Free entropy and property T factors

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Edited by Richard V. Kadison, University of Pennsylvania, PA, and approved May 19, 2000 (received for review April 26, 2000)

We show that a large class of finite factors has free entropy dimension less than or equal to one. This class includes all prime factors and many property T factors.

In the early 1980s, D. Voiculescu (1) developed a noncommutative probability theory over von Neumann algebras, the algebras introduced and studied by F. J. Murray and J. von Neumann (2–6). von Neumann algebras are quantized measure spaces, which are the basis for the study of noncommutative analysis and geometry (7).

A *von Neumann algebra* is a strong-operator closed self-adjoint subalgebra of the algebra of all bounded linear transformations on a Hilbert space. *Factors* are von Neumann algebras whose centers consist of scalar multiples of the identity. They are the building blocks from which all the von Neumann algebras are constructed.

F. J. Murray and J. von Neumann (2) classified factors by means of a relative dimension function. *Finite factors* are those for which this dimension function has a finite range. For finite factors, this dimension function gives rise to a unique tracial state. In general, a von Neumann algebra admitting a faithful normal trace is said to be *finite*. Infinite-dimensional finite factors are called factors of type II_1. The trace is the analogue of classical integration and finite factors are basic noncommutative measure spaces.

With classical independence replaced by "free independence" on noncommutative spaces, Voiculescu develops the theory of free probability. The free central limit theorem states that the analogous averaging of free independent random variables obeys a Semicircular Law instead of a Gaussian Law in the classical case. Voiculescu and others use this theory to answer several important old questions in the theory of von Neumann algebras.

Recently, Voiculescu introduced a notion of "free entropy" (8), an analogue of classical entropy and Fisher's information measure. Associated with free entropy, he defined a free entropy dimension which, in some sense, measures the "noncommutative dimension" of a space (or, the minimal number of generators needed in the case of factors). Free entropy has become a very powerful tool in the study of von Neumann algebras since its recent introduction (9–11). Of course, some basic questions concerning free entropy and free entropy dimension have appeared. For example, it is not yet known whether the free entropy dimension of a set of generators for a von Neumann algebra depends on the set. Is this dimension an *invariant* of the algebra? That is currently the major problem of the theory. For finite injective (or hyperfinite) von Neumann algebras, Voiculescu shows that his free entropy dimension is an invariant. He also shows that it is an invariant for self-adjoint operator algebras with a faithful trace (as applied to sets of *algebraic* generators). It seems quite plausible that it is an invariant for C*-algebras with a faithful trace, as well. For von Neumann algebras, this becomes somewhat more speculative, the most severe test involving its application to factors with the Connes–Jones property T (12).

For some years, now, Voiculescu has set, as a primary objective for study, the determination of free entropy dimension for sets of generators in property T factors. He has proved the first result in ref. 13, showing that some property T factors have generators with free entropy dimension not exceeding 1. Thus with respect to free entropy dimension and certain generators, property T factors behave differently from free group factors. Of course, the first question to ask after this is whether there are generators in those same property T factors whose free entropy dimension is greater than 1. We answer that question by showing that *every* set of generators of a large class of factors of type II_1, including those considered in ref. 13 and all nonprime factors (11), have free entropy dimension not exceeding 1 if there is at least one set of generators that satisfies a "cyclic" commuting relation.

We describe below, briefly, the construction of finite factors by using regular representations of discrete groups. Then we define free entropy and free entropy dimension and list some of their basic properties. Finally, we state our main results and outline the proofs.

MATHEMATICS

Definitions

There are two main classes of examples of von Neumann algebras introduced by Murray and von Neumann (3, 5). One is obtained from the "group-measure space construction," the other is based on the regular representation of a (discrete) group G (with unit e). The second class is the one needed in this note.

The Hilbert space $\mathcal{H}$ is $l^2(G)$. We assume that G is countable so that $\mathcal{H}$ is separable. For each g in G, let L_g denote the left translation of functions in $l^2(G)$ by g^{-1}. Then $g \to L_g$ is a faithful unitary representation of G on $\mathcal{H}$. Let $\mathcal{L}_G$ be the von Neumann algebra generated by $\{L_g : g \in G\}$. Similarly, let R_g be the right translation by g on $l^2(G)$ and $\mathcal{R}_G$ be the von Neumann algebra generated by $\{R_g : g \in G\}$. Then the commutant $\mathcal{L}'_G$ of $\mathcal{L}_G$ is equal to $\mathcal{R}_G$ and $\mathcal{R}'_G = \mathcal{L}_G$. The function u_g that is 1 at the group element g and 0 elsewhere is a cyclic trace vector for $\mathcal{L}_G$ (and $\mathcal{R}_G$). In general, $\mathcal{L}_G$ and $\mathcal{R}_G$ are finite von Neumann algebras. They are factors (of type II_1) precisely when each conjugacy class in G (other than that of e) is infinite. In this case we say that G is an *infinite conjugacy class* (i.c.c.) group.

Specific examples of such II_1 factors result from choosing for G any of the free groups F_n on n generators ($n \geq 2$), or the permutation group Π of integers $\mathbf{Z}$ (consisting of those permutations that leave fixed all but a finite subset of $\mathbf{Z}$). Murray and von Neumann (4) prove that $\mathcal{L}_{F_n}$ and $\mathcal{L}_{\Pi}$ are not * isomorphic to each other. A factor is *hyperfinite* if it is the ultraweak closure of the ascending union of a family of finite-dimensional self-adjoint subalgebras. In fact, $\mathcal{L}_{\Pi}$ is the *unique* hyperfinite factor of type II_1; it is contained in any factor of type II_1. When $G = SL_n(\mathbf{Z})$, $n \geq 3$, $\mathcal{L}_{SL_n(\mathbf{Z})}$ is a von Neumann algebra with property T (12) (it is a factor when n is odd). Property T factors are not isomorphic to factors arising from free groups or Π (ref. 14 or 15). The classification of all von Neumann algebras has been reduced to the case of factors of type II_1 in a certain sense. Some basic problems concerning the free group factors remain unsolved. Among them is the one which asks whether the factors arising from free groups on different number of generators are isomorphic. This

This paper was submitted directly (track II) to the PNAS office.

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is one of the motivating questions that led to the development of free probability theory.

Recall that a *W*-probability space* is a pair $(\mathcal{M}, \tau)$ consisting of a von Neumann algebra $\mathcal{M}$ and a faithful normal state τ. The most important case is when the state is a trace ($\tau(AB) = \tau(BA)$) and, hence, $\mathcal{M}$ is a finite von Neumann algebra. Elements in $\mathcal{M}$ are called *random variables*. With A in $\mathcal{M}$, $\|A\|_2$ denotes the L^2-norm (or trace-norm) of A with respect to τ (that is, $\|A\|_2^2 = \tau(A^*A)$). A family $\mathcal{M}_\iota$, $\iota \in \mathbf{I}$, of von Neumann subalgebras of $\mathcal{M}$ are *free* (or *free independent*) with respect to the trace τ if $\tau(A_1A_2\cdots A_n) = 0$ whenever $A_j \in \mathcal{M}_{\iota_j}$, $\iota_1 \neq \cdots \neq \iota_n$ and $\tau(A_j) = 0$ for $1 \le j \le n$ and every n in $\mathbf{N}$. A family of subsets (or elements) of $\mathcal{M}$ are said to be *free* if the von Neumann subalgebras they generate are free.

Voiculescu proves the free central limit theorem: *Suppose A_j, $j = 1, 2, \ldots$, are free random variables such that $\tau(A_j) = 0$. Denote by $\frac{r^2}{4}$ the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \tau(A_j^2)$. Assume further that $\sup_{j,k} |\tau(A_j^k)| < \infty$. Then the distribution of $(A_1 + \cdots + A_n)/\sqrt{n}$ converges pointwise to the semicircular law with center at 0 and radius r*. He also generalizes Wigner's semicircular law for large random matrices to n-tuples of large random matrices and proves the asymptotic freeness in limit as the degree of the matrices tends to infinity. By using techniques developed in proving these results, he shows that the free group factor on two generators is isomorphic to the 2×2 matrix algebra with entries in the free group factor on five generators. This is the first major break through in the study of the isomorphism problem of free group factors.

Now we describe briefly the definition of free entropy. Let $M_k(\mathbf{C})$, or simply M_k, be the $k \times k$ matrix algebra with entries in $\mathbf{C}$, and τ_k be the normalized trace on M_k, i.e., $\tau_k = k^{-1}\mathrm{Tr}$, where Tr is the usual trace. Let M_k^{sa} denote the set of all self-adjoint matrices in M_k and $(M_k^{\mathrm{sa}})^n$ denote the (vector-space) direct sum of n copies of M_k^{sa}. We denote by $\| \ \|_e$ the euclidean norm on $(M_k^{\mathrm{sa}})^n$ given by $\|(A_1, \ldots, A_n)\|_e^2 = \mathrm{Tr}(A_1^2 + \cdots + A_n^2) = k\tau_k(A_1^2 + \cdots + A_n^2)$, for $(A_1, \ldots, A_n)$ in $(M_k^{\mathrm{sa}})^n$. We use $\| \ \|_2$ to denote the trace norm induced by τ_k. Let Λ be Lebesgue measure on $(M_k^{\mathrm{sa}})^n$ induced by the euclidean norm $\| \ \|_e$.

Suppose $X_1, \ldots, X_n$ are self-adjoint elements in a W*-probability space $(\mathcal{M}, \tau)$. The free entropy of $X_1, \ldots, X_n$, denoted by $\chi(X_1, \ldots, X_n)$, is defined in the following.

For any positive ε and R, and any k, m in $\mathbf{N}$, let $\Gamma_R(X_1, \ldots, X_n; m, k, \varepsilon)$ be the subset of $(M_k^{\mathrm{sa}})^n$ consisting of all $(A_1, \ldots, A_n)$ in $(M_k^{\mathrm{sa}})^n$ such that $\|A_j\| \le R$, $1 \le j \le n$, and

$$|\tau_k(A_{\iota_1}\cdots A_{\iota_q}) - \tau(X_{\iota_1}\cdots X_{\iota_q})| < \varepsilon$$

for all $1 \le \iota_1, \ldots, \iota_q \le n$ and all q with $0 \le q \le m$. Then

$$\chi(X_1, \ldots, X_n) = \sup_{R>0}\inf_{m,\varepsilon>0}\limsup_{k\to\infty}\left(k^{-2}\log\Lambda(\Gamma_R(X_1, \ldots, X_n; m, k, \varepsilon)) + \frac{n}{2}\log k\right).$$

The modified free entropy (or simply, free entropy) $\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$ of $X_1, \ldots, X_n$ *in the presence of* $Y_1, \ldots, Y_p$ is given in the following.

Let $X_1, \ldots, X_n, Y_1, \ldots, Y_p$, $n \ge 1$, $p \ge 0$, be self-adjoint random variables in $(\mathcal{M}, \tau)$, $\Gamma_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \varepsilon)$ be the image of the projection of

$$\Gamma_R(X_1, \ldots, X_n, Y_1, \ldots, Y_p; m, k, \varepsilon)$$

(defined above) onto its first n components. Then

$$\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p) = \sup_{R>0}\inf_{m,\varepsilon>0}\limsup_{k\to\infty}\left(k^{-2}\log\Lambda(\Gamma_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \varepsilon)) + \frac{n}{2}\log k\right).$$

When $Y_1, \ldots, Y_p$ are non-self-adjoint elements, $\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$ may be identified with $\chi(X_1, \ldots, X_n : A_1, \ldots, A_p, B_1, \ldots, B_p)$, where $A_j = Y_j + Y_j^*$ and $B_j = -i(Y_j - Y_j^*)$ for each j.

The (modified) free entropy dimension $\delta(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$ is defined by

$$\delta(X_1, \ldots, X_n : Y_1, \ldots, Y_p) = n + \limsup_{\varepsilon\to 0}\frac{\chi(X_1+\varepsilon S_1, \ldots, X_n+\varepsilon S_n : S_1, \ldots, S_n, Y_1, \ldots, Y_p)}{|\log\varepsilon|},$$

where $\{S_1, \ldots, S_n\}$ is a free semicircular family and the two sets $\{X_1, \ldots, X_n, Y_1, \ldots, Y_p\}$ and $\{S_1, \ldots, S_n\}$ are free.

We list some basic properties of free entropy and free entropy dimension.

(*i*) $\chi(X_1, \ldots, X_n) \le \frac{n}{2}\log(2\pi e C^2 n^{-1})$, where $C = \tau(X_1^2 + \cdots + X_n^2)^{1/2}$.

(*ii*) $\chi(X) = \int\int \log|s - t|d\mu(s)d\mu(t) + \frac{3}{4} + \frac{1}{2}\log 2\pi$, where μ is the (measure on the spectrum of X corresponding to the) distribution of X.

(*iii*) $\chi(X_1, \ldots, X_n) = \chi(X_1) + \cdots + \chi(X_n)$ when $X_1, \ldots, X_n$ are free.

(*iv*) $\delta(S_1, \ldots, S_n) = n$ when $S_1, \ldots, S_n$ are free semi-circular elements. (Note that the free group factor on n generators is generated by n free semicircular elements.)

Main Results

The assumptions of the following theorem are satisfied when the algebra $\mathcal{M}$ is a tensor product of two factors of type II_1, or the group von Neumann algebra arising from $SL_n(\mathbf{Z})$, $n \ge 3$, or certain amalgamated free product. One of the consequences of our theorem says that the free group factors are not isomorphic to any factors that satisfy the assumptions of the theorem.

MAIN THEOREM. *Let $(\mathcal{M}, \tau)$ be a W*-probability space. Suppose $\mathcal{M}$ is generated by its subalgebras $\mathcal{N}_1, \ldots, \mathcal{N}_r$, and, for each $\mathcal{N}_j$, $1 \le j \le r$, there is a non-atomic self-adjoint subalgebra $\mathcal{A}_j$ of $\mathcal{N}_j$ such that $\mathcal{A}_j$ commutes with $\mathcal{N}_{j+1}$ (here we let $\mathcal{N}_{r+1}$ be $\mathcal{N}_1$). Assume that $X_1, X_2, \ldots, X_n$ are self-adjoint elements in $\mathcal{M}$ that generate $\mathcal{M}$ as a von Neumann algebra. Then $\delta(X_1, \ldots, X_n) \le 1$.*

The above theorem is a consequence of the following result on free entropy.

PROPOSITION. *Suppose $(\mathcal{M}, \tau)$ is a W*-probability space. We assume that $\mathcal{M}$ contains subalgebras $\mathcal{N}_1, \ldots, \mathcal{N}_r$, and, for each $\mathcal{N}_j$, $1 \le j \le r$, there is a Haar unitary element U_j' in $\mathcal{N}_j$ such that U_j' commutes with $\mathcal{N}_{j+1}$ (again, $\mathcal{N}_{r+1}$ is $\mathcal{N}_1$). Let $X_1, X_2, \ldots, X_n$ be self-adjoint elements in $\mathcal{M}$. Suppose there are unitary elements $V_{j1}, V_{j2}, \ldots, V_{jl_j}$ in $\mathcal{N}_j$, for $j = 1, 2, \ldots, r$, $l_j \in \mathbf{N}$, and polynomials $\varphi_j(x_1, x_2, \ldots, x_l)$ in $\mathbf{C}\langle x_1, x_2, \ldots, x_l\rangle$, where $l = l_1 + l_2 + \ldots + l_r$, such that*

$$\|X_j - \varphi_j(V_{11}, \ldots, V_{1l_1}, \ldots, V_{r1}, \ldots, V_{rl_r})\|_2 \le \frac{1}{2}\omega, \quad j = 1, 2, \ldots, n,$$

for some positive number ω. Then

$$\chi(X_1,\ldots,X_n:V_{11},\ldots,V_{rl_r},U_1',\ldots,U_r') \le \log(4aC)+\frac{n}{2}\log(n\pi)+n\log 3+\frac{n}{2} + (n-1-\omega)\log\omega,$$

where C ($\le e^{3\pi}$) is a constant and $a=\max_{1\le j\le n}\|X_j\|_2+1$.

In the following, we outline the proof of the proposition. Note that the free entropy of $X_1,\ldots,X_n$ is the limit of the logarithm of the volume of $\Gamma_R(X_1,\cdots,X_n:\cdots;k,m,\varepsilon)$. We will get a good upper bound for the volume by showing that the covering number of $\Gamma_R(X_1,\ldots,X_n:\cdots;m,k,\varepsilon)$ (by small balls) is small. When X_j is approximated by $\varphi_j(V_{11},\ldots,V_{1l_1},\ldots,V_{r1},\ldots,V_{rl_r})$, the elements in $\Gamma_R(X_1,\ldots,X_n:\cdots;m,k,\varepsilon)$ are approximated by $\varphi_j(G_{11},\ldots,G_{rl_r})$ where G_{st} values are matrices approximating V_{st} values in moments. First we reduce the covering of $\Gamma_R(X_1,\ldots,X_n:\cdots;m,k,\varepsilon)$ to the covering on the ranges of φ_j values, which then depend on the domains of φ_j values. When passing the commuting relations on U_j' and $V_{j+1,t}$ values to their corresponding matricial approximants, we show that the G_{st} values lie in a lower dimensional submanifold of $M_k(\mathbf{C})^l$ ($l=l_1+\cdots,l_r$), so the covering number for this submanifold is small. The following lemmas describe the details of these steps. Before stating them, we introduce some notation.

Let p be a large prime number, k a large integer such that $\frac{k}{p}$ is an integer. Let W_0 be the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & e^{2\pi i/p} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{2\pi i(p-1)/p}\end{pmatrix}$$

in $M_p(\mathbf{C})$, W_1 the matrix

$$\begin{pmatrix} I_{\frac{k}{p}} & 0 & \cdots & 0\\ 0 & e^{2\pi i/p}I_{\frac{k}{p}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{2\pi i(p-1)/p}I_{\frac{k}{p}}\end{pmatrix}$$

in $M_k(\mathbf{C})$ (here $I_{\frac{k}{p}}$ denotes the identity matrix in $M_{\frac{k}{p}}$ and $M_k=M_p\otimes M_{\frac{k}{p}}$). Let c_0 be the distance between two adjacent pth roots of unity, i.e., $c_0=|1-e^{2\pi i/p}|$. Clearly, $|e^{2\pi is/p}-e^{2\pi it/p}|\ge c_0\ge\pi/p$ when $s,t\in\mathbf{Z}, s\neq t$ and $0\le s,t\le p-1$. For any given $D>0$, $\omega>0, r,l\in\mathbf{N}$, let δ_{r+1} be $\omega/(8D\sqrt{l})$, define, successively, $\delta_r,\ldots,\delta_1$ by the following equations

$$\delta_{r+1}=\frac{17p\delta_r}{\pi},\quad\ldots,\quad\delta_2=\frac{3p\delta_1}{\pi}.$$

That is,

$$\delta_{r+1}=\frac{\omega}{8D\sqrt{l}},\quad\delta_r=\frac{\pi\omega}{17p\cdot 8D\sqrt{l}},\quad\ldots,\qquad \delta_1=\frac{\pi^r\omega}{(17p)^{r-1}\cdot 3p\cdot 8D\sqrt{l}}$$

for the given positive numbers D, l, ω and r.

For the given D, l and ω, δ_1 is defined by the above equation. Then ε_0 will be the positive number in 2.2 and 2.3 in ref. 16 determined by this δ_1 (here we only use cases when $N=2$ in ref. 16).

We choose an arbitrary ε_1 with $0<\varepsilon_1<\delta_1$. Then we have the following.

LEMMA 1. *Assume that G_1, G_2 in $M_k(\mathbf{C})$ satisfy the following*

$$\|G_1\|,\|G_2\|\le 1;\quad \|W_1-G_1\|_2\le\delta_1,\quad \|G_1G_2-G_2G_1\|_2\le\varepsilon_1.$$

Let

$$\mathcal{B}(W_1,\delta_1)=\{G\in M_k(\mathbf{C})\mid\|G\|_2\le 1,\ \|W_1G-GW_1\|_2\le 3\delta_1\}$$
$$\Omega(W_1,\delta_1,\delta_2)=\{H\in M_k(\mathbf{C})\mid\|G-H\|_2\le\delta_2,\ \text{for some } G\in\mathcal{B}(W_1,\delta_1)\}.$$

Then G_2 is in $\mathcal{B}(W_1,\delta_1)$ and the covering number $\mu(\Omega(W_1,\delta_1,\delta_2), 2\sqrt{2}\delta_2)$ for $\Omega(W_1,\delta_1,\delta_2)$ with balls of radius $2\sqrt{2}\delta_2$ satisfies $\mu(\Omega(W_1,\delta_1,\delta_2),2\sqrt{2}\delta_2)\le(3/\delta_2)^{2k^2/p}$, where the metric on $\Omega(W_1,\delta_1,\delta_2)$ is given by the normalized trace norm $\|\ \|_2$ on $M_k(\mathbf{C})$.

We will use $T(W_1,\delta_1,\delta_2)$ to denote the set that has the minimal cardinality of covering balls for $\Omega(W_1,\delta_1,\delta_2)$ with radius $2\sqrt{2}\delta_2$. So the cardinality of $T(W_1,\delta_1,\delta_2)$ is bounded by $(\frac{3}{\delta_2})^{2k^2/p}$. Denote by $T_1(W_1,\delta_1,\delta_2)$, the subset of $T(W_1,\delta_1,\delta_2)$, consisting of balls that contain a conjugating matrix U^*W_1U of W_1 for some U in $\mathcal{U}(k)$. We shall enlarge the balls in $T_1(W_1,\delta_1,\delta_2)$ so that each original ball is contained in the ball centered at some U^*W_1U with radius $8\delta_2$. Then each of the balls in $T_1(W_1,\delta_1,\delta_2)$ contains the original ball in $T(W_1,\delta_1,\delta_2)$ containing U^*W_1U. Define, also, $T_2(W_1,\delta_1,\delta_2)$ to be the subset of $T(W_1,\delta_1,\delta_2)$ consisting of balls that contain some unitary matrix. We, again, enlarge the balls in $T_2(W_1,\delta_1,\delta_2)$ the same way as for $T_1(W_1,\delta_1,\delta_2)$ so that each of its balls is centered at a unitary matrix with radius $8\delta_2$. We use $\mathbf{B}(V,a)$ to denote the (closed) ball centered at V (in $M_k(\mathbf{C})$) with radius a (with respect to the normalized trace norm $\|\ \|_2$).

Now, for any given ball in $T_1(W_1,\delta_1,\delta_2)$, let W_2 be its center and $W_2=U^*W_1U$ for some U in $\mathcal{U}(k)$. Replacing $W_1,\delta_1,3\delta_1$, and δ_2 in the above lemma by $W_2, 8\delta_2, 17\delta_2$, and δ_3, respectively, we have the following.

LEMMA 2. *Assume that G_2, G_3 in $M_k(\mathbf{C})$ satisfy the following*

$$\|G_2\|,\|G_3\|\le 1;\quad \|W_2-G_2\|_2\le 8\delta_2,$$
$$\|G_2G_3-G_3G_2\|_2\le\varepsilon_1,\qquad(0<\varepsilon_1\le\delta_1<8\delta_2).$$

Let

$$\mathcal{B}(W_2,\delta_2)=\{G\in M_k(\mathbf{C})\mid\|G\|_2\le 1,\ \|W_2G-GW_2\|_2\le 17\delta_2\}$$
$$\Omega(W_2,\delta_2,\delta_3)=\{H\in M_k(\mathbf{C})\mid\|G-H\|_2\le\delta_3,\ \text{for some } G\in\mathcal{B}(W_2,\delta_2)\}.$$

Then G_3 is in $\mathcal{B}(W_2,\delta_2)$ and $\mu(\Omega(W_2,\delta_2,\delta_3),2\sqrt{2}\delta_3)\le(3/\delta_3)^{2k^2/p}$.

Similarly, we will have corresponding $T_1(W_2,\delta_2,\delta_3)$ and $T_2(W_2,\delta_2,\delta_3)$ as described preceding *Lemma 2*. Continuing this process, we obtain, for each $j\ge 1$,

$$\mathcal{B}(W_j,\delta_j),\quad\Omega(W_j,\delta_j,\delta_{j+1}),\quad T(W_j,\delta_j,\delta_{j+1}),$$

and corresponding subsets $T_1(W_j,\delta_j,\delta_{j+1})$ and $T_2(W_j,\delta_j,\delta_{j+1})$ of $T(W_j,\delta_j,\delta_{j+1})$ that will satisfy conditions similar to those of the case when $j=2$. In particular, the cardinality of $T(W_j,\delta_j,\delta_{j+1})$ is bounded by $(\frac{3}{\delta_j})^{2k^2/p}$, for $j=1,\ldots,r$.

We use ε_0, again, to denote $\min\{\varepsilon_0,\delta_1,\varepsilon_1\}$. Now ε_0 satisfies the assumptions in *Lemmas 1* and *2* (in place of ε_1).

LEMMA 3. *Suppose $G_1,\ldots,G_r,S_{11},\ldots,S_{1l_1},\ldots,S_{r1},\ldots,S_{rl_r}$ are unitary matrices in $M_k(\mathbf{C})$ satisfying the following*:

(1) $\|G_jG_{j+1}-G_{j+1}G_j\|_2\le\varepsilon_0$;
(2) $\|G_1-W_1\|_2\le\delta_1$;
(3) *for each $j>1$, there is a unitary matrix U_j such that $\|G_j-U_jW_1U_j^*\|_2\le\delta_j$;*
(4) $\|G_{j+1}S_{jl}-S_{jl}G_{j+1}\|_2\le\varepsilon_0$ *and* $\|G_1S_{rl}-S_{rl}G_1\|_2\le\varepsilon_0$;

MATHEMATICS

(5) *there is a unitary matrix* V_{jl} *such that* $\|S_{jl} - V_{jl}\|_2 \le \delta_1$; *for* $j = 1, \ldots, r$ *and* $l = 1, \ldots, l_j$. *Then there is a family of balls* $\{\mathbf{B}(Y_{jl}, 8\delta_{j+2})\}$ (*for* $j = 1, \ldots, r-1$) *from* $T_2(W_{j+1}, \delta_{j+1}, \delta_{j+2})$ *such that* $\|S_{jl} - Y_{jl}\|_2 \le 8\delta_{j+2}$ *and a family of balls* $\{\mathbf{B}(Y_{rl}, 8\delta_2)\}$ *from* $T_2(W_1, \delta_1, \delta_2)$, *such that* $\|S_{rl} - Y_{rl}\|_2 \le 8\delta_2$.

Since, for each j, $1 \le j \le r$, $\{Y_{jl} : l = 1, \ldots, l_j\}$ are centers of balls in the set $\{T_2(W_{j+1}, \delta_{j+1}, \delta_{j+2})\}$, we know that the total number of choices for $\{Y_{jl} : j = 1, \ldots, r;\ l = 1, \ldots, l_j\}$ is bounded by $(3/\delta_2)^{2k^2 l_1/p} \cdots (3/\delta_{r+1})^{2k^2 l_r/p}$.

Some of the key ideas of the proof of the proposition are similar to those used in the proof of Theorem 2.1 in ref. 11. First we choose a non-atomic self-adjoint subalgebra $\mathcal{A}_j$ of $\mathcal{N}_j$ (generated by U_j'). Since $\mathcal{A}_j$, for $1 \le j \le r$, is non-atomic, there is a unitary U_j in $\mathcal{A}_j$ so that its distribution is the same as that of W_1, i.e., $\tau(U_j^s) = \tau_k(W_1^s),\ s = 1, 2, \ldots$.

From Proposition 6.3 in ref. 9, we know that

$$\chi(X_1, \ldots, X_n : U_1', \ldots, U_r', V_{11}, \ldots, V_{rl_r}) \le \chi(X_1, \ldots, X_n : U_1, \ldots, U_r, V_{11}, \ldots, V_{rl_r}).$$

By the definition of $\chi(X_1, \ldots, X_n : U_1, \ldots, U_r, V_{11}, \ldots, V_{rl_r})$, we know that it is the limit of the logarithm of the volume of

$$\Gamma_R(X_1, \ldots, X_n : \{U_j\}_{j=1}^r, \{V_{jt}\}_{j=1,\ldots,r;t=1,\ldots,l_j}; k, m, \varepsilon).$$

Our goal is to estimate the covering numbers of $\Gamma_R(X_1, \ldots, X_n : \{U_j\}, \{V_{jt}\}; k, m, \varepsilon)$ in the euclidean space $(M_k^{sa})^n$. First, we study elements in $\Gamma_R(X_1, \ldots, X_n, \{U_j\}, \{V_{jt}\}; k, m, \varepsilon)$, then we project down to its first n components. From ref. 8, we know that it is enough to choose R so that $R > \max_j(\|X_j\| + 1)$.

Let

$$(A_1, \ldots, A_n, G_1', \ldots, G_r', S_{11}', \ldots, S_{1l_1}', \ldots, S_{r1}', \ldots, S_{rl_r}')$$

be an arbitrary element in $\Gamma_R(X_1, \ldots, X_n, \{U_j\}, \{V_{jt}\}; k, m, \varepsilon)$. When k, m are large and ε small enough, we have that

$$\|A_j - \varphi_j(S_{11}', \ldots, S_{1l_1}', \ldots, S_{r1}', \ldots, S_{rl_r}')\|_2 \le \frac{2}{3}\omega,$$

for $j = 1, 2, \ldots, n$.

Since the U_j values are unitary elements of finite order and V_{jt} values are unitary, we may choose unitary matrices G_j (with the same distribution as that of U_j) and unitary matrices S_{jt} in $\mathcal{U}(k)$ such that

$$\|A_j - \varphi_j(S_{11}, \ldots, S_{1l_1}, \ldots, S_{r1}, \ldots, S_{rl_r})\|_2 \le \omega,$$

for $j = 1, 2, \ldots, n$. Moreover, with m large and ε small enough, we have

$$\|G_j G_{j+1} - G_{j+1} G_j\|_2 \le \varepsilon_0, \quad \|G_{j+1} S_{jl} - S_{jl} G_{j+1}\|_2 \le \varepsilon_0,$$
$$\|G_1 S_{rl} - S_{rl} G_1\|_2 \le \varepsilon_0.$$

Except for (2) in *Lemma 3*, G_j values and S_{jt} values satisfy all other assumptions. Choose a unitary matrix U in $M_k(\mathbf{C})$ such that $U^* G_1 U = W_1$. Thus, with respect to any given $\delta_1 > 0$, when k and m are large enough and $\varepsilon (\le \varepsilon_0)$ sufficiently small, $U^* G_j U$ and $U^* S_{jt} U$ satisfy all the assumptions in *Lemma 3.3*. Taking conjugation by U on A_j values we also have that

$$\|U^* A_j U - \varphi_j(U^* S_{11} U, \ldots, U^* S_{1l_1} U, \ldots, U^* S_{r1} U, \ldots, U^* S_{rl_r} U)\|_2 \le \omega,$$

for $j = 1, 2, \ldots, n$.

By *Lemma 3*, for each pair (j, t), $1 \le j \le r-1$ and $1 \le t \le l_j$, there is an element Y_{jt}, the center of a ball in $T_2(W_{j+1}, \delta_{j+1}, \delta_{j+2})$, such that

$$\|U^* S_{jt} U - Y_{jt}\|_2 \le 8\delta_{j+2},$$

and an element Y_{rt} (for $1 \le t \le l_r$), the center of a ball in $T_2(W_1, \delta_1, \delta_2)$, such that

$$\|U^* S_{rt} U - Y_{rt}\|_2 \le 8\delta_2.$$

Choose a γ-net $(U_\alpha)_{\alpha \in \mathbf{T}(k)}$ in $\mathcal{U}(k)$ with respect to the operator norm such that $|\mathbf{T}(k)| \le (C/\gamma)^{k^2}$ for each k in $\mathbf{N}$, where C is a constant. We choose γ to be $\frac{\omega}{2a}$, where a is given in the theorem.

Hence there is an α in $\mathbf{T}(k)$ such that $\|U - U_\alpha\| \le \gamma$. Then

$$\|U_\alpha^* A_j U_\alpha - U^* A_j U\| \le \omega, \qquad j = 1, \ldots, n.$$

Let Φ be the map defined in Lemma 1.2 in ref. 11 and D the constant $D(\Phi)$. Then we have that

$$\begin{aligned}
\|U_\alpha^* A_j U_\alpha - \varphi_j(Y_{11}, \ldots, Y_{rl_r})\|_e &\le \|U_\alpha^* A_j U_\alpha - U^* A_j U\|_e \\
&\quad + \|U^* A_j U - \varphi_j(U^* S_{11} U, \ldots, U^* S_{1l_1} U, \ldots, U^* S_{r1} U, \ldots, U^* S_{rl_r} U)\|_e \\
&\quad + \|\varphi_j(U^* S_{11} U, \ldots, U^* S_{1l_1} U, \ldots, U^* S_{r1} U, \ldots, U^* S_{rl_r} U) - \varphi_j(Y_{11}, \ldots, Y_{rl_r})\|_e \\
&\le 2k^{1/2}\omega + D\sqrt{\sum_{jt} \|U^* S_{jt} U - Y_{jt}\|_e^2} \\
&\le 2k^{1/2}\omega + Dk^{1/2}\sqrt{l_1(8\delta_3)^2 + l_2(8\delta_4)^2 + \ldots + l_{r-1}(8\delta_{r+1})^2 + l_r(8\delta_2)^2} \\
&\le 2k^{1/2}\omega + 8Dk^{1/2}\sqrt{l}\,\delta_{r+1} \\
&\le 3k^{1/2}\omega
\end{aligned}$$

where $l = l_1 + l_2 + \ldots + l_n$ and $\delta_{r+1} = \omega/(8D\sqrt{l})$.

Thus

$$\|(U_\alpha^* A_1 U_\alpha, \ldots, U_\alpha^* A_n U_\alpha) - (\varphi_1(Y_{11}, \ldots, Y_{rl_r}), \ldots, \varphi_n(Y_{11}, \ldots, Y_{rl_r}))\|_e \le 3\omega\sqrt{nk}.$$

Let $\mathbf{B}(\Phi(Y_{11}, \ldots, Y_{rl_r}), 3\omega\sqrt{nk})$ be the ball in $(M_k^{sa})^n$ of radius $3\omega\sqrt{nk}$ with center $(\varphi_1(Y_{11}, \ldots, Y_{rl_r}), \ldots, \varphi_n(Y_{11}, \ldots, Y_{rl_r}))$. Then the volume of $\mathbf{B}(\Phi(Y_{11}, \ldots, Y_{rl_r}), 3\omega\sqrt{nk})$ is $\pi^{\frac{1}{2}nk^2}\Gamma(1 + \frac{1}{2}nk^2)^{-1}(3\omega\sqrt{nk})^{nk^2}$, and

$$(A_1, \ldots, A_n) \in (U_\alpha)^{(n)} \mathbf{B}(\Phi(Y_{11}, \ldots, Y_{rl_r}), 3\omega\sqrt{nk})(U_\alpha^*)^{(n)},$$

where $(U_\alpha)^{(n)}$ is $(U_\alpha, \ldots, U_\alpha)$ in $(M_k)^n$ and $\Gamma(\cdot)$ is the Γ-function.

By counting the number of all such balls $(U_\alpha)^{(n)}\mathbf{B}(\Phi(Y_{11}, \ldots, Y_{rl_r}), 3\omega\sqrt{nk})(U_\alpha^*)^{(n)}$, we know that it is bounded by

$$|\mathbf{T}(k)| \left(\frac{3}{\delta_2}\right)^{2k^2(l_r+1)/p} \left(\frac{3}{\delta_3}\right)^{2k^2(l_1+1)/p} \cdots \left(\frac{3}{\delta_{r+1}}\right)^{2k^2(l_{r-1}+1)/p}.$$

Thus

$$\begin{aligned}
&\Lambda(\Gamma_R(X_1, \ldots, X_n : U_1, \ldots, U_r, V_{11}, \ldots, V_{rl_r}; m, k, \varepsilon)) \\
&\le |\mathbf{T}(k)| \left(\frac{3}{\delta_2}\right)^{2k^2(l_r+1)/p} \left(\frac{3}{\delta_3}\right)^{2k^2(l_1+1)/p} \cdots \\
&\quad \times \left(\frac{3}{\delta_{r+1}}\right)^{2k^2(l_{r-1}+1)/p} \pi^{\frac{1}{2}nk^2} \cdot \Gamma\left(1 + \frac{1}{2}nk^2\right)^{-1} (3\omega\sqrt{nk})^{nk^2} \\
&\le \left(\frac{3}{\delta_2}\right)^{2k^2(l+r)/p} \left(\frac{C}{\gamma}\right)^{k^2} \pi^{\frac{1}{2}nk^2} \Gamma\left(1 + \frac{1}{2}nk^2\right)^{-1} (3\omega\sqrt{nk})^{nk^2}
\end{aligned}$$

$$\leq \left(\frac{2(17p)^{r-1}D\sqrt{l}}{\pi^{r-1}\omega}\right)^{2k^2(l+r)/p}\left(\frac{2aC}{\omega}\right)^{k^2}\pi^{\frac{1}{2}nk^2}$$
$$\times\,\Gamma\left(1+\frac{1}{2}nk^2\right)^{-1}(3\omega\sqrt{nk})^{nk^2}.$$

Hence

$$\chi(X_1,\ldots,X_n:U_1,\ldots,U_r,V_{11},\ldots,V_{rl_r})$$
$$\leq \limsup_{k\to\infty}\Big(k^{-2}\log\Lambda(\Gamma_R(X_1,\ldots,X_n: G_1,\ldots,G_r,V_{11},\ldots,V_{rl_r};m,k,\varepsilon))+\frac{n}{2}\log k\Big)$$
$$\leq \frac{2(l+r)}{p}\log\left(\frac{2(17p)^{r-1}D\sqrt{l}}{\pi^{r-1}\omega}\right)+\log\left(\frac{2aC}{\omega}\right)+\frac{n}{2}\log(n\pi)$$
$$+\,n\log 3+n\log\omega$$
$$+\limsup_{k\to\infty}\left(n\log k-k^{-2}\log\Gamma\left(1+\frac{nk^2}{2}\right)\right).$$

For the given ω, D, l and r, the above inequalities hold for any p. We may choose p large enough so that

$$\frac{2(l+r)}{p}\log\left(\frac{2(17p)^{r-1}D\sqrt{l}}{\pi^{r-1}\omega}\right)\leq\log 2$$

(since $\frac{\log p}{p}\to 0$ when $p\to\infty$).

Thus we have

$$\chi(X_1,\ldots,X_n:U_1',\ldots,U_r',V_{11},\ldots,V_{rl_r})$$
$$\leq \log(4aC)+\frac{n}{2}\log(n\pi)+n\log 3+\frac{n}{2}$$
$$+\,(n-1-\omega)\log\omega.$$

This finishes the proof of the proposition. The proofs of the three lemmas are straight forward. We omit the details here.

This work was supported by the National Science Foundation.

1. Voiculescu, D., Dykema, K. & Nica, A. (1992) *Free Random Variables,* CRM Monograph Series (Am. Math. Soc., Providence, RI), Vol. 1.
2. Murray, F. J. & von Neumann, J. (1936) Ann. Math. **37,** 116–229.
3. Murray, F. J. & von Neumann, J. (1937) *Trans. Am. Math. Soc.* **41,** 208–248.
4. Murray, F. J. & von Neumann, J. (1943) *Ann. Math.* **44,** 716–808.
5. von Neumann, J. (1930) *Math. Ann.* **102,** 370–427.
6. von Neumann, J. (1940) *Ann. Math.* **41,** 94–161.
7. Connes, A. (1994) *Noncommutative Geometry* (Academic, San Diego).
8. Voiculescu, D. (1994) *Invent. Math.* **118,** 411–440.
9. Voiculescu, D. (1996) *Geom. Funct. Anal.* **6,** 172–199.
10. Ge, L. (1997) *Amer. J. Math.* **119,** 467–485.
11. Ge, L., (1998) *Ann. Math.* **147,** 143–157.
12. Connes, A. & Jones, V. (1985) *Bull. London Math. Soc.* **17,** 57–62.
13. Voiculescu, D. (1999) *J. Reine Angew. Math.* **514,** 113–118.
14. Popa, S. (1986) Correspondences, preprint.
15. Popa, S. (1993) *Invent. Math.* **111,** 375–405.
16. Hadwin, D. (1998) *Contemp. Math.* **228,** 111–131.