathbb{Z}$. It remains to show that $(\gamma_3) - (\gamma_5)$ hold for Q. The convergence already established shows for all $i \in \mathbb{Z}$, as $m \to \infty$.

$$q_m(t_i(q_m)) \to Q(\tilde{t}_i) \in z_i^{\sigma_i}. \quad \textbf{[15]}$$

Therefore, by **[15]** and (γ_3) for q_m

$$Q(t) \in \bar{\mathcal{R}}_{i+1}, \; t \in [\tilde{t}_i, \tilde{t}_{i+1}], \; i \geq 0$$

$$Q(t) \in \bar{\mathcal{R}}_i, \; t \in [\tilde{t}_i, \tilde{t}_{i+1}], \; i \leq -1 \quad \textbf{[16]}$$

and $(\gamma_3) - (\gamma_4)$ holds for Q with $t_i(Q) = \tilde{t}_i$. Finally, as $m \to \infty$,

$$z_i^{\sigma_i}(s_i(q_m)) = q_m(t_i(q_m)) \to Q(\tilde{t}_i) \equiv z_i^{\sigma_i}(\tilde{s}_i). \qquad \textbf{[17]}$$

The latter equality defines $\tilde{s}_i$ and implies

$$s_i(q_m) \to \tilde{s}_i \qquad \textbf{[18]}$$

as $m \to \infty$. Now, (γ_5) for q_m and **[18]** gives (γ_5) for Q so $Q \in \Gamma_\sigma$.

Next, it must be shown that $J(Q) < \infty$ and $J(Q) = c_\sigma$. There is an $M > 0$ such that

$$J(q_m) \leq M. \qquad \textbf{[19]}$$

For $q = q_m$, write

$$J(q) \equiv J^+(q) + J^-(q), \qquad \textbf{[20]}$$

where $J^+(q)$ denotes the sum over those $a_i(q)$ such that $a_i(q) \geq 0$. Note that the definition of $a_i(q)$ implies $a_i(q) < 0$ is only possible when $i \geq \ell^+ + 1$ or $i \leq -\ell^- - 1$. By **[11]** and **[19]**,

$$J^+(q) = J(q) - J^-(q) \leq M + M_0 \qquad \textbf{[21]}$$

and therefore

$$\sum_{i \in \mathbb{Z}\setminus\{0\}} |a_i(q)| \leq M + 2M_0 \equiv M_1 \qquad \textbf{[22]}$$

Hence, for any $n \in \mathbb{N}$ with, e.g., $n > \ell^+ + \ell^-$,

$$\int_{t_{-n}(q_m)}^{t_n(q_m)} L(q_m)dt \leq M_1 + (2n - \ell^+ - \ell^-)c_k. \qquad \textbf{[23]}$$

This implies

$$\int_{\tilde{t}_{-n}}^{\tilde{t}_n} L(Q)dt \leq M_1 + (2n - \ell^+ - \ell^-)c_k$$

or equivalently

$$\sum_{-n}^{n} a_i(Q) \leq M_1 \qquad \textbf{[24]}$$

Hence, $J(Q) < \infty$ via **[24]**.

A variant of arguments from refs. 4–6 now shows $J(Q) = c_\sigma$. Indeed, let $\varepsilon > 0$. There is an $m_0 = m_0(\varepsilon)$ such that $m \geq m_0$,

$$J(q_m) \leq c_\sigma + \varepsilon. \qquad \textbf{[25]}$$

Further, choose $j = j(\varepsilon)$ so that

$$J(Q) \leq \sum_{-j}^{j} a_i(Q) + \varepsilon. \qquad \textbf{[26]}$$

It can also be assumed that, for $m \geq m_0$,

$$\int_{t_{-j}(Q)}^{t_j(Q)} L(Q)dt \leq \int_{t_{-j}(q_m)}^{t_j(q_m)} L(q_m)dt + \varepsilon \qquad \textbf{[27]}$$

Therefore, by **[25–27]** and **[11]**,

$$\begin{aligned} J(Q) &\leq \sum_{-j}^{j} a_i(q_m) + 2\varepsilon \\ &\leq c_\sigma - \sum_{|i|>j} a_i(q_m) + 3\varepsilon \\ &\leq c_\sigma + 3\varepsilon + \int_0^{s_j(q_m)} L(z_0^r)dt + \int_0^{s_{-j}(q_m)} L(z_0^p)dt \end{aligned} \qquad \textbf{[28]}$$

The constraints on q_m established in the paragraph containing **[13]** imply

$$s_j(q_m) \leq s_j(\tau_{\ell^+}H^r) \leq \varepsilon;\ s_{-j}(q_m) \leq s_{-j}(\tau_{-\ell^-}H^p) \leq \varepsilon \qquad \textbf{[29]}$$

for j sufficiently large. Now, **[28–29]** yield $J(Q) = c_\sigma$.

That Q is a solution of (LS) follows from simple local minimization and comparison arguments as in Proposition 5.4 of ref. 4.

Remark 30 *For* $\mathrm{i} \geq \ell^+$, $\mathrm{s_{i+1}(Q) < s_i(Q)}$; *for* $\mathrm{i} \leq -\ell^-$, $\mathrm{s_{i-1}(Q) < s_i(Q)}$. *Indeed, if equality holds in the* + *case, excising* $\mathrm{Q}|_{\mathrm{t_i}}^{\mathrm{t_{i+1}}}$ *from* Q *and gluing* $\mathrm{Q}|_{-\infty}^{\mathrm{t_i}}$ *to* $(\mathrm{Q} - \mathrm{k})|_{\mathrm{t_{i+1}}}^{\infty}$ *yields* $\mathrm{Q}^* \in \Gamma_\sigma$ *with* $\mathrm{J(Q^*) < J(Q)}$, *a contradiction, unless* $\mathrm{Q}|_{\mathrm{t_i}}^{\mathrm{t_{i+1}}}$ *coincides with* $\mathrm{v^r}$. *But, because* Q *is a solution of* (LS), *this is impossible.*

To complete the proof of Theorem 4, it must be shown that Q is a minimal solution of (LS). Suppose $x < y$. We claim $Q|_x^y$ minimizes $\int L(\cdot)dt$ over the class of $W^{1,2}$ curves with the same end points as $Q|_x^y$ and that cross the $z_i^\pm$ in the order given by σ. Indeed, let w denote the minimizer of this variational problem. It suffices to prove that Q^*, the curve obtained by replacing $Q|_x^y$ by w, belongs to Γ_σ, for then

$$J(Q^*) \leq J(Q). \qquad \textbf{[31]}$$

Therefore, there must be equality in **[31]**, and Q^* is a solution of (LS). But Q and Q^* coincide on an open set, so uniqueness of solutions of (LS) implies $Q \equiv Q^*$.

To verify that Q^* satisfies $(\gamma_1) - (\gamma_5)$, note that the range of w lies in X_σ via the minimality properties of the boundary curves of X_σ. Hence, (γ_1) holds. Parametrizing Q^* appropriately gives (γ_2). There is a finite set of $z_j^\pm$ that $Q|_x^y$ intersects $z_j^\pm$. Because w is a solution of (LS), there is a natural corresponding set of $t_j(w)$, namely $t_j(w)$ is the unique (via the minimality of $z_j^\pm$) value of t at which w intersects $z_j^\pm$. Thus, Q^* satisfies (γ_3), and minimality arguments imply (γ_4). Suppose (γ_5) fails, e.g., for $i > 0$. Then, $s_{i+1}(Q^*) > s_i(Q^*)$ for some smallest i. Because $s_j(Q^*) \to 0$ as $j \to \infty$, there is a smallest $j > i + 1$ such that $s_j(Q^*) \leq s_i(Q^*)$. If $s_j = s_i$, excise $Q^*|_{t_i(Q^*)}^{t_j(Q^*)}$ from Q^* and glue $Q^*|_{-\infty}^{t_i}$ to $(Q^* - k)|_{t_j}^{\infty}$, obtaining $\hat{Q} \in \Gamma_\sigma$ with $J(\hat{Q}) < J(Q^*) \leq J(Q)$, a contradiction. If $s_j < s_i$, define $P(t) = Q^*(t) + k$, $t \geq t_i(Q^*)$. Suppose for convenience that $r = +$. Because $s_{i+1} > s_i$, $P(t_i)$ lies between $Q^*(\mathbb{R})$ and $v^+(\mathbb{R})$ while $P(t_{j-1})$ lies between $Q^*(\mathbb{R})$ and the portion of ∂X_σ given by appropriate segments of $\{\tau_\ell H^-\}$. Therefore, there is a $t^* \in (t_i, t_{j-1})$ such that $P(t^*) \in Q^*(\mathbb{R})$; i.e., $Q^*(t^*) + k = Q(\tilde{t})$. Excising $Q^*|_{t^*}^{\tilde{t}}$ from Q^* and arguing as for $s_j = s_i$ yields $\hat{Q}$ such that $J(\hat{Q}) < J(Q^*)$. Possibly, (γ_5) still fails for $\hat{Q}$, but, repeating the above argument a finite number of times yields $\hat{Q} \in \Gamma_\sigma$ such that $J(\hat{Q}) < J(Q)$, a contradiction. This Q is a minimal solution of (LS), and Theorem 4 is proved.

Proof of Theorem 3. Let $\sigma \in \Sigma$. Define $d_m = (d_{m_i})_{i\in\mathbb{Z}} \in \Sigma$ as follows: $d_{mi} = \sigma_i$, $|i| \leq m$; $d_{mi} = \sigma_m$, $i \geq m$; $d_{mi} = \sigma_{-m}$, $i \leq -m$. Then, $d_m \in \Sigma^{\sigma_{-m},\sigma_{-m}}$, so, by Theorem 4, there is a minimal solution $Q_m \in \Gamma_{d_m}$ of (LS). The form of d_m and X_{d_m} together with (LS) imply the functions Q_m are bounded in C^2_{loc} and therefore converge in C^2_{loc} to $Q_\sigma \in X_\sigma$. It readily follows that Q_σ is a minimal solution of (LS) of homotopy type σ, and the proof is complete.

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