Compact stellarators with modular coils

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Compact stellarator designs with modular coils and only two or three field periods are now available; these designs have both good stability and quasiaxial symmetry providing adequate transport for a magnetic fusion reactor. If the bootstrap current assumes theoretically predicted values a three field period configuration is optimal, but if that net current turns out to be lower, a device with two periods and just 12 modular coils might be better. There are also attractive designs with quasihelical symmetry and four or five periods whose properties depend less on the bootstrap current. Good performance requires that there be a satisfactory magnetic well in the vacuum field, which is a property lacking in a stellarator–tokamak hybrid that has been proposed for a proof of principle experiment. In this paper, we present an analysis of stability for these configurations that is based on a mountain pass theorem asserting that, if two solutions of the problem of magnetohydrodynamic equilibrium can be found, then there has to be an unstable solution. We compare results of our theory of equilibrium, stability, and transport with recently announced measurements from the large LHD experiment in Japan.

We shall be concerned with the theory of stellarators and tokamaks, which are toroidal devices for the confinement of plasma in magnetic fusion research (1). Three-dimensional computer codes have been run efficiently to design compact stellarators with modular coils whose physical properties compare favorably with those of tokamaks. Quasiaxial symmetry (QAS) of the magnetic spectrum ensures that there will be satisfactory confinement of hot particles at reactor conditions. However, the similarity to tokamaks that are prone to current driven disruptions has caused concern in the fusion community about the magnetohydrodynamic stabilty of the new configurations. It is primarily the issue of stability that we address in the present paper by applying the NSTAB computer code, which has been benchmarked carefully against both experimental measurements and other mathematical methods (2, 3). We study the equilibrium and stability of stellarators that have been proposed for a proof of principle experiment to help select one with a good prospect for success, and to put the results in perspective we also examine a model of the LHD configuration.†

Bifurcated Equilibria and Nonlinear Stability

Linear stability of stellarators is hard to treat numerically because the equilibrium problem in three dimensions does not in general possess smooth solutions without islands or current sheets. Accurate spectral computations involve approximating divergent Fourier series that can only be summed by filtering them in ways that are not easy to specify in advance. The difficulty becomes especially damaging if the nonexistence of a solution of finite difference equations has resulted in failure to drive the residuals toward zero in the numerical calculations (2, 4). We circumvent this problem of singularities by applying the minimax theory of critical points to replace the question of stability by the question whether there are bifurcated equilibria. Thus, we apply an intuitively appealing mountain pass theorem for the energy landscape to assert that, if several solutions of a conservation form of equations derived from the variational principle of magnetohydrodynamics can be found, then there must be at least one that is unstable. Nonlinear estimates of pressure limits obtained in this fashion tend to be as much as 1% larger than standard predictions for ballooning modes (5). A similar discrepancy occurs in comparisons of other computational work with ballooning theory, so something may be wrong with the way the local analysis is done.

To explain our method better, we recall that an elementary form of the mountain pass theorem states that, if a function of several variables, such as the potential energy of a mechanical system, has two minima, or equilibria, then it also has another critical point that can be described as a saddle point. To locate the saddle point one first draws a curve that joins the two minima on the surface defined by the function and finds a maximum on the curve. Then one chooses the curve so as to minimize that maximum. The minimax point obtained this way is the desired saddle point, or mountain pass, at which the corresponding mechanical equilibrium is unstable.

The design of advanced stellarators is based on analysis of the Fourier expansion

$$\frac{1}{B^2} = \sum B_{mn}(s)\cos(m\theta - n\phi)$$

of the magnetic field strength in terms of renormalized flux coordinates s, θ, and ϕ. The corresponding spectrum of Fourier coefficients B_{mn} controls most physical properties of the configuration (6). They are in turn directly related to Fourier coefficients Δ_{mn} in a representation,

$$r + iz \; = \; e^{iu} \sum \Delta_{mn} e^{-imu+inv},$$

of the plasma surface, where r, ϑ, and z are cylindrical coordinates and u and $v = Q\vartheta$ are poloidal and toroidal angles. Good equilibrium, stability, and transport are achieved by a reduction in the size of B_{mn} for selected values of the indices m and n. At the same time, there is to be little change in the location of the plasma or in the rotational transform ι as the dimensionless measure $\beta = 2p/B^2$ of the pressure p increases. We call the spectrum B_{mn} quasisymmetric if a single row B_{m0} or diagonal B_{mm} of elements becomes dominant.

Experimental work is in progress to assess quasihelically symmetric stellarators with four or five field periods as a fusion concept (7, 8). Recently we have developed a Modular Helias-like Heliac (MHH2) with only $Q = 2$ field periods that has the low plasma aspect ratio $A = 2.5$, has good QAS, and has an external magnetic field generated by just 12 modular coils. Nonlinear stability seems to be satisfactory up to an average β of 5%, and islands near a dangerous resonance at $\iota/2 = 1/4$ are suppressed by controlling the corresponding term B_{41} in the magnetic spectrum. However, if the bootstrap current assumes values predicted by transport theory this configuration might not be optimal for a proof of principle experiment because of that resonance (9).

APPLIED PHYSICAL SCIENCES

Abbreviations: LHD, Large Helical Device; QAS, quasiaxial symmetry; MHH2, Modular Helias-like Heliac; NIFS, National Institute for Fusion Science; NBI, Neutral Beam Injection; MHD, magnetohydrodynamics.

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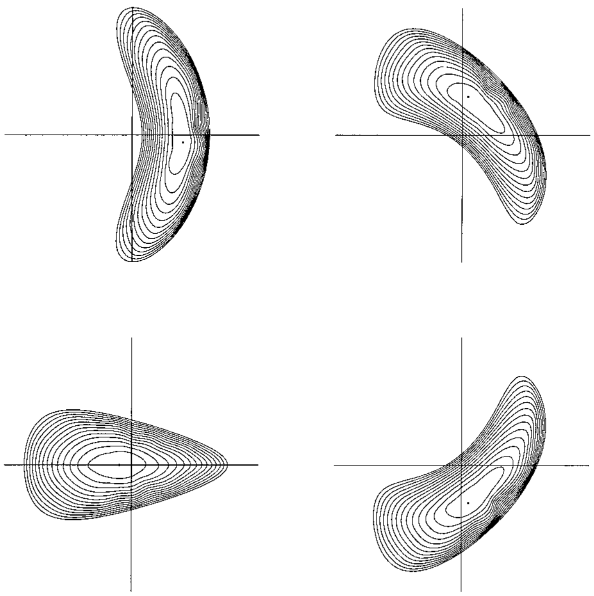

Fig. 1. Asymmetrical cross sections of a bifurcated equilibrium of the MHH2 stellarator at $\beta = 0.06$ with net current putting the rotational transform in the range $0.60 > \iota > 0.56$. The existence of more than one equilibrium ensures that there is a minimax solution at these conditions which is unstable. Notice the ballooning structure of the mode, which is localized in a region of bad curvature and decays dramatically along a magnetic line in as little as half a field period.

There are runs of the NSTAB computer code showing that when the rotational transform of the MHH2 is increased to account for a contribution from bootstrap current, the most dangerous magnetohydrodynamic modes remain stable for β below 5%. Fig. 1 displays a bifurcated solution with a broad pressure profile of the form $p = p_0(1 - s^{1.5})^{1.5}$ establishing instability of a mode with ballooning structure at larger β. The shape of the magnetic surfaces indicates that there is little problem with the existence of the equilibrium.

A set of just 12 modular coils has been designed to generate the external magnetic field for the MHH2 stellarator of low aspect ratio $A = 2.5$. A control surface like that of the plasma boundary was obtained on which filaments specifying the coils would have minimal curvature and twist. The Biot–Savart law was used to calculate a current distribution,

$$\varphi = 5v/\pi + \sum \varphi_{mn} \sin(mu - nv),$$

on that surface without superfluous oscillations introduced by least squares approximation of the equilibrium field. It was possible to arrive at a robust configuration with ample spacing among the filaments and with a sufficient gap between the coils and the separatrix.

A new code called NWIND has been applied to find shape factors Δ_{mn} specifying a surface that fits closely to some magnetic line traversing the separatrix. This enabled us to further modify the coils so as to obtain a better match with the original equilibrium computed at $\beta = 0$. In Table 1, we present Fourier coefficients defining this solution of the problem, which we believe to be suitable for the construction of a proof of principle experiment. Fig. 2 displays the three-dimensional geometry of the coils. The data indicate how Fourier series have been filtered to get rid of extraneous harmonics. Special attention has been given to the choice of resonant terms to avoid trouble with magnetic islands, but there is evidence that a similar device of higher aspect ratio might have magnetic surfaces that are more robust because the coefficient B_{41} becomes smaller than 0.1%

Table 1. Fourier coefficients Δ^b_{mn} defining the boundary of the plasma, Fourier coefficients Δ^c_{mn} defining the control surface for the coils, and Fourier coefficients φ_{mn} defining the Biot–Savart current distribution for the MHH2 stellarator

m	n	Δ^b_{mn}	Δ^c_{mn}	φ_{mn}
−2	−1	0.000	−0.200	0.000
−1	−1	0.170	0.320	0.000
−1	0	0.150	0.210	0.000
−1	1	−0.010	0.000	0.000
0	0	1.000	1.790	0.000
0	1	0.000	0.020	−0.591
0	2	0.000	0.000	−0.137
0	3	0.000	0.000	−0.101
1	−2	0.000	0.000	−0.013
1	−1	0.050	0.060	−0.039
1	0	2.500	2.700	0.237
1	1	0.160	0.090	0.644
1	2	0.010	0.030	0.100
1	3	−0.010	0.000	−0.076
2	0	−0.060	−0.090	0.145
2	1	−0.450	−0.260	−0.769
2	2	−0.060	0.000	−0.050
3	0	0.000	0.000	−0.053
3	1	−0.040	0.000	0.173
3	2	0.090	0.100	0.181
3	3	0.000	0.000	−0.176
3	4	0.000	0.000	0.038
4	0	0.020	0.020	−0.054
4	1	0.030	0.100	0.012
4	2	−0.020	−0.030	−0.067
4	3	−0.020	−0.020	−0.013
4	4	0.000	0.000	0.068
5	1	0.000	0.000	−0.075
5	2	0.000	0.000	0.020
5	3	0.000	0.000	0.006
6	1	0.000	0.000	0.018
6	3	0.000	0.000	0.017

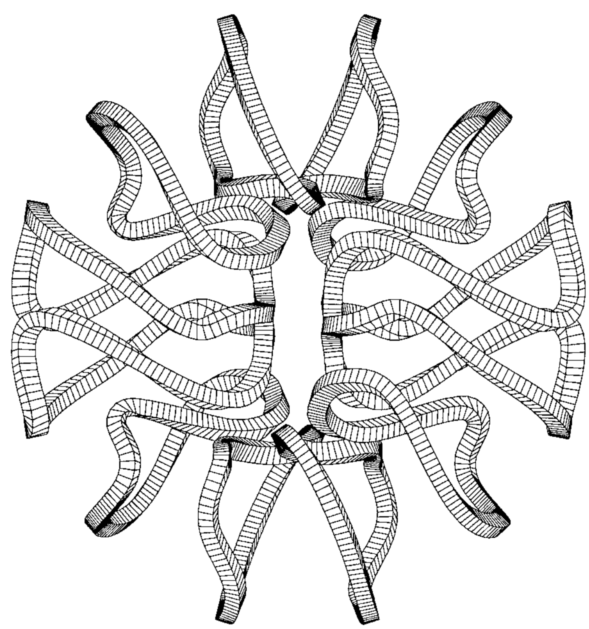

Fig. 2. Twelve modular coils generating the external magnetic field of the MHH2 stellarator. Fourier series for the filaments defining the coils have been filtered judiciously to eliminate superfluous curvature and twist.

of the magnetic field strength. In an experiment, one would presumably add auxiliary coils to control the $m = 4$, $n = 1$ island that can appear when there is net current so that $\iota/2 = 1/4$ inside the plasma.

Larger values of the bootstrap current can be exploited in QAS stellarators with three field periods to contribute rotational transform improving the equilibrium. The QAS3 is a configuration of this kind that we have designed as a stellarator–tokamak hybrid (5, 6). The plasma has aspect ratio $A = 4$ and 24 modular coils are required to produce the external field. In runs of the NSTAB computer code, symmetric cross sections of the flux surfaces establish nonlinear stability up to an average β of 6%, where the solution is unique, and there is no damaging ripple in the surfaces, so neither the equilibrium nor the stability β limit is exceeded.

The mountain pass analysis of stability is most convincing when a bifurcated solution with asymmetrical flux surfaces can be calculated that is fully converged. That is easy to accomplish at a β of 1% for a configuration modeled on one proposed by the fusion community for a proof of principle stellarator experiment (10). Like the QAS3, it has three field periods, and its aspect ratio is only $A = 3.4$, but unfortunately there is a 15% magnetic hill in the vacuum field when no plasma is present. Despite a nested surface hypothesis, the NSTAB calculation of the vacuum solution shows that it has an $m = 2$, $n = 0$ island shaped like a peanut at the magnetic axis. Further computations of bifurcated equilibria suggest that there may be trouble accessing a second stability regime in this configuration, which does not seem to be robust. Better results can be obtained if there is reversed shear generated by bootstrap current. Some of the trouble with stability comes from trying to operate the system as a stellarator–tokamak hybrid at low β when there is too much induced current.

For a concept exploration experiment that would be less ambitious, the fusion community has designed a quasiomnigeneous stellarator that has three field periods and an aspect ratio 3.5, but is closer to quasihelical than quasiaxial symmetry (10). The geometry of the separatrix is not unlike that of the MHH2, but the vacuum magnetic field does not have such a good well. We have run the NSTAB code for a model of this configuration, and a solution has been found at a β of 5% with flux surfaces that have a rectangular shape because the resonant term B_{41} in the magnetic spectrum is not small at $\iota/3 = 1/4$. This indicates that there may be a problem with the equilibrium at high β.

Conventional Stellarators

To validate our theory, numerical studies have been made of the CHS stellarator constructed at the National Institute for Fusion Science (NIFS) in Japan so that NSTAB calculations can be compared with the experimental measurements that are available (6, 11). When the vertical field is adjusted to shift the magnetic axis inward, the stability limit drops to a β of 0.5%, which was the highest value observed in the CHS experiment for such a case. With the magnetic axis in an outward position the equilibrium and stability limit seems to be near a β of 3%, but that conclusion has some sensitivity to geometry and profiles.

More recently, measurements have been announced from the much larger LHD experiment.† Broad pressure profiles were obtained, and values of average β have been observed between 2% and 3% with the magnetic axis at an inward location with a large radius of 3.6 m. Even nonlinear NSTAB calculations led us to expect a lower β limit in such circumstances. However, Fig. 3 displays a bifurcated equilibrium with $p = p_0(1 - s^{1.8})^{1.2}$ that models the new data and shows that the most unstable mode remains localized near a resonant surface, so it is relatively harmless. An island chain becomes visible that demonstrates the ability of the NSTAB code to simulate this phenomenon despite a nested surface hypothesis that has been made. Fig. 4 describes convergence of the bifurcated solution. More specific predictions of a β limit based on these computations require one to decide more precisely when the amplitude of the ripple in the magnetic surfaces is big enough to imply destruction of the plasma. Our overall conclusion is that some predictions of both local and global stability theory may have been too pessimistic and

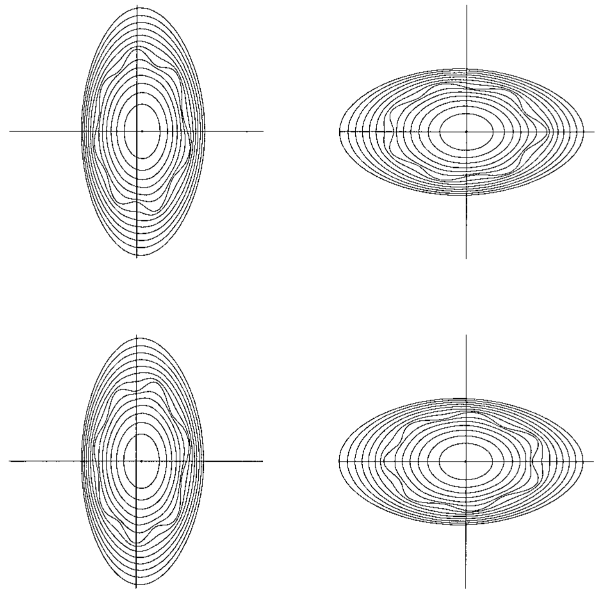

Fig. 3. Poincaré sections over two field periods of the magnetic surfaces of a bifurcated LHD equilibrium with a broad pressure profile at a β of 2.5% in the range observed by the experiment. The ripple at a resonance where $\iota = 5/7$ represents a nonlinearly saturated unstable mode that does not damage the solution significantly. This example shows how the NSTAB code captures small islands.

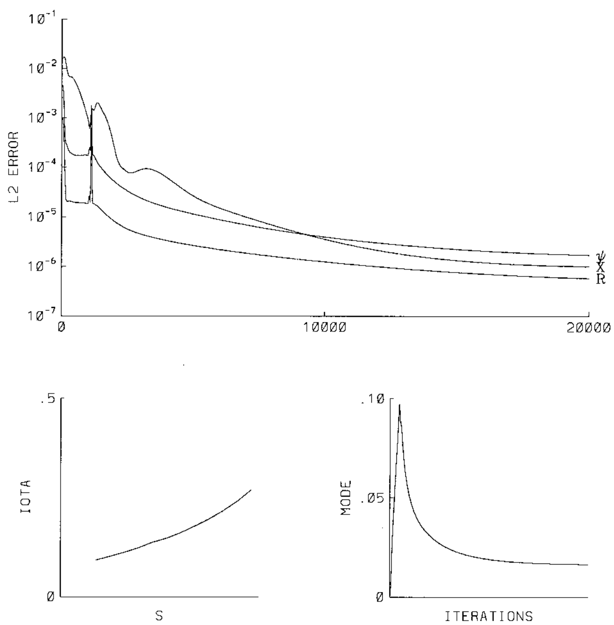

Fig. 4. Plots of the residuals, the rotational transform, and a Fourier coefficient of the localized $m = 7$, $n = 5$ mode as functions of the iteration cycle in the NSTAB calculation of a bifurcated LHD equilibrium shown in Fig. 3. Convergence of the iterative scheme to a solution without helical symmetry is demonstrated by the numerical data.

caution is called for in interpreting the results of ideal MHD calculations.

The convergence of the NSTAB runs we have described for the LHD stellarator is very good. However, for the compact MHH2 configuration of aspect ratio $A = 2.5$, the numerical calculations are much more difficult. It becomes necessary to rezone the coordinate system carefully and to rotate it one circuit in the poloidal direction as the toroidal angle increases through a field period. To implement the rotation, we replace u by $u + v$ in our formula for the plasma surface. Too few harmonics do not provide satisfactory resolution in the spectral method, and too many result in divergence of the preconditioned iterative scheme that is employed. Most of our conclusions have been based on using Fourier terms of degree $m \leqslant 20$ in the poloidal direction and of degree $n \leqslant 16$ in the toroidal direction. Because the physics of the low aspect ratio MHH2 is interesting, further study of the mathematical problems that arise is called for. More specifically, one must ask whether estimates of ballooning stability use equilibria that have been computed with sufficient accuracy.

Transport of Ions and Electrons

The most effective way to calculate transport in stellarators is to apply the Monte Carlo method, which is motivated by the problem of tracking orbits. We have implemented a test particle model from kinetic theory which consists of placing a particle in the plasma and asking how long it takes to leave. Because the effect of a single particle on macroscopic quantities is negligible, the electromagnetic field and the flow field of the background remain fixed, so the momentum of the test particle need not be conserved during collisions. Statistical estimates of the confinement time are calculated by our TRAN computer code, which treats the problem as a random walk among drift surfaces whose complicated geometry plays an important role (12). Details of the geometry are handled computationally by an appropriate ordinary differential equation solver that tracks guiding center orbits.

In plasma equilibria where the mean free path extends hundreds of times around the torus, conservation of momentum is not a plausible mechanism to account for quasineutrality, which is required to solve for the magnetic field. In our work, we introduce three-dimensional perturbations of the electrostatic potential

$$\Phi = P_{00}(s) + \sum P_{mn} \cos(m\theta - n\phi)$$

that simulate turbulence triggered by the displacement current. In the TRAN code, we compute not only the ion confinement time but also the electron confinement time, which enables us to determine the coefficients P_{mn} from quasineutrality (6, 12). Thus, a truncated quasineutrality condition rather than Ohm's law is used to determine Φ.

Numerical calculations show that two-dimensional models do not adequately explain experimental measurements of transport in a plasma. The Monte Carlo method takes into account the complex geometry of the drift surfaces and computes neoclassical transport dominated by complicated banana orbits. In our work, the three-dimensional perturbations of Φ are selected iteratively to impose a truncated version of the quasineutrality requirement. Whenever this is accomplished successfully, it turns out that the numerical results agree well with observations. The physics of the model would be more consistent if we chose the flow velocity $\mathbf{U}$ to ensure conservation of momentum in the collision operator, just as we find the electrostatic potential Φ by imposing quasineutrality. However, that would lead to a far more tedious computation that does not in practice affect the results enough to justify the effort, so we simply put $\mathbf{U} = 0$. Thus the final formulation of our method is based on both theoretical and practical considerations (12).

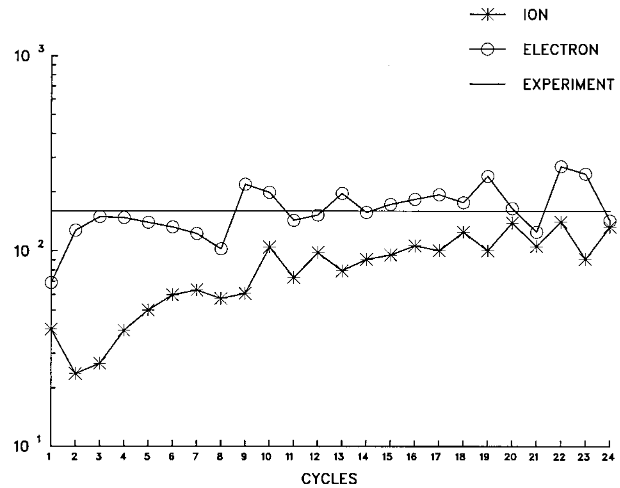

Fig. 5. TRAN calculation of the energy confinement time τ_E in milliseconds for an NBI shot of the LHD experiment using a quasineutrality algorithm to determine the electric potential Φ. Oscillations of Φ along the magnetic lines model turbulence and anomalous transport so that good agreement with the measured value is obtained. If the oscillations are suppressed the computed value increases by a factor of two.

The TRAN implementation of the quasineutrality algorithm is only effective at low collision frequencies such that the first few harmonics P_{mn} suffice to approximate the electrostatic potential Φ. Hence only tokamak data were used in our first comparisons of the theory with experiment. That was perhaps unconvincing because as a model of turbulence, we introduced bifurcated three-dimensional equilibria whose physical relevance was not universally recognized. Recently measurements of confinement time in the LHD stellarator at NIFS have been announced that are at the required level of collisionality, so a more satisfactory comparison becomes feasible.[†] More specifically, we have run the TRAN code to model an observation using NBI (Neutral Beam Injection) heating that produced a peak electron temperature of 3.3 keV and a peak ion temperature of 3.1 keV at density 1.5×10^{13} cm^{-3}. The central magnetic field was 2.8 tesla, the radius of the magnetic axis was 3.6 m, and the small radius of the plasma was 65 cm. Both the experimental measurement and the mathematical computation gave the value $\tau_E = 160$ ms for the energy confinement time, with a margin of error comparable to uncertainties connected with an estimate near 2 of the effective charge number Z, which occurs in TRAN formulas for the collision freqency (12). This preliminary result is displayed in Fig. 5, but a more detailed analysis would be desirable to validate the theory.

We are developing the MHH2 as a reactor concept with large radius 8 m, plasma radius 3.2 m, and a gap of 2 m between the separatrix of the plasma and the filaments defining the modular coils, which are to have relatively small curvature. There is enough flexibility in the system to allow for corrections that may become necessary after more detailed studies of alpha particle confinement and the divertor are made. Monte Carlo simulations of the MHH2 reactor using the TRAN computer code show that the energy confinement time τ_E scales like $\rho_L^{-2.5}$, where ρ_L is the ion gyroradius measured in units of the plasma radius. More specifically, with a magnetic field of 5 tesla in the plasma, average temperature 14 keV and average density 1.4×10^{14} cm^{-3}, we estimate that $\tau_E = 3$ s. For a more conventional stellarator like the LHD the corresponding estimate of τ_E turns out to be significantly smaller so that the size of the reactor becomes quite large. This is because asymmetric Fourier coefficients B_{mn} in the spectrum for a conventional stellarator are so much bigger than those of an optimized helias.

Conclusions

New QAS configurations with two or three field periods called the MHH2 and the QAS3 have been found for a proof of principle stellarator experiment. Our mathematical theory, which captures islands by calculating weak solutions of the MHD (magnetohydrodynamic) equilibrium equations, suggests that these are better candidates than those developed by other methods with less resolution. We have collaborated with NIFS to design a modular stellarator experiment based on the new MHH2 configuration. In recent LHD experiments, very good measurements of the energy confinement time have been reported at low collision frequencies near those of a reactor. Monte Carlo simulations suggest that in the more compact MHH2, a peak temperature as high as 3 keV might be attained at density 10^{13} cm^{-3} for a magnetic field of 1.5 tesla and a major radius of only 1.5 m. To raise the temperature above 5 keV in a conventional stellarator like the LHD may require major heating power, but no comparable difficulty should occur in the MHH2 because two-dimensional quasisymmetry of the magnetic spectrum is predicted to provide good transport at low collisionality.

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