A unified and universal explanation for Lévy laws and $1/f$ noises

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Lévy laws and $1/f$ noises are shown to emerge uniquely and universally from a general model of systems which superimpose the transmissions of many independent stochastic signals. The signals are considered to follow, statistically, a common—yet arbitrary—generic signal pattern which may be either stationary or dissipative. Each signal is considered to have its own random transmission amplitude and frequency. We characterize the amplitude-frequency randomizations which render the system output's stationary law and power-spectrum universal—i.e., independent of the underlying generic signal pattern. The classes of universal stationary laws and power spectra are shown to coincide, respectively, with the classes of Lévy laws and $1/f$ noises—thus providing a unified and universal explanation for the ubiquity of these classes of "anomalous statistics" in various fields of science and engineering.

anomalous statistics | universality | Poissonian randomizations | shot noise

Anomalous statistics are ubiquitously observed in many fields of science and engineering, and their exploration has drawn major interest in recent years by both experimentalists and theoreticians (1–3). In the context of stationary stochastic processes and signals, one can observe either amplitudinal or temporal anomalous statistics.

Amplitudinal anomalous statistics are manifested by wide process fluctuations, and are referred to as the Noah effect (4). Quantitatively, amplitudinal anomalous statistics are characterized by "heavy-tailed" stationary laws (5)—stationary laws whose distribution tails follow asymptotic power-law decay. The quintessential proxy of amplitudinal anomalous statistics is the class of Lévy laws (6, 7). This class of probability laws emerges from the Central Limit Theorem as the universal scaling limits of sums of independent and identically distributed (IID) random variables with infinite variance (8, 9).

Temporal anomalous statistics are manifested by long process memory and are referred to as the Joseph effect (4). Quantitatively, temporal anomalous statistics are characterized by long-range correlations (10, 11)—autocorrelation functions following asymptotic power-law decay. The quintessential proxy of temporal anomalous statistics is the class of $1/f$ noises (12–14)—stationary processes with power-law power spectra.

Stationary stochastic processes and signals are prevalent across many fields of science and engineering. Examples of such processes and signals include the intensity of solar luminosity, the sales of a consumer product, and transmissions sent through communication channels. In many systems, a very large collection of microscopic stationary stochastic inputs are aggregated up to form a macroscopic stationary system output. In the context of the aforementioned examples, consider, respectively, the luminosity of a galaxy, the sales of a large department store, and transmissions sent through a central communication router.

In this research, we consider systems whose outputs are superpositions of many stationary stochastic inputs—all inputs being, statistically, of the same stationary signal pattern but with different amplitudes and frequencies. We further consider the amplitudes and frequencies of the inputs to be random, and address the following universality question: Is it possible to randomize the inputs' amplitudes and frequencies so that the system outputs' stationary law, or power spectrum, be independent of the inputs' stationary signal pattern?

Randomizations rendering the system outputs' stationary laws independent of the inputs' stationary signal patterns are termed "amplitude-universal," and randomizations rendering the system outputs' power spectra independent of the inputs' stationary signal patterns are termed "temporal-universal." In this research, we prove the existence of both amplitude-universal and temporal-universal randomizations and explicitly characterize these randomization classes. We further prove that in the case of amplitude-universal randomizations, the system outputs' stationary laws coincide with the class of Lévy laws, and in the case of temporal-universal randomizations the system outputs' stationary laws coincide with the class of $1/f$ noises. This research thus establishes a unified and universal explanation to the ubiquity of Lévy laws and $1/f$ noises.

The remainder of this article is organized as follows. In Section 2, the aforementioned system model and results are presented in detail. In Section 3, the aforementioned results are shown to hold also in systems whose outputs are superpositions of many dissipative stochastic inputs which appear and vanish randomly in time—all inputs being, statistically, of the same dissipative signal pattern, but with different amplitudes and frequencies. A discussion and a conclusion of the results follows, respectively, in Sections 4 and 5. For proofs, the readers are referred to the *SI Appendix*.

Section 2: The Stationary Superposition Model

Consider a stochastic process $Y = (Y(t))_{-\infty<t<\infty}$ which is a superposition of a countable collection of stationary stochastic signals transmitted from independent transmission sources. Assume that the transmission sources produce statistically identical stochastic signals—albeit with different amplitudes and frequencies—which are IID copies of a generic stationary signal pattern $X = (X(t))_{-\infty<t<\infty}$ with zero mean and short-range correlations.* The stochastic process Y is thus given by

$$Y(t) = \sum_k a_k X_k(\omega_k t) \qquad [1]$$

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*Recall that a stationary stochastic process is said to have short-range correlations if either of the following equivalent conditions hold (10, 11): (*i*) The process's autocovariance is an integrable function (over the real line); or (*ii*) the process's power spectrum is a bounded function (over the real line).

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where (*i*) a_k is the (real-valued) transmission amplitude of source k; (*ii*) ω_k is the (positive-valued) transmission frequency of source k; and (*iii*) $X_k = (X_k(t))_{-\infty<t<\infty}$ is the stochastic signal pattern transmitted by source k—an IID copy of the generic stationary signal pattern X.

The stationary superposition model of Eq. **1** establishes a stochastic map $X \mapsto Y$, which maps the input stationary signal pattern X to the output stochastic process Y. The output is a stationary stochastic process, and hence its amplitudinal statistics are governed by its stationary law, and its temporal statistics are governed by its power spectrum. Our aim is to find and characterize randomizations of the amplitude-frequency pairs $\{(a_k, \omega_k)\}_k$ which render either the stationary law of the output process Y, or its power spectrum, independent of the input signal pattern X.

To illustrate things, consider Eq. **1** as a system whose amplitude-frequency pairs $\{(a_k, \omega_k)\}_k$ are randomized by the system engineer, whereas the signal pattern X is chosen arbitrarily by the system users. The engineer's goal is to design a system whose output statistics (either the stationary law or the power spectrum of the output process Y) are universal, regardless of the choice of the system users. Can this goal be accomplished? And, if yes, what are the universal randomizations, and what are the resulting universal output statistics? These are the questions we address in this research.

Section 2.1: Randomizations. The common randomization method of the amplitude-frequency pairs $\{(a_k, \omega_k)\}_k$ is the "IID randomization"—i.e. setting the parameters to be a sequence of IID random variables. Yet another randomization method is the "Poissonian randomization"—i.e. setting the amplitude-frequency pairs to be the random points of a Poisson process (15). The IID randomization is a special case of the Poissonian randomization, and the later method goes beyond the realm of the former method.

Poissonian randomizations are a major statistical model for the random scattering of points in general domains (15) and have a wide spectrum of applications ranging from insurance and finance (16) to queueing systems (17). In recent years, we applied Poissonian randomizations in various topics in statistical physics—obtaining results which are unattainable by IID randomizations. Examples include nonlinear shot noise systems (18), fractality in the context of random populations (19–21), and statistical resilience of random populations to the action of random perturbations (22).

Henceforth, we consider the amplitude-frequency pairs $\{(a_k, \omega_k)\}_k$ to be the random points of a Poisson process, scattered on the upper half-plane $\mathcal{H} = (-\infty, \infty) \times (0, \infty)$ with intensity $\lambda(x,y)$. Informally, this Poissonian randomization means that a transmission source with amplitude-frequency pair (a, ω) belonging to the infinitesimal rectangle $(x, x+dx) \times (y, y+dy)$ exists with probability $\lambda(x,y)dxdy$. More precisely, this Poissonian randomization means, as defined by ref. 15, that (*i*) the number of transmission sources with amplitude-frequency pairs residing in a subdomain D of the upper half-plane $\mathcal{H}$ is Poisson-distributed with mean $\int\int_D \lambda(x,y)dxdy$; and (*ii*) the number of transmission sources with amplitude-frequency pairs residing in disjoint subdomains of the upper half-plane $\mathcal{H}$ are independent random variables. The Poissonian intensity $\lambda(x,y)$ governs the randomization of the amplitude-frequency pairs $\{(a_k, \omega_k)\}_k$.

Section 2.2: Amplitudinal Universality and Lévy Laws. Consider the stationary law of the output process Y. We seek Poissonian intensities $\lambda(x,y)$ for which the choice of the signal pattern X affects the output's stationary law only by a scale factor—and term such intensities "amplitude-universal." Analysis shows that the intensity $\lambda(x,y)$ is amplitude-universal if and only if the function $\phi(x) = \int_0^\infty \lambda(x,y)dy$ $(-\infty < x < \infty)$ is a power law (see *SI Appendix*). More specifically, if $\phi(x) = c_1|x|^{-1-\alpha}$, then the Fourier transform of the output's stationary law is given by

$$\langle \exp(i\theta Y(t)) \rangle = \exp(-c_2|\theta|^\alpha) \qquad [2]$$

where θ is the (real-valued) Fourier variable and: (*i*) c_1 is a positive-valued coefficient; (*ii*) α is an exponent taking values in the range $0 < \alpha < 2$; and (*iii*) c_2 is a positive-valued coefficient depending on the coefficient c_1, the exponent α, and the signal pattern X.

The Fourier transform of Eq. **2** characterizes the class of Lévy laws (6, 7). Hence, we conclude with the following result: The universal stationary laws of the output process Y—being independent, up to a scale factor, of the input signal pattern X—are Lévy laws. Lévy laws arise from the Central Limit Theorem as the universal scaling limits of sums of IID random variables with infinite variance (8, 9) and are ubiquitously observed in anomalous transport (2, 23, 24). Lévy laws are of infinite variance, and are heavy-tailed (5)—i.e., their distribution tails follow an asymptotic power-law decay with exponent α. Thus, the output process Y, when randomized by amplitude-universal Poissonian intensities, exhibits the Noah effect (4).

The simplest example of amplitude-universal intensities is $\lambda(x,y) = c_1|x|^{-1-\alpha}\delta(y-1)$, where $\delta(\cdot)$ is the Dirac delta function. This example represents the case where the frequencies $\{\omega_k\}_k$ are all set to equal unity ($\omega_k = 1$). Another example is $\lambda(x,y) = c_1|x|^{-1-\alpha}P(y)$, where $P(\cdot)$ is a probability density function defined on the positive half-line. This example represents the case where the frequencies $\{\omega_k\}_k$ are IID random variables whose distribution is governed by the density $P(\cdot)$. An example which couples x and y is $\lambda(x,y) = c_1|x|^{-1-\alpha}Q(x;y)$, where $Q(x;\cdot)$ is (for each $x > 0$) a probability density function defined on the positive half-line. This example represents the case where the frequency ω_k is a random variable whose distribution is conditional on the amplitude a_k and is governed by the density $Q(a_k;\cdot)$. Yet another example that couples x and y is $\lambda(x,y) = c_1 P(|x|^\varepsilon y)|x|^{\varepsilon-1-\alpha}$ where $P(\cdot)$ is a probability density function defined on the positive half-line, and where ε is a real-valued exponent.

Section 2.3: Temporal Universality and $1/f$ Noises. Consider the power spectrum of the output process Y. We seek Poissonian intensities $\lambda(x,y)$ for which the choice of the signal pattern X affects the output's power spectrum only by a scale factor—and term such intensities "temporal-universal." Analysis shows that the intensity $\lambda(x,y)$ is temporal-universal if and only if the function $\psi(y) = \int_{-\infty}^\infty x^2\lambda(x,y)dx$ $(y > 0)$ is a power law (see *SI Appendix*). More specifically, if $\psi(y) = c_3 y^{-\beta}$, then the output's power spectrum is given by

$$\int_{-\infty}^{\infty} \langle Y(0)Y(t) \rangle \exp(ift)dt = \frac{c_4}{|f|^\beta} \qquad [3]$$

where f is the (real-valued) power spectrum variable, and (*i*) c_3 is a positive-valued coefficient; (*ii*) β is an exponent taking values in the range $0 < \beta < 1$; and (*iii*) c_4 is a real-valued coefficient depending on the coefficient c_3, the exponent β, and the signal pattern X.

The power spectrum of Eq. **3** characterizes the class of $1/f$ noises ("flicker noises") (12–14). Hence, we conclude with the following result: The universal power spectra of the output process Y—being independent, up to a scale factor, of the input signal pattern X—are $1/f$ noise power spectra. The autocovariance functions corresponding to the power spectra of Eq. **3** are "long-range correlated" (10, 11)—following a power-law decay with exponent $1-\beta$. Thus, the output process Y, when randomized by temporal-universal Poissonian intensities, exhibits the Joseph effect (4).

The output process Y, when randomized by temporal-universal Poissonian intensities, is an irregular stochastic process. Its underlying superposition-aggregate—the right-hand side of Eq. **1**—is divergent, and this is why it is referred to as noise, rather than as process. However, the corresponding integrated output process $Z = (Z(t))_{t\geq 0}$—given by $Z(t) = \int_0^t Y(t')dt'$ $(t \geq 0)$—is a regular stochastic process. Moreover, analysis shows that the mean square displacement of the integrated output process follows a superlinear power-law growth (see *SI Appendix*)

$$\langle Z(t)^2 \rangle = c_5 t^{1+\beta} \quad [4]$$

($t \geq 0$), where c_5 is a positive-valued coefficient depending on the coefficient c_4 and on the exponent β. Superlinear power-law growth of the mean square displacement is the hallmark of superdiffusive transport (1, 25). As an illustrative comparison, consider the case of diffusion where: (*i*) Y is a white noise, an irregular stochastic process; (*ii*) Z is a Brownian motion, a regular stochastic process; and (*iii*) the diffusion mean square displacement $\langle Z(t)^2 \rangle$ follows a linear growth rather than a super-linear power-law growth.

The simplest example of temporal-universal intensities is $\lambda(x,y) = c_3\delta(x-1)y^{-\beta}$, where $\delta(\cdot)$ is the Dirac delta function. This example represents the case where the amplitudes $\{a_k\}_k$ are all set to equal unity ($a_k = 1$). Another example is $\lambda(x,y) = c_3 P(x) y^{-\beta}$, where $P(\cdot)$ is a probability density function defined on the real line with a second moment equaling unity. This example represents the case where the amplitudes $\{a_k\}_k$ are IID random variables whose distribution is governed by the density $P(\cdot)$. An example which couples x and y is $\lambda(x,y) = c_3 P(xy^{\varepsilon}) y^{3\varepsilon-\beta}$ where $P(\cdot)$ is a probability density function defined on the real line with a second moment equaling unity, and where ε is a real-valued exponent.

Section 3: The Dissipative Superposition Model

So far, we have considered the superposition of stationary stochastic processes. However, the universal emergence of Lévy laws and $1/f$ noises—as established for the stationary superposition model of Section 2—holds also in the case of dissipative systems.

Consider a stochastic process $Y = (Y(t))_{-\infty<t<\infty}$ which is a superposition of a countable collection of independent dissipative stochastic signals initiating and vanishing randomly in time. Assume that the dissipative stochastic signals are IID copies—albeit with different initiation epochs, amplitudes, and frequencies—of a generic dissipative signal pattern $X = (X(t))_{t\geq 0}$ with zero mean.[†] The stochastic process Y is thus given by

$$Y(t) = \sum_{\tau_k \leq t} a_k X_k(\omega_k(t-\tau_k)) \quad [5]$$

where (*i*) τ_k is the (real-valued) initiation epoch of signal k; (*ii*) a_k is the (real-valued) amplitude of signal k; (*iii*) ω_k is the (positive-valued) frequency of signal k; and (*iv*) $X_k = (X_k(t))_{t\geq 0}$ is the stochastic pattern of signal k—an IID copy of the generic dissipative signal pattern X.

The superposition model of Eq. **5** is the dissipative analogue of the stationary superposition model of Eq. **1**—replacing the always-present stationary signal of source k (in Eq. **1**) by a dissipative signal which initiates at the random time τ_k and thereafter decays to zero in Eq. **5**. The stationary superposition model of Section 2 represents a static setting in which the collection of transmission sources—which produce the superimposed output process Y—is fixed and unchanging. The dissipative superposition model of this section represents a dynamic setting in which the superimposed signals appear and vanish randomly as time progresses.

The dissipative superposition model can be illustrated as a general shot noise system in which external shocks impact the system randomly in time and thereafter dissipate: Shock k hits the system at time τ_k with amplitude a_k. After impact, shock k decays with frequency ω_k according to the dissipative signal pattern X_k. We note that the special case of deterministic dissipative signal patterns—i.e., $X(t) = h(t)$ ($t \geq 0$), where $h(\cdot)$ is an "impulse-response function" decaying to zero (26)—corresponds to the class of linear shot noise systems (27–29).

As in the stationary superposition model of Section 2, we consider a Poissonian randomization of the parameters $\{(\tau_k, a_k, \omega_k)\}_k$. Namely, $\{(\tau_k, a_k, \omega_k)\}_k$ are considered the random points of a Poisson process, scattered on the half-space $(-\infty,\infty) \times (-\infty,\infty) \times (0,\infty)$ with a time-homogeneous intensity $\lambda(t,x,y) = \eta(x,y)$. In the shot noise illustration, this Poissonian randomization means that shots with amplitude a and frequency ω arrive to the system, randomly in time, according to the time-homogeneous Poissonian rate $\eta(a,\omega)$.

The dissipative superposition model of Eq. **5** establishes a stochastic map $X \mapsto Y$, which maps the input dissipative signal pattern X to the output stochastic process Y. Moreover, the Poissonian randomization of the parameters $\{(\tau_k, a_k, \omega_k)\}_k$—governed by the time-homogeneous Poissonian rate $\eta(x,y)$—renders the output Y a stationary stochastic process.

The definitions of amplitudinal universality and temporal universality—in the context of the dissipative superposition model of Eq. **5**—are identical to the stationary superposition model of Section 2. We define the Poissonian inflow rate $\eta(x,y)$ as (*i*) "amplitude-universal" if the choice of the dissipative signal pattern X affects the output's stationary law only by a scale factor; and (*ii*) "temporal-universal" if the choice of the dissipative signal pattern X affects the output's power spectrum only by a scale factor.

The stationary superposition model results of the previous section remain valid for the dissipative superposition model of this section—once replacing the Poissonian intensity $\lambda(x,y)$ by the function $\eta(x,y)/y$. Specifically (see *SI Appendix*), the result of Eq. **2**—amplitudinal universality of Lévy laws—holds with $\phi(x) = \int_0^\infty [\eta(x,y)/y]dy$ ($-\infty < x < \infty$), and the result of Eq. **3**—temporal universality of $1/f$ noise power spectra—holds with $\psi(y) = \int_{-\infty}^\infty x^2[\eta(x,y)/y]dx$ ($y > 0$).

Section 4: Discussion

Both the stationary superposition model of Section 2 and the dissipative superposition model of Section 3 are prevalent in systems carrying heavy information traffic. Examples include throughputs in major transmission and communication channels and loads on large data and communication servers. For the sake of illustration, if the stochastic process Y represents the load level on a given server, then (*i*) in the stationary superposition model, the indexing k represents a fixed population of customers connected to the server—e.g., the subscribers of a large Internet provider, and (*ii*) in the dissipative superposition model, the indexing k represents demands arriving randomly to the server—e.g., the entries to a popular Internet site.

A substantial body of empirical evidence indicates that Internet-age information traffic displays both amplitudinal and temporal anomalous statistics (30–33). Trying to explain the long-range correlations observed in information traffic, a superposition model of "on-off sources" was introduced and analyzed in (34). An on-off source produces a signal which is an alternating renewal process with values 1 and 0—the value 1 representing transmission and the value 0 representing no transmission. The scaling limits of aggregates of independent on-off sources (with infinite variance) were shown to converge, stochastically, to fractional Brownian noise (35), hence displaying the Joseph effect.

The on-off model presented in (34) is one among a class of superposition models leading to fractional Brownian noise and the Joseph effect. This class of models considers the stochastic scaling limits of aggregates of the form

$$Y_n(t) = \sum_{k=1}^{n} X_k(t) \quad [6]$$

PHYSICS

[†] By "dissipative" we mean that the signal pattern X decays to zero stochastically, at a sufficiently rapid pace, as $t \to \infty$. The precise definition is given in the *SI Appendix*.

$(t \geq 0)$, where the summands $X_k = (X_k(t))_{t\geq 0}$ $(k = 1, 2, \ldots)$ are IID copies of a generic signal process $X = (X(t))_{t\geq 0}$. In this class, each model deals with a specific underlying generic signal X—e.g., on-off processes (34), renewal processes (36, 37), persistent random walks (38), and Ornstein–Uhlenbeck processes (39).

The model established in this research is fundamentally different of the aforementioned superposition model: considering the randomization (via random transmission amplitudes and frequencies) of the superimposed signals rather than their stochastic scaling limits; considering arbitrary underlying generic signal processes rather than a specific one; and seeking amplitudinal-universality and temporal-universality rather than setting as goal to obtain a fractional Brownian noise scaling limit.

In this research, we sought randomizations of the transmission amplitude-frequency pairs that yield output statistics which are independent of the input signals (X). To that end, we had to allow for arbitrary system inputs and had no a priori idea what universal statistics, if any, would turn out. At no stage have we aimed at obtaining Lévy laws and $1/f$ noises, let alone obtaining them from a superposition of a specific input signal. Rather, Lévy laws and $1/f$ noises emerged as the answers to the amplitudinal-universality and temporal-universality questions we embarked from.

The exponent β of the $1/f$ noises we obtained—in both the stationary superposition model of Section 2 and the dissipative superposition model of Section 3—takes values in the range $0 < \beta < 1$. In real life however, the exponent of observed $1/f$ noises is typically in the range $0 < \beta < 2$ (40). This discrepancy stems from a couple of reasons: (*i*) Our model considers stationary output stochastic processes; and (*ii*) the definition we use for the power spectrum of the stationary output stochastic process Y is the Fourier transform of its autocovariance function (Eq. **3**). Indeed, the power-spectra definition we applied allows only for $1/f$ noises with exponent in the range $0 < \beta < 1$. To obtain $1/f$ noises with exponent in the range $0 < \beta < 2$, nonstationary stochastic processes need be considered as well as a different definition of power spectra.[‡] In forthcoming research (41), we consider the universal generation of statistically self-similar stochastic processes, using a Poissonian parameters-randomization analogous to the one applied in the dissipative superposition model of *Section 3: The Dissipative Superposition Model*. As a consequence, the research (41) establishes a universal mechanism for the generation of $1/f$ noises with an exponent in the range of $\beta > 1$–the "nonstationary side" of the $1/f$ noise regime.

Section 5: Conclusions

In this letter, we considered countable superpositions of independent stochastic signals in which each signal has its own amplitude and frequency, and all signals share a statistically generic signal pattern X. The setting could be either static or dynamic. In the former setting, the signals are stationary processes transmitted by a static collection of transmission sources. In the latter setting, the signals are dissipative processes appearing and vanishing randomly and dynamically as time progresses. Both settings generate an aggregate superimposed system output Y, which is a stationary stochastic process.

Considering Poissonian randomizations of the amplitudes and frequencies of the superimposed signals, we sought randomizations yielding system outputs Y whose statistics are independent of the signal pattern X. Analysis showed that (*i*) the only amplitude-universal statistics—outputs' stationary laws which are independent, up to a scale factor, of the signal pattern X—are Lévy laws, and (*ii*) the only temporal-universal statistics—outputs' power spectra which are independent, up to a scale factor, of the signal pattern X—are $1/f$ noise power spectra. We emphasize that both results are unattainable by IID randomizations.

The signal pattern X represents the microscopic structure of the superposition system, whereas the process Y represents the system's macroscopic structure. The results obtained assert that Lévy laws and $1/f$ noises are, respectively, the unique amplitudinal and temporal statistics which are universal—i.e., attainable at the system's macro level regardless of the system's microstructure. This research provides a unified and universal explanation to the ubiquity of Lévy laws and $1/f$ noises—the quintessential proxies of anomalous statistics—in a wide array of natural and engineered systems.

[‡]Specifically, the power spectrum of a general finite-variance stochastic process $\xi = (\xi(t))_{t\geq 0}$ is given by the definition $S_\xi(f) := \lim_{T\to\infty} \frac{1}{T}\langle|\int_0^T \exp(ift)\xi(t)dt|^2\rangle$ (f real).

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