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Two Modern Developments in Matching Theory

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Matching theory is a mathematical theory of choice behavior, parts of which have been shown to hold in natural human environments and to have important therapeutic applications. Two modern developments in matching theory are discussed in this article. The first is the mathematical description of behavior in asymmetrical choice situations, which are situations where different reinforcers and/or different behaviors are associated with concurrently available response alternatives. Most choice situations in natural human environments are probably asymmetrical. The second development in matching theory is the mathematical description of a tendency toward indifferent responding in all choice situations. Behavior in asymmetrical choice situations and the tendency toward indifferent responding in all choice situations can be described by modifications of the matching equations, which change the equations from lines into power functions. These modern forms have been extraordinarily successful in describing behavior in choice situations, and are the forms most likely to accurately describe human behavior in naturally occurring environments.

Key words: choice behavior, applied science, mathematical theory, asymmetrical situations, natural environments

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Matching theory is a mathematical theory of choice behavior. It is based on laboratory experimentation, most of which has involved nonhuman organisms exhibiting well-defined behaviors in controlled environments. Twenty years after pigeons' keypecking was first found to conform to the matching equation (Herrnstein, 1961), it was noted in the literature that matching theory was relevant to the concerns of applied behavior analysts (McDowell, 1981). Since then, several authors have commented on the importance of matching theory for understanding human behavior, and for developing effective therapeutic interventions (Epling & Pierce, 1983; McDowell, 1982; Myerson & Hale, 1984). In spite of these efforts, matching theory remains relatively unknown among applied behavior analysts.

The purpose of the present article is to continue an earlier attempt to explain matching theory to applied scientists. In a previous article (McDowell, 1988) the rudiments of the theory were presented, its conceptual significance was discussed, and laboratory research supporting the theory was reviewed. Applied research bearing on matching theory's validity in natural human environments was also reviewed, as were therapeutic applications of the theory. In the earlier article, the theory was presented in a form that was current in the early 1970s. But in the past 20 or so years a large amount of research had led to important changes in matching theory. Two of these changes are the topic of the present article. Before discussing them, the basics of matching theory will be reviewed.

REVIEW OF MATCHING THEORY

When an individual is able to exhibit a variety of behaviors and in fact exhibits one behavior to the exclusion of the others, then the individual is said to have made a choice. As an example, consider an accountant who may read a newspaper instead of working on a client's tax return, or instead of preparing a presentation for an upcoming meeting, or instead of phoning the Internal Revenue Service to check on the progress of an audit.

Choice is usually continuous, which means that at any moment an individual may stop engaging in a particular behavior and start engaging in some other behavior. For example, at any time the accountant may put down the news-

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paper and start working on the client's taxes; while working on the taxes, the accountant may at any moment return to reading the newspaper, and so on. In addition, there are often occasional consequences for engaging in the various behaviors. For example, one of the accountant's partners may make a critical remark about reading the paper on company time. The interesting problem is to determine how the accountant's distribution of behavior across the available response alternatives is governed.

Choice behavior is studied in the laboratory by means of concurrent schedules of reinforcement. A concurrent schedule consists of two or more individual, or component, schedules that are available to the organism at the same time. This type of schedule can be arranged in a pigeon chamber that has two keys to peck. The pigeon's pecking on one key is analogous to the accountant reading the newspaper, and its pecking on the other key can be considered analogous to the accountant doing anything other than reading the newspaper. Choice is continuous in this laboratory procedure because at any moment the pigeon may stop pecking one key and start pecking the other. Occasional consequences for pecking each key can be arranged by programming intermittent schedules of reinforcement on each key. Again, the interesting problem is to determine how the pigeon's distribution of behavior across the available response alternatives is governed.

Basic research has answered the question of how choice behavior is governed. Organisms distribute their behavior among concurrently available response alternatives in the same proportion that reinforcements are distributed among the alternatives. Take the simplest case of a pigeon working on a two-key concurrent schedule. If R_1 represents the rate of pecking the first key, R_2 represents the rate of pecking the second key, and r_1 and r_2 represent the rates of reinforcement obtained from pecking each key, then this answer can be written algebraically as

$$\frac{R_1}{R_1 + R_2} = \frac{r_1}{r_1 + r_2}. \qquad (1)$$

Equation 1 is known as the matching equation, and is so called because the proportion of responses on a given alternative (the left side of the equation) equals, or matches, the proportion of reinforcements obtained from that alternative (the right side of the equation). This relationship also holds if time spent responding, rather than response rate, is measured. If T_1 represents the time spent responding on the first alternative and T_2 represents the time spent responding on the second alternative, then the matching equation can be written as

$$\frac{T_1}{T_1 + T_2} = \frac{r_1}{r_1 + r_2}. \qquad (2)$$

Equation 2 is sometimes said to represent time-allocation matching (Baum & Rachlin, 1969). Time-allocation matching greatly enhances the applicability of the account because, unlike brief key pecks and lever presses, many behaviors extend in time and cannot be partitioned easily into units for counting (which is necessary in order to determine their rate of occurrence). Reading the newspaper and working on tax returns are examples of behaviors that extend in time.

Equations 1 and 2 constitute a deterministic, mathematical account of choice behavior. The accuracy of forms of these equations has been confirmed extensively in laboratory experiments with pigeons, rats, crows, cows, squirrel monkeys, and humans, whose key pecking, lever pressing, treadle pressing, chain pulling, standing, talking, and macrosaccadic eye movements were reinforced with food, water, intracranial brain stimulation, shock avoidance, money, opportunities to listen to comedy records, or opportunities to view sexually interesting slides (McDowell, 1988). Forms of Equations 1 and 2 typically accounted for about 90% of the data variance in experiments like these (McDowell, 1988). In other words, the equations have been found to be valid and precise descriptions of choice behavior.

In a very influential paper, Herrnstein (1970) extended the applicability of the matching equations, and in so doing greatly increased their significance. He argued that all behavior could be conceptualized as choice behavior. Obviously, an organism can exhibit a variety of behaviors in any environment. Even in a one-key experimental chamber, a pigeon is not limited to key pecking; it may also preen, flap its wings, roost, and so on. It will be helpful to consider this one-key chamber more carefully. The rate of pecking the key can be represented by R, and the aggregate rate of exhibiting behaviors other than pecking the key can be represented by R_e. In other words, if we are interested in key pecking as the target behavior, then R_e represents the aggregate rate of all "extraneous," or nontarget, behavior. The rate of reinforcement obtained for pecking the key can be represented by r, and the aggregate rate of reinforcement obtained for exhibiting the extraneous behaviors can be represented by r_e. Equation 1, the matching equation, can be written for a one-key environment conceptualized in this way as

$$\frac{R}{R + R_e} = \frac{r}{r + r_e}. \qquad (3)$$

The two alternatives are pecking the key, and doing anything other than peck the key. Herrnstein (1970) assumed that the total rate of behavior, which includes the rate of key pecking and the rate of exhibiting all extraneous behaviors, was constant. In other words, he assumed that $R + R_e = k$, where k is a constant. If k is substituted for $R + R_e$, the above equation becomes

$$\frac{R}{k} = \frac{r}{r + r_e},$$

and solving for R, the rate of key pecking, yields

$$R = \frac{kr}{r + r_e}. \qquad (3)$$

Equation 3 expresses the absolute rate, R, of the target behavior as a hyperbolic function of the absolute rate, r, of contingent reinforcement obtained for exhibiting the target behavior. This equation, which has been referred to as Herrnstein's hyperbola, is concave downward in the first quadrant, approaches k asymptotically, and approaches k more quickly the smaller the value of r_e (McDowell, 1988). Equation 3 has been referred to as a quantitative statement of the law of effect (e.g., de Villiers, 1977) because it expresses quantitatively the relationship between behavior (R) and reinforcement (r).

Just as Equation 3 was developed from Equation 1, a time-allocation form can be developed from Equation 2. The steps in the calculation are the same. The result is

$$T = \frac{kr}{r + r_e}, \qquad (4)$$

where T represents the time spent engaging in the target behavior, and $k = T + T_e$, where T_e represents the aggregate amount of time spent engaging in all extraneous (i.e., nontarget) behaviors.

Equations 3 and 4 take the straightforward mathematical descriptions of data embodied in Equations 1 and 2 and turn them into a theory. The theory depends on conceptualizing all behavior as choice behavior to which the matching equations (Equations 1 and 2) apply, and on assuming that the total rate of behavior, $R + R_e$, is constant.

Equations 3 and 4 are especially important because they entail a novel understanding of the effects of reinforcement on behavior. The equations assert that behavior is determined not only by contingent reinforcement (r), but also by all other reinforcement provided by the environment (r_e). For example, Equation 3 asserts that the rate of a target behavior (R) will change when r_e changes, even though the rate of reinforcement contingent on the target behavior (r) has not changed. More specifically, when r_e increases (i.e., when extraneous reinforcement is added to the environment), Equation 3 requires the rate of the target behavior to decrease, and when r_e decreases (i.e., when extraneous reinforcement is withdrawn from the environment), Equation 3 requires the rate of the target behavior to increase. According to matching theory, the effect of contingent reinforcement can be understood only in terms of the overall context of reinforcement in which it occurs.

The hyperbolic form of Equation 3 has been extensively confirmed in laboratory experiments with human and nonhuman subjects (McDowell, 1988). The equation typically accounts for a large percentage of the variance in the response rate data. The interpretation of r_e as the rate of extraneous reinforcement has also been confirmed. That is, experiments have shown that when the rate of reinforcement contingent on a target behavior is held constant, the rate of the target behavior varies with the rate of extraneous reinforcement in the manner required by Equation 3 (McDowell, 1988).

As discussed in detail in the earlier article (McDowell, 1988), available evidence indicates that matching theory holds in natural human environments as well as in the laboratory, and that it has useful therapeutic applications.

Having reviewed the basics of matching theory, it is now possible to understand modern developments in the theory. Two such developments will be discussed here. The first deals with the mathematical description of behavior in asymmetrical choice situations, and the second deals with the mathematical description of indifferent responding.

ASYMMETRY

The data that originally led to the matching equation (Equation 1) were obtained from pigeons working on two-key concurrent schedules in standard experimental chambers (Herrnstein, 1961). The two response alternatives were identical. One key was located on the left side of the chamber, the other was located on the right side. Reinforcements arranged for responding on the two keys were also the same. Each reinforcement, whether for a peck on the left key or for a peck on the right key, consisted of a brief presentation of a hopper filled with grain. This type of concurrent schedule could be said to arrange a choice situation that is symmetrical across response alternatives. A symmetrical choice situation is one where the behaviors required on each alternative are the same, and the reinforcements obtained for responding on each alternative are the same.

What happens when a choice situation is asymmetrical? Asymmetrical choice situations can be arranged by, for example, making one key harder to peck than the other (a quantitative difference between the two response alternatives), or requiring key pecking as one of the response alternatives and treadle pressing as the other (a qualitative difference). Asymmetrical choice situations can also be arranged by manipulating the parameters of reinforcement. For example, the food hopper could be presented for a longer period of time on one response alternative than on the other (a quantitative difference between reinforcers), or buckwheat could be used as the reinforcer on one response alternative and wheat could be used as the reinforcer on the other (a qualitative difference).

It seems likely that asymmetrical choice situations would be the rule in natural human environments. In the case of the accountant, for example, reading a newspaper and working on a tax return are qualitatively different behaviors. Another example is a classroom where correct arithmetic performance is reinforced with tokens and disruptive behavior is inadvertently reinforced with attention from the teacher. In this concurrent schedule the behaviors differ qualitatively and the reinforcers differ qualitatively.

Distortion of Matching Produced by Violations of Symmetry

Research has shown that violations of symmetry distort the matching relationship. Fortunately, the distortion can be described mathematically and, as will be explained later, when this description is superimposed on Equations 1 and 2, highly accurate accounts of the data are again obtained.

The top panel of Figure 1 shows how the distortion produced by violations of symmetry appears in data from concurrent schedules. The response proportion, which is the left side of Equation 1, is plotted along the y axis, and the reinforcement proportion, which is the right side of Equation 1, is plotted along the x axis. The heavy line, which is a plot of Equation 1, represents perfect matching. Notice that Equation 1 is a simple mathematical form, namely, a straight line with y intercept equal to zero and slope equal to unity. In other words, Equation 1 is a specific instance of the general line, $y = mx + c$, where, in this instance, y represents the response proportion, x represents the reinforcement proportion, $c = 0$, and $m = 1$. When choice is symmetrical, plots of observed response proportions against obtained reinforcement proportions fall along the straight matching line. When symmetry is violated, the plots bow away from the matching line (Baum, 1974), as indicated by the curves in the top panel of Figure 1.

Violations of symmetry are said to bias responding. If the data fall along an upward-bowing curve (using the coordinates of the top panel of Figure 1), then responding is said to be biased in favor of the first response alternative. Matching requires the proportion of responses to equal the proportion of reinforcements: if a pigeon obtains 25% of its reinforcements from the first alternative, then it must allocate 25% of its pecks to that alternative. But if responding is biased in favor of the first alternative, the bird will allocate more than 25% of its pecks to that alternative, as shown by the upward-bowing curve. The downward-bowing curve in the top panel of Figure 1 shows where the data would fall if there were a smaller amount of bias and if it were in favor of the second alternative. To say that responding is biased in favor of a particular alternative is to say that regardless of the reinforcement proportion obtained from that alternative (with the restriction that it be greater than zero and less than unity), more behavior is allocated to it, and less behavior is allocated to the other alternative, than required by matching. As the bias in favor of one alternative increases, the bowing of the data away from the matching line becomes more extreme.

The downward-bowing curve in the top panel of Figure 1 might represent data

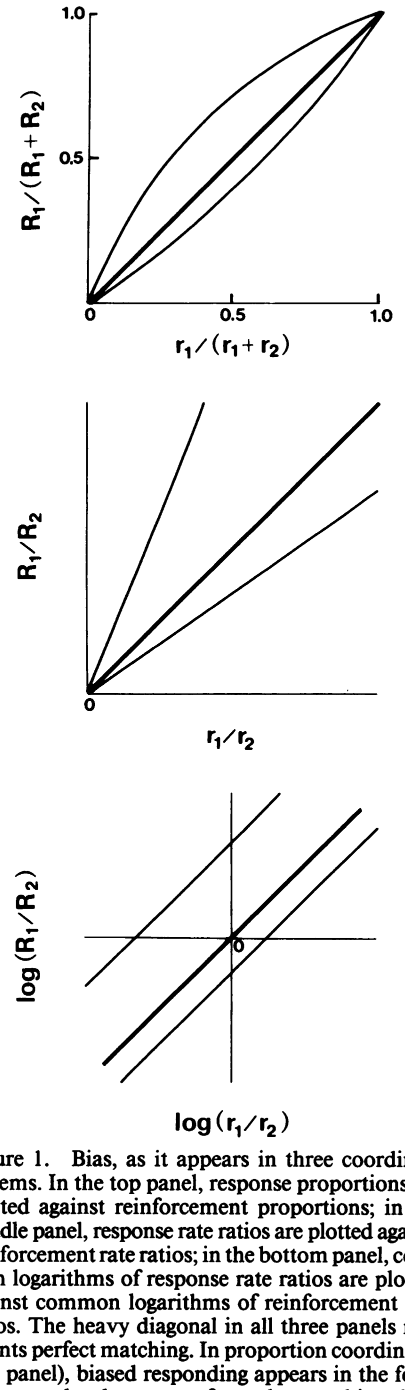

Figure 1. Bias, as it appears in three coordinate systems. In the top panel, response proportions are plotted against reinforcement proportions; in the middle panel, response rate ratios are plotted against reinforcement rate ratios; in the bottom panel, common logarithms of response rate ratios are plotted against common logarithms of reinforcement rate ratios. The heavy diagonal in all three panels represents perfect matching. In proportion coordinates (top panel), biased responding appears in the form of curves that bow away from the matching diagonal. In ratio coordinates (middle panel), biased responding appears in the form of lines with varying slopes and constant intercepts equal to zero. In logarithmic coordinates (bottom panel), biased responding appears in the form of lines with varying intercepts and constant slopes equal to unity.

from a choice situation where the first key is harder to peck than the second. This would produce biased responding in favor of the second key across the entire range of reinforcement proportions. The response portion on the first key would still vary with the reinforcement proportion obtained from that key, but it would vary more sluggishly, as the sagging curve indicates. The upward-bowing curve in the top panel of Figure 1 might represent data from a choice situation where the duration of the hopper presentation is longer on the first response alternative than on the second. This would produce biased responding in favor of the first alternative across the entire range of reinforcement proportions.

Mathematical Description of Biased Responding

The mathematical description of the distortion produced by violations of symmetry is problematic. Exactly what kind of curve is this? Rather than attempting to select some arbitrary function form, matching researchers have solved the problem of describing the distortion by reexamining the matching equations themselves (Baum, 1974; Staddon, 1968). It turns out that Equation 1, which is reproduced here for convenience,

$$\frac{R_1}{R_1 + R_2} = \frac{r_1}{r_1 + r_2}, \tag{1}$$

can be expressed in a form that permits a straightforward description of the distortion produced by violations of symmetry.

It is well known that if a legitimate operation is performed on one side of an equation, the same operation must be performed on the other side of the equation in order to preserve the equality. Hence, we may take the reciprocal of one side of Equation 1, provided that we also take the reciprocal of the other side:

$$\frac{R_1 + R_2}{R_1} = \frac{r_1 + r_2}{r_1}.$$

Separating the terms on each side of this expression yields

$$\frac{R_1}{R_1} + \frac{R_2}{R_1} = \frac{r_1}{r_1} + \frac{r_2}{r_1}.$$

Notice that the first term on the left side of this equation and the first term on the right side are both equal to unity:

$$1 + \frac{R_2}{R_1} = 1 + \frac{r_2}{r_1}.$$

Subtracting unity from both sides of the equation leaves

$$\frac{R_2}{R_1} = \frac{r_2}{r_1},$$

and taking the reciprocal of both sides gives

$$\frac{R_1}{R_2} = \frac{r_1}{r_2}. \tag{5}$$

Equation 5 expresses the matching relationship in terms of ratios of response and reinforcement rates, whereas Equation 1 expresses it in terms of proportions of responses and reinforcements. These two methods of expressing the matching relationship are equivalent, as the above calculation demonstrates. That is, if Equation 1 is true, then Equation 5 must also be true. Notice that, like Equation 1, Equation 5 is a straight line, $y = mx + c$, with y intercept (c) equal to zero and slope (m) equal to unity. The difference is that for Equation 1, y and x represent proportions of responses and reinforcements, whereas for Equation 5 they represent ratios of response and reinforcement rates.

The middle panel of Figure 1 shows the matching relationship plotted in terms of response and reinforcement rate ratios. The heavy straight line is Equation 5, which can be referred to as the ratio form of the matching equation. The advantage of using this form is that distortions of the ratio form that are produced by violations of symmetry tend to fall along straight lines, rather than curved lines, and this simplifies their mathematical description. As illustrated in the figure, when the matching relationship is expressed in ratio form, the distorted relationship remains a straight line with y intercept equal to zero, but with slope not equal to unity. When bias is in favor of the first alternative, the slope is greater than unity, and when bias is in favor of the second alternative, the slope is less than unity (in the coordinates of the middle panel of Figure 1). Hence, responding in symmetrical choice situations, and in all possible asymmetrical choice situations, can be described by the line

$$\frac{R_1}{R_2} = b\frac{r_1}{r_2}, \tag{6}$$

where b, which is the slope of the line, can be referred to as the bias parameter. If the choice situation is symmetrical, $b = 1$, and perfect matching (Equation 5) holds. If the choice situation is asymmetrical, then the bias parameter takes on some value other than unity.

Just as Equation 6 was obtained from Equation 1, it is possible to obtain a time-allocation form from Equation 2. The steps in the calculation are the same. The result is

$$\frac{T_1}{T_2} = b\frac{r_1}{r_2}. \tag{7}$$

To understand Equations 6 and 7 fully, it may be helpful to consider an example. Suppose that a student's disruptive and on-task behavior in a classroom are both reinforced by attention from the teacher. This is a concurrent schedule where one alternative is disruptive behavior, the other is on-task behavior, and responding on both alternatives is reinforced intermittently by attention from the teacher. Choice is asymmetrical in this case because the two behaviors are different. If strict matching held, the proportion of time spent engaging in disruptive behavior would equal the proportion of reinforcements obtained from disruptive behavior, and the proportion of time spent engaging in on-task behavior would equal the proportion of reinforcements obtained from on-task behavior (Equation 2). But suppose the ratio of the times spent engaging in the two behaviors (where the first alternative is disruptive behavior) is plotted against the ratio of reinforcement rates obtained for engaging in the two behaviors, and the data points are found to fall along a line with y intercept equal to zero and slope greater than one (similar to the steepest line in the middle panel of Figure 1). This would mean that the student's responding was governed by matching but was biased in favor of the disruptive behavior. Fitting a straight line to the data and obtaining its slope would give an estimate of the bias (i.e., the value of b). Bias in favor of the disruptive behavior means, among other things, that the student would spend more time engaging in disruptive behavior than in on-task behavior, all else being equal.

Logarithmic Forms

Equations 6 and 7 represent a straightforward solution to the problem of describing the matching relationship distorted by violations of symmetry. These equations can also be expressed in logarithmic forms, which are especially useful. Before discussing these forms, some of the properties of common logarithms will be reviewed.

A common logarithm is an alternative method of expressing a number. Every number has a common logarithm. For example, the common logarithms, or common logs, of 1, 10, 100, and 1,000 are 0, 1, 2, and 3 respectively. The common logs of every number between 1 and 10 lie between 0 and 1; the common logs of every number between 10 and 100 lie between 1 and 2, and so on. The advantage of using logarithms is that they simplify arithmetic operations. Specifically, logarithms change multiplication and division into addition and subtraction. Consider the product, $10 \times 100 = 1{,}000$. This product can be expressed in logarithms as the sum $1 + 2 = 3$. That is, the common log of 10 (1), plus the common log of 100 (2), equals the common log of 1,000 (3). As this example illustrates, numbers can be multiplied by adding their logarithms and finding the number that corresponds to the sum. Stated another way, the logarithm of a product is the sum of the logarithms of its separate factors.

The exact relationship between a number, N, and its common logarithm, log

N, is $N = 10^{\log N}$. This is the definition of a logarithm and can be stated in words as follows: the common logarithm of any number, N, is the power to which 10 must be raised in order to obtain that number. For example, 2 is the common log of 100 because 10 must be raised to the second power to obtain 100. Similarly, 3 is the common log of 1,000 because 10 must be raised to the third power to obtain 1,000. The definition of common logarithms is built into most hand-held calculators. The "log" button applies the definition to find the log of the number that appears in the display, and the "10^x" button applies the definition in the other direction to find the number that corresponds to the log that appears in the display.

Taking the common logs of both sides of Equation 6 gives

$$\log\left(\frac{R_1}{R_2}\right) = \log\left(b\frac{r_1}{r_2}\right).$$

Notice that the right side of this expression is the logarithm of a product. We have just seen that the logarithm of a product is equal to the sum ot the logarithms of its individual factors. Hence, we can write this expression as

$$\log\left(\frac{R_1}{R_2}\right) = \log\left(\frac{r_1}{r_2}\right) + \log(b). \quad (8)$$

Equation 8 is the logarithmic form of Equation 6. Like Equation 6 it is an instance of the general straight line, $y = mx + c$. There are three differences between the lines represented by Equations 8 and 6. First, in Equation 8, y and x represent the logarithms of the response and reinforcement rate ratios, whereas in Equation 6, y and x represent the response and reinforcement rate ratios themselves. Second, the slope of Equation 8 is unity, whereas the slope of Equation 6 is the bias parameter, b. And third, the y intercept of Equation 8 is the logarithm of the bias parameter, whereas the y intercept of Equation 6 is zero. To summarize, Equation 8 has a constant slope equal to unity, and describes bias by changes in its y intercept, whereas Equation 6 has a constant y intercept equal to zero, and describes bias by changes in its slope.

Just as Equation 8 was obtained by taking the logarithms of both sides of Equation 6, Equation 6 can be reobtained from Equation 8 by reversing this process. Reversing the process means finding the numbers to which the common logarithms correspond. As already noted, a common log and the number, N, to which it corresponds are related by $N = 10^{\log N}$; hence, we must "exponentiate" in order to obtain the number. That is, we must raise 10 to a power equal to the common log. If both sides of Equation 8 are exponentiated, we have

$$10^{\log(R_1/R_2)} = 10^{\log(r_1/r_2)+\log(b)}.$$

The left side of this equation is equal to R_1/R_2, but the exponent on the right is the sum of two terms. The right side can be simplified by recalling one of the rules of exponents, namely, that $10^{c+d} = 10^c 10^d$. This rule can be expressed in words as follows: when multiplying numbers with the same base (in this case, 10), add the exponents. For example, $10^{1+2} = 10^1 10^2$, that is, $1{,}000(10^{1+2})$ equals $10(10^1)$ times $100(10^2)$. Applying this rule to the above expression produces

$$10^{\log(R_1/R_2)} = 10^{\log(r_1/r_2)}10^{\log(b)}.$$

The definition of common logarithms allows us to rewrite this as

$$\frac{R_1}{R_2} = b\frac{r_1}{r_2},$$

which is Equation 6. Hence, Equation 8 can be obtained by taking the logarithms of both sides of Equation 6, and Equation 6 can be obtained by exponentiating both sides of Equation 8.

The bottom panel of Figure 1 illustrates the important features of Equation 8. The logarithms of the response and reinforcement rate ratios are plotted along the y and x axes. The heavy straight line represents perfect matching, that is, it is a plot of Equation 8 with $b = 1$. If $b = 1$, then $\log(b) = 0$; hence, when choice is symmetrical, the y intercept of Equation 8 is zero. The upper line in the bottom panel has a slope of 1 but a y intercept greater than 0 (i.e., $b > 1$, and so bias favors the first alternative), and the lower line has a slope of 1 but an intercept less than 0 (i.e., $b < 1$, and so bias favors the second alternative).

Equations 6 and 8 are different, but equivalent, ways of describing behavior in symmetrical and asymmetrical choice situations. Just as a logarithmic form of biased response-rate matching was obtained from Equation 6, a logarithmic form of biased time-allocation matching can be obtained from Equation 7. The utility of the logarithmic forms will become clearer in our discussion of another modern development in matching theory, namely, the mathematical description of indifference.

INDIFFERENCE

Distortions of the matching relationship that are produced by violations of symmetry are, in a general sense, predictable. That is, if the reinforcers on the two alternatives differ, or if the behaviors required on the two alternatives differ, then the expected distortion of the matching relationship is very likely to appear. In addition, Equations 6 and 7 and their logarithmic forms are very likely to describe the distorted matching relationship well. There is another kind of distortion of the matching relationship, however, that is more problematic because it has no obvious explanation. It is produced by the tendency of responding to deviate from perfect matching in the direction of indifference. This tendency seems to occur to at least some degree in most choice situations (Baum, 1979; Myers & Myers, 1977; Wearden & Burgess, 1982).

Deviation from Matching Produced by Tendency Toward Indifference

The deviation of responding in the direction of indifference is illustrated in the top panel of Figure 2. Response proportions are plotted against reinforcement proportions, and the heavy straight line represents perfect matching, that is, it is a plot of Equation 1. If a pigeon allocated half its pecks to the first alternative and half its pecks to the second alternative regardless of the proportion of reinforcements obtained from each alternative, then its response proportions would fall along the broken horizontal line in the top panel of Figure 2, and it could be said to be indifferent between the two response alternatives. Indifference between two alternatives means that the same amount of behavior is allocated to each alternative regardless of the consequences.

The typical outcome of a concurrent-schedule experiment is a slight deviation from matching in the direction of indifference. This is shown by the curve closest to the matching diagonal in the top panel of Figure 2. The other curve in the panel shows a larger deviation from matching in the direction of indifference. Deviations like these are often said to represent undermatching. Undermatching is responding that is not as extreme as that required by matching. For example, if 25% of the reinforcers are obtained from the first alternative, then matching requires 25% of the responses to be allocated to that alternative. But if there is a deviation in the direction of indifference, more than 25% of the responses will be allocated to the first alternative. Similarly, if 75% of the reinforcers are obtained on the first alternative, then matching requires 75% of the responses to be allocated to that alternative. But if there is a deviation in the direction of indifference, less than 75% of the responses will be allocated to the first alternative. Undermatching refers to behavior that tends toward indifference, and so is not as extreme as that required by matching.

Mathematical Description of Undermatching

Unlike bias, the causes of undermatching are unclear. Nevertheless, it is possible to describe this deviation mathematically, and to incorporate it into the matching equation. As in the case of bias, when data are plotted as response and reinforcement proportions (top panel of Figure 2), undermatching appears as a curve of unknown form. When the data are plotted as ratios of response and reinforcement rates (Equation 5), as in the

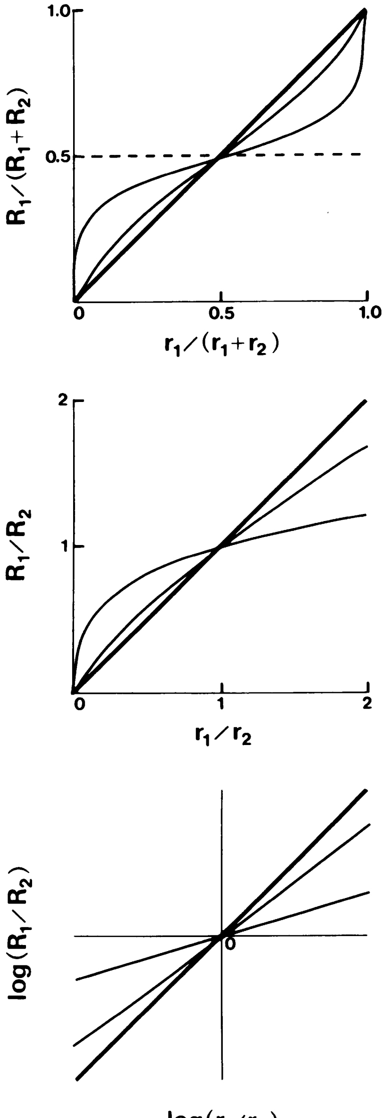

Figure 2. Undermatching, as it appears in three coordinate systems. In the top panel, response proportions are plotted against reinforcement proportions; in the middle panel, response rate ratios are plotted against reinforcement rate ratios; in the bottom panel, common logarithms of response rate ratios are plotted against common logarithms of reinforcement rate ratios. The heavy diagonal in all three panels represents perfect matching. In proportion coordinates (top panel), undermatching appears in the form of sigmoidal curves that deviate from the matching diagonal in the direction of indifference. Indifferent responding is represented by the dashed horizontal line. In ratio coordinates (middle panel), undermatching appears in the form of negatively accelerated curves that pass through the origin. In logarithmic coordinates (bottom panel), undermatching appears in the form of lines with varying slopes less than unity, and constant intercepts equal to zero.

middle panel of Figure 2, the deviation is still a curve. The curves in this panel are simpler than those in the top panel, but their exact forms are still unknown. The bottom panel of Figure 2 shows how the deviation appears when the data are plotted as logarithms of response and reinforcement rate ratios. In these coordinates, data that show undermatching tend to fall along straight lines. The slopes of these lines are less than unity, and the slopes decrease as undermatching becomes more severe.

Because data that show undermatching fall along straight lines in logarithmic coordinates, it is relatively simple to describe undermatching mathematically. It is only necessary to allow the slope of Equation 8 to take on values less than unity. If a represents the slope of the line, then Equation 8 can be written

$$\log\left(\frac{R_1}{R_2}\right) = a \log\left(\frac{r_1}{r_2}\right) + \log(b). \quad (9)$$

For the lines plotted in the bottom panel of Figure 2 there is no bias, which means that b equals unity and $\log(b)$, the y intercept of Equation 9, equals zero. For the matching line (the heavy diagonal), $a = 1$; for the other two lines, which represent data showing two different degrees of undermatching, $a < 1$. Thus, Equation 9 describes undermatching by letting a vary below unity.

Equation 9 is the logarithmic form of an equation that can describe perfect matching ($a = b = 1$), biased matching (b not equal to one), undermatching ($a < 1$), and any combination of biased matching and undermatching. Estimates of a and b for real data sets are typically obtained by fitting a straight line to the logarithms of the response and reinforcement rate ratios using simple linear regression techniques. The slope, a, of the regression line gives the degree of undermatching, and its intercept, $\log(b)$, when converted into an ordinary number, gives the extent and direction of the bias.

Equation 9 was obtained from Equation 8, which was in turn obtained from Equation 6. A time-allocation form can be obtained from Equation 7 in the same way. The result is

$$\log\left(\frac{T_1}{T_2}\right) = a \log\left(\frac{r_1}{r_2}\right) + \log(b). \quad (10)$$

Like Equation 9, Equation 10 describes bias by changes in its intercept, and undermatching by changes in its slope.

Power Function Forms

What are the equations of which Equations 9 and 10 are the logarithmic forms? We know that Equation 8 is the logarithmic form of Equation 6, and that a similar logarithmic form can be obtained from Equation 7. It was also shown that Equation 6 could be obtained by exponentiating both sides of Equation 8. In the same way, Equation 7 can be obtained by exponentiating both sides of its logarithmic form. It follows that we can find the equations of which Equations 9 and 10 are the logarithmic forms by exponentiating both sides of these equations. Beginning with Equation 9, exponentiation produces

$$10^{\log(R_1/R_2)} = 10^{a\log(r_1/r_2)+\log(b)}.$$

Applying the rule of exponents reviewed earlier gives

$$10^{\log(R_1/R_2)} = 10^{a\log(r_1/r_2)}10^{\log(b)}.$$

Evidently, the left side of this equation is R_1/R_2, and the second factor on the right is b. But the first factor on the right is complicated by the appearance of the additional factor, a, in the exponent. The right side of this equation can be simplified by recalling a second rule of exponents, namely,

$$(10^c)^d = 10^{cd}.$$

For example,

$$(10^3)^2 = 10^{(3)(2)} = 10^6.$$

That is, 1,000 squared equals 10^6, or 1,000,000. Applying this rule to the above expression allows us to write

$$10^{\log(R_1/R_2)} = [10^{\log(r_1/r_2)}]^a 10^{\log(b)}.$$

And, applying the definition of common logarithms, this expression becomes

$$\frac{R_1}{R_2} = b\left(\frac{r_1}{r_2}\right)^a. \quad (11)$$

Equation 11 is the equation of which Equation 9 is the logarithmic form. Of course, the bias parameter, b, appears as a factor on the right, just as it does in Equation 6. The slope, a, of the logarithmic form appears in Equation 11 as an exponent on the reinforcement rate ratio. Evidently, Equation 11 describes bias by allowing the factor, b, to vary above and below unity, and describes undermatching by allowing the exponent, a, to vary below unity.

Equation 11 is a specific instance of a general form known as a power function. The general power function is written $y = cx^d$, where y and x are the dependent and independent variables, d is an exponent on the independent variable, and c is a factor multiplying the exponentiated independent variable. Just as Equation 11 was obtained from Equation 9, a time-allocation form can be obtained in the same way from Equation 10. The result is the power function,

$$\frac{T_1}{T_2} = b\left(\frac{r_1}{r_2}\right)^a. \quad (12)$$

Equations 11 and 12 are the modern versions of the original matching equations (Equations 1 and 2). They can describe perfect matching, and any degree and combination of bias and undermatching. Their logarithmic forms (Equations 9 and 10) are used to obtain estimates of the factor, b, and the exponent, a, from real data sets. As already noted, the logarithmic forms are used for this purpose because they are straight lines, and so can be fitted to the logarithms of the response rate and reinforcement rate ratios using simple linear regression techniques.

Modern discussions of matching in

multialternative environments usually refer to Equations 11 and 12, rather than to their logarithmic equivalents, because the former equations are free of nonarithmetic transformations of the data. Nevertheless, the logarithmic forms are essential when it becomes necessary to estimate a and b from real data sets.

APPLIED RELEVANCE OF BIAS AND UNDERMATCHING

Like many other aspects of matching theory (McDowell, 1988), bias and undermatching are important for applied work. Because these phenomena are regularly observed in laboratory environments, it is reasonable to suppose that they will also be observed in natural human environments. Obviously, knowledge of bias and undermatching, and of their mathematical description, will enable applied scientists to understand and deal effectively with the distortions of the matching relationship that are likely to be found in naturalistic settings.

The phenomenon of bias has special relevance to applied work. As suggested earlier, bias results from differences in reinforcer and response values. That is, some reinforcers are more highly valued than others, some behaviors are more highly valued than others, and these differences give rise to biased responding. Matching theory's mathematical treatment of bias permits differences in value to be studied quantitatively because the bias parameter in Equations 11 and 12 can be considered to be the ratio of the values of different reinforcers or different behaviors (McDowell, 1987). This application of matching theory has been discussed in a variety of theoretical contexts (e.g., McDowell, 1987) and has been carried out in the basic laboratory (Cliffe & Parry, 1980; Miller, 1976). For example, Cliffe and Parry (1980) studied the lever pressing of a male sex offender. The subject worked on concurrent schedules of reinforcement, where reinforcement consisted of opportunities to observe sexually arousing slides of adult females, adult males, or young girls. Equation 11 was found to govern the subject's behavior, and different pairs of reinforcers produced different degrees of bias. Cliffe and Parry used the empirically obtained bias parameters to calculate relative measures of the values of the three reinforcing stimuli. In the same way, relative measures of different reinforcers and different behaviors can be obtained in natural environments, assuming that Equations 11 and 12 hold in these environments. Such measures would answer important questions such as whether one type of reinforcer for a particular individual (e.g., tokens) is more powerful than another (e.g., praise), and by how much.

CONCLUSION

Equations 11 and 12 are the equations that have been so successful in describing data from concurrent schedules (McDowell, 1988). The parameter-free simplicity of the original matching equations is lost in these modern forms, but their advantage is an extraordinarily wide range of applicability (Baum, 1979; McDowell, 1988; Wearden & Burgess, 1982). The equations constitute a deterministic, mathematical account of choice behavior that is highly accurate. There is no doubt that this is a remarkable achievement for basic behavior analysis.

As mentioned earlier, biased responding is not generally considered a threat to matching because many of the conditions that produce it are known (Baum, 1974). Undermatching, on the other hand, is more problematic, because its cause is unknown. A small degree of undermatching is frequently observed in response rate data from concurrent schedules. Undermatching may also be common when time allocation is measured, although this is a matter of dispute (Baum, 1979, 1983; Mullins, Agunwamba, & Donohoe, 1982; Wearden & Burgess, 1982).

Although Equations 11 and 12 are known to describe the choice behavior of many species, including humans, in the laboratory (McDowell, 1988), it is not known whether these equations describe human behavior in natural environments. Available data indicate that the single alternative form, Equation 3, holds in natural human environments (McDowell, 1988), and it seems reasonable to suppose that Equations 11 and 12 also hold. Nevertheless, research is needed in this area. Besides testing the validity of Equations 11 and 12, the objectives of such work include determining the extent of undermatching in natural environments and, as noted earlier, studying how different behaviors and different reinforcers affect bias. Martens & Houk (1989) have developed experimental procedures that may be generally useful in this type of naturalistic research.

The developments discussed in this article bring the multialternative forms of matching theory up to date. But there have also been modern developments in the single alternative forms (Equations 3 and 4), including serious challenges to the general validity of matching theory (e.g., McDowell & Wood, 1984, 1985). So far, the theory has withstood these challenges (McDowell, 1986), but empirical and mathematical work on competing theories continues (e.g., Killeen, 1982; McDowell & Wixted, 1988). The next few years should be an interesting time for matching theory, and for mathematical accounts of behavior in general. Applied workers may benefit from keeping abreast of developments in this area of basic research.

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