

# Improving efficiency and robustness of the doubly robust estimator for a population mean with incomplete data

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## SUMMARY

Considerable recent interest has focused on doubly robust estimators for a population mean response in the presence of incomplete data, which involve models for both the propensity score and the regression of outcome on covariates. The usual doubly robust estimator may yield severely biased inferences if neither of these models is correctly specified and can exhibit nonnegligible bias if the estimated propensity score is close to zero for some observations. We propose alternative doubly robust estimators that achieve comparable or improved performance relative to existing methods, even with some estimated propensity scores close to zero.

*Some key words:* Causal inference; Enhanced propensity score model; Missing at random; No unmeasured confounders; Outcome regression.

## 1. INTRODUCTION

The challenge of estimating a population mean response on the basis of incomplete data arises in many settings. Nonresponse in sample surveys or dropout and noncompliance in clinical trials may lead to missing outcomes for some subjects; likewise, making causal inference on a treatment mean may be viewed as a missing data problem, where potential outcomes under treatment are missing for subjects actually observed to receive control (Kang & Schafer, 2007). In these situations, unless the missingness mechanism is completely at random (Rubin, 1976), it is well known that the naive sample mean based on the complete cases is a biased estimator.

If missing data can reasonably be assumed missing at random, or, equivalently, if the no unmeasured confounders assumption (Rosenbaum & Rubin, 1983; Robins et al., 2000) is tenable when making causal inference from observational data, popular approaches include estimation based on a posited outcome regression model for the relationship between response and covariates and methods that use fitted models for the propensity score, the probability of the response being observed given covariates (Rosenbaum & Rubin, 1983), such as stratification or matching (Rosenbaum & Rubin, 1984; Rubin & Thomas, 1996; Lunceford & Davidian, 2004) and inverse probability weighting of responses (Robins et al., 1994; Rosenbaum, 1987; Lunceford & Davidian, 2004). These methods require correct specification of the model for outcome regression or propensity score, respectively. Robins et al. (1994) identified a class of augmented inverse probability weighted estimators that involve modelling both the outcome regression and propensity score, with the efficient member of the class obtained when both models are correct. Scharfstein et al. (1999) noted that estimators in this class are doubly robust in that they are consistent for the true population mean even if one of the outcome regression or propensity score models, but not both, is misspecified. Given the protection afforded by this

property, these estimators have been advocated for routine use (Bang & Robins, 2005). However, Kang & Schafer (2007) demonstrated via simulation that the usual doubly robust estimator can be severely biased when both models are misspecified, even if they are nearly correct, and that bias is especially problematic when some estimated propensity scores are close to zero, yielding very large weights. Estimation based on an outcome regression model only performed much better under misspecification in the Kang–Schafer simulation scenario, leading the authors to warn against the use of doubly robust estimators in practice. Tan (2006) discussed alternative approaches to constructing doubly robust estimators that may alleviate some of these difficulties. In this paper, we propose doubly robust estimators that may yield improved performance relative to existing competitors.

## 2. EXISTING DOUBLY ROBUST ESTIMATORS

As in Kang & Schafer (2007), we consider the standard missing data set-up; the spirit of the developments is equally relevant to the causal inference context. Consider  $n$  subjects drawn at random from a population of interest, where the ideal, full data are  $(Y_i, X_i)$  ( $i = 1, \dots, n$ ) independent and identically distributed across  $i$ ;  $Y_i$  is the response or outcome; and  $X_i$  is a vector of covariates. As in § 1,  $Y_i$  is not available for all subjects; thus, the data actually observed are independent and identically distributed  $(R_i Y_i, R_i, X_i)$  ( $i = 1, \dots, n$ ), where  $R_i = 1$  or 0 as  $Y_i$  is observed or missing. The goal is to estimate the population mean,  $\mu = E(Y)$ , on the basis of these observed data. Throughout, assume that responses are missing at random (Rubin, 1978) in that  $Y_i$  and  $R_i$  are conditionally independent given  $X_i$ .

The propensity score is  $\text{pr}(R = 1 | X)$ ; denote the true propensity score as  $\pi_0(X)$ . Ordinarily,  $\pi_0(X)$  is unknown, and it is customary to posit a parametric model; for example, a logistic regression model  $\pi(X, \gamma) = \{1 + \exp(\tilde{X}^T \gamma)\}^{-1}$ ,  $\tilde{X} = (1, X^T)^T$ . Letting  $\hat{\gamma}$  denote the maximum likelihood estimator for  $\gamma$  based on  $(R_i, X_i)$  ( $i = 1, \dots, n$ ), it is straightforward to show (Lunceford & Davidian, 2004) that the inverse probability weighted estimators

$$\hat{\mu}_{\text{IPW1}} = n^{-1} \sum_{i=1}^n \frac{R_i Y_i}{\pi(X_i, \hat{\gamma})} \quad \text{and} \quad \hat{\mu}_{\text{IPW2}} = \left\{ \sum_{i=1}^n \frac{R_i}{\pi(X_i, \hat{\gamma})} \right\}^{-1} \sum_{i=1}^n \frac{R_i Y_i}{\pi(X_i, \hat{\gamma})} \quad (1)$$

are consistent for  $\mu$  if  $\pi(X, \gamma)$  is correctly specified; that is,  $\pi_0(X) = \pi(X, \gamma_0)$  for some  $\gamma_0$ .

Alternatively, because under missing at random  $E\{E(Y | R = 1, X)\} = E\{E(Y | X)\} = E(Y)$ , letting  $m_0(X)$  denote the true outcome regression  $E(Y | X)$ , it is natural to adopt a model  $m(X, \beta)$  for  $m_0(X)$ , estimate  $\beta$  by some  $\hat{\beta}$  using the complete cases  $\{i : R_i = 1\}$  and estimate  $\mu$  by

$$\hat{\mu}_{\text{OR}} = n^{-1} \sum_{i=1}^n m(X_i, \hat{\beta}), \quad (2)$$

which is consistent for  $\mu$  if  $m(X, \beta)$  is correctly specified; that is,  $m_0(X) = m(X, \beta_0)$  for some  $\beta_0$ , and if  $\hat{\beta}$  is consistent for  $\beta_0$ . Because  $\hat{\beta}$  is based only on the complete cases, if the distributions of  $X$  conditional on  $R = 1$  and  $R = 0$  differ, (2) involves extrapolation.

From Robins et al. (1994) and Tsiatis & Davidian (2007), all estimators for  $\mu$  that are consistent and asymptotically normal when the propensity score model is correct are asymptotically equivalent to an estimator of the form

$$n^{-1} \sum_{i=1}^n \left\{ \frac{R_i Y_i}{\pi(X_i, \hat{\gamma})} + \frac{R_i - \pi(X_i, \hat{\gamma})}{\pi(X_i, \hat{\gamma})} h(X_i) \right\} \quad (3)$$

for arbitrary  $h(X)$ . Estimators in class (3) are referred to as augmented inverse probability weighted because they have the form of  $\hat{\mu}_{IPW1}$  in (1) plus an augmentation term depending on  $h(X)$ ;  $\hat{\mu}_{IPW1}$  is obtained when  $h(X) \equiv 0$ . From Robins et al. (1994), the estimator with the smallest asymptotic variance among those in class (3), so with  $\pi(X, \gamma)$  correct, is

$$\hat{\mu}_{DR} = n^{-1} \sum_{i=1}^n \left\{ \frac{R_i Y_i}{\pi(X_i, \hat{\gamma})} - \frac{R_i - \pi(X_i, \hat{\gamma})}{\pi(X_i, \hat{\gamma})} m(X_i, \hat{\beta}) \right\}, \tag{4}$$

taking  $h(X_i) = -m(X_i, \hat{\beta})$ , where  $m(X, \beta)$  is correctly specified, and  $\hat{\beta}$  is consistent for  $\beta_0$ . Scharfstein et al. (1999) noted that  $\hat{\mu}_{DR}$  remains consistent if only one of the outcome regression model  $m(X, \beta)$  or the propensity score model  $\pi(X, \gamma)$  is correctly specified, but is inconsistent if both are misspecified; this property is referred to as double robustness. If  $m(X, \beta)$  is correct, then  $\hat{\mu}_{OR}$  is at least as efficient as  $\hat{\mu}_{DR}$  (Tan, 2007) but is inconsistent otherwise, while double robustness of (4) affords protection against such misspecification.

The estimator (4), with  $\gamma$  estimated by maximum likelihood and  $\beta$  estimated by ordinary or iteratively reweighted least squares is generally regarded as the usual doubly robust estimator. Kang & Schafer (2007) and Tan (2006) identified alternative doubly robust estimators, all involving models for the propensity score and outcome regression and some appearing to have forms outside the augmented class (3). The former authors attributed poor performance of (4) when the propensity or both models are misspecified in part to inverse weighting by the propensity score. Tsiatis & Davidian (2007) noted that such alternative estimators can be rewritten in the form (3) and used semiparametric theory to argue that poor performance when one or the other model is incorrect may be partly a consequence of the method used to estimate  $\beta$ . In § 3, we identify doubly robust estimators from this perspective. When both models are correct and  $\gamma$  is estimated by maximum likelihood, all doubly robust estimators are consistent with the same asymptotic variance; moreover, the asymptotic properties do not depend on the method used to estimate  $\beta$  (Tan, 2007; Tsiatis & Davidian, 2007).

### 3. ALTERNATIVE DOUBLY ROBUST ESTIMATORS

In this section, we focus on estimation of  $\beta$  in a posited outcome regression model  $m(X, \beta)$ , possibly nonlinear in  $\beta$ , to identify doubly robust estimators with desirable properties. To fix ideas, we consider first a fully specified propensity score model  $\pi(X)$ , say, involving no unknown parameters; we relax this shortly. Suppose, for some estimator  $\hat{\beta}$  for  $\beta$ , we estimate  $\mu$  by

$$n^{-1} \sum_{i=1}^n \left\{ \frac{R_i Y_i}{\pi(X_i)} - \frac{R_i - \pi(X_i)}{\pi(X_i)} m(X_i, \hat{\beta}) \right\}. \tag{5}$$

We now examine how to estimate  $\beta$  to achieve the estimator for  $\mu$  of form (5) that is (i) doubly robust and, (ii) if the propensity score is correctly specified, has smallest asymptotic variance among all estimators for  $\mu$  of form (5) using  $m(X, \beta)$ , even if  $m(X, \beta)$  is incorrect.

Suppose first that the propensity score is correct,  $\pi(X) = \pi_0(X)$ , but  $m(X, \beta)$  may or may not be correctly specified. It is straightforward to show that using any estimator  $\hat{\beta}$  in (5) leads to a consistent estimator for  $\mu$  whose asymptotic variance is the same as that of

$$n^{-1} \sum_{i=1}^n \left\{ \frac{R_i Y_i}{\pi_0(X_i)} - \frac{R_i - \pi_0(X_i)}{\pi_0(X_i)} m(X_i, \beta^*) \right\}, \tag{6}$$

where  $\beta^*$  is the limit in probability of  $\hat{\beta}$ . Using the formula  $\text{var}(\cdot) = E\{\text{var}(\cdot | X, Y)\} + \text{var}\{E(\cdot | X, Y)\}$ , this variance is proportional to

$$\text{var} \left\{ \frac{RY}{\pi_0(X)} - \frac{R - \pi_0(X)}{\pi_0(X)} m(X, \beta^*) \right\} = E \left[ \frac{1 - \pi_0(X)}{\pi_0(X)} \{Y - m(X, \beta^*)\}^2 \right] + \text{var}(Y). \tag{7}$$

A natural objective is to identify the value of  $\beta^*$ , and corresponding estimators  $\hat{\beta}$  converging in probability to it, such that (7) is minimized whether or not  $m(X, \beta)$  is correct. Letting  $m_\beta(X, \beta) = \partial/\partial\beta\{m(X, \beta)\}$ , note that (7) is minimized by choosing  $\beta^*$  as the solution to  $E\{[1 - \pi_0(X)]\pi_0^{-1}(X)\{Y - m(X, \beta^*)\}m_\beta(X, \beta^*)\} = 0$ , or equivalently

$$E \left[ \frac{1 - \pi_0(X)}{\pi_0(X)} \{m_0(X) - m(X, \beta^*)\}m_\beta(X, \beta^*) \right] = 0; \tag{8}$$

denote this value by  $\beta_{\text{opt}}^*$ . Note that  $\beta_{\text{opt}}^* = \beta_0$  when  $m(X, \beta)$  is correctly specified.

Consider first the ordinary least-squares estimator for  $\beta$ ,  $\hat{\beta}_1$ , say, solving

$$n^{-1} \sum_{i=1}^n R_i \{Y - m(X_i, \beta)\} m_\beta(X_i, \beta) = 0, \tag{9}$$

based on the complete cases. If the propensity score is correct,  $\pi(X) = \pi_0(X)$ , but  $m(X, \beta) \neq m_0(X)$  for any  $\beta$ , then the left-hand side of (9) converges in probability to

$$E [\pi_0(X) \{m_0(X) - m(X, \beta)\} m_\beta(X, \beta)]. \tag{10}$$

Then  $\hat{\beta}_1$  converges in probability to the value  $\beta_1$  such that (10) equals zero; however, comparing (10) to (8) shows  $\beta_1 \neq \beta_{\text{opt}}^*$ . If the propensity score is incorrect, but the outcome regression model is correct, so that  $m(X, \beta_0) = m_0(X)$  for some  $\beta_0$ , then the left-hand side of (9) again converges to (10), and  $\beta_1 = \beta_0$ , so that  $\hat{\beta}_1$  converges in probability to  $\beta_0$ . Thus, the estimator (5) for  $\mu$  using  $\hat{\beta}_1$  is doubly robust but does not achieve the minimum variance when the outcome regression model is misspecified. Estimation of  $\beta$  by solving (9) would most likely be undertaken with continuous  $Y$ ; a similar result holds if  $\beta$  is estimated via iteratively reweighted least squares, as in the case of a generalized linear model  $m(X, \beta)$ .

Suppose we consider instead estimating  $\beta$  by minimizing the empirical variance of (5),  $n^{-2} \sum_{i=1}^n [R_i Y_i \pi^{-1}(X_i) - \{R_i - \pi(X_i)\} \pi^{-1}(X_i) m(X_i, \beta)]^2$  in  $\beta$ , leading to  $\hat{\beta}_2$  solving

$$n^{-1} \sum_{i=1}^n \frac{R_i - \pi(X_i)}{\pi(X_i)} \left\{ \frac{R_i Y_i}{\pi(X_i)} - \frac{R_i - \pi(X_i)}{\pi(X_i)} m(X_i, \beta) \right\} m_\beta(X_i, \beta) = 0. \tag{11}$$

If the propensity score is correct but  $m(X, \beta) \neq m_0(X)$  for any  $\beta$ , then the left-hand side of (11) converges in probability to an expression of the form (8). Thus, it follows that  $\hat{\beta}_2$  converges in probability to  $\beta_{\text{opt}}^*$ . When the propensity score is incorrect but the outcome regression model is correct, algebra shows that the left-hand side of (11) converges to

$$E \left( \left[ \frac{\pi_0(X)\{1 - \pi(X)\}}{\pi^2(X)} m_0(X) - \left\{ \frac{\pi_0(X) - 2\pi_0(X)\pi(X) + \pi^2(X)}{\pi^2(X)} m(X, \beta) \right\} \right] m_\beta(X, \beta) \right).$$

The value of  $\beta$  setting this equal to zero, to which  $\hat{\beta}_2$  converges in probability, is clearly not  $\beta_0$ . Thus, the estimator (5) using  $\hat{\beta}_2$  achieves minimum variance but is not doubly robust.

These calculations show that using familiar or seemingly intuitive techniques to estimate  $\beta$  for use in (5) leads to estimators for  $\mu$  that meet one of conditions (i) or (ii), but not both. To satisfy

(i) and (ii) simultaneously, we consider  $\hat{\beta}_3$  to be the solution to

$$n^{-1} \sum_{i=1}^n \frac{R_i}{\pi(X_i)} \frac{1 - \pi(X_i)}{\pi(X_i)} \{Y_i - m(X_i, \beta)\} m_{\beta}(X_i, \beta) = 0, \tag{12}$$

which may be viewed as weighted least squares based on complete cases with weights  $\{1 - \pi(X_i)\}/\pi^2(X_i)$ . When the propensity score is correct but the outcome regression is not, like that of (11), the left-hand side of (12) converges in probability to an expression of the form (8), and hence  $\hat{\beta}_3$  converges in probability to  $\beta_{\text{opt}}^*$ . When the outcome regression is correctly specified and the propensity score is not, the left-hand side of (12) converges to  $E[\pi_0(X)\{1 - \pi(X)\}\pi^{-2}(X)\{m_0(X) - m(X, \beta)\}m_{\beta}(X, \beta)]$ , which equals zero when  $\beta = \beta_0$ , so that  $\hat{\beta}_3$  converges in probability to  $\beta_0$ . Thus, the estimator (5) for  $\mu$  with  $\hat{\beta} = \hat{\beta}_3$  is doubly robust and achieves minimum asymptotic variance even if  $m(X, \beta)$  is misspecified.

In practice, a parametric propensity score model  $\pi(X, \gamma)$  would be posited. Here, we cannot use the above results directly to find an estimator for  $\mu$  of the form of  $\hat{\mu}_{\text{DR}}$  in (4), where  $\hat{\gamma}$  is the maximum likelihood estimator for binary regression, that satisfies conditions (i) and (ii). There is an effect of estimating  $\gamma$  that must be taken into account, so that finding  $\hat{\beta}$  converging to the minimizer of (7), which assumes  $\pi(X)$  is fully specified, does not necessarily lead to minimum asymptotic variance under a model  $\pi(X, \gamma)$  with  $\gamma$  estimated. However, we may exploit the insights gained from the foregoing results, as we now demonstrate.

Let  $S_{\gamma}(R, X, \gamma) = \{R - \pi(X, \gamma)\}[\pi(X, \gamma)\{1 - \pi(X, \gamma)\}]^{-1}\pi_{\gamma}(X, \gamma)$  be the score for  $\gamma$ , where  $\pi_{\gamma}(X, \gamma) = \partial/\partial\gamma\{\pi(X, \gamma)\}$ . From the point of view of semiparametric theory, the elements of the class of influence functions (Tsiatis, 2006, chapter 3) corresponding to estimators for  $\mu$  of the form (5), with fully and correctly specified  $\pi(X)$  but possibly incorrect  $m(X, \beta)$  and using  $\hat{\beta}$  converging in probability to some  $\beta^*$ , have the form  $RY/\pi_0(X) - [\{R - \pi_0(X)\}/\pi_0(X)]m(X, \beta^*) - \mu$ . The influence functions corresponding to estimators of the form (4) when  $\pi(X, \gamma)$  is correctly specified, so that  $\pi(X, \gamma_0) = \pi_0(X)$  for some  $\gamma_0$ , have the form

$$\frac{RY}{\pi_0(X)} - \frac{R - \pi_0(X)}{\pi_0(X)}m(X, \beta^*) - \Gamma_0^{\top}(\beta^*)\Sigma_{\gamma\gamma,0}^{-1}S(\gamma, R, X, \gamma_0) - \mu \tag{13}$$

and this equals

$$\frac{RY}{\pi_0(X)} - \frac{R - \pi_0(X)}{\pi_0(X)} \left\{ m(X, \beta^*) + \Gamma_0^{\top}(\beta^*)\Sigma_{\gamma\gamma,0}^{-1} \frac{\pi_{\gamma,0}(X)}{1 - \pi_0(X)} \right\} - \mu, \tag{14}$$

where  $\pi_{\gamma,0}(X) = \pi_{\gamma}(X, \gamma_0)$ ,

$$\begin{aligned} \Gamma_0(\beta^*) &= E[\pi_{\gamma,0}(X)\{m_0(X) - m(X, \beta^*)\}/\pi_0(X)], \\ \Sigma_{\gamma\gamma,0} &= E(\pi_{\gamma,0}(X)\pi_{\gamma,0}^{\top}(X)/[\pi_0(X)\{1 - \pi_0(X)\}]), \end{aligned}$$

with  $\Sigma_{\gamma\gamma,0}$  assumed nonsingular. The influence functions (13) thus involve an additional term due to estimation of  $\gamma$ , the projection onto the propensity score tangent space, the linear space spanned by the score (Tsiatis, 2006, Theorem 9.1). Because the influence function of an estimator dictates its asymptotic variance, we would like to find  $\hat{\beta}$  to substitute in (4) converging to  $\beta_{\text{opt}}^{**}$ , say, that minimizes the variance of (13). We do this by considering a class of influence functions containing class (13), with elements

$$\frac{RY}{\pi_0(X)} - \frac{R - \pi_0(X)}{\pi_0(X)}m(X, \beta^*) - c^{*\top}S(\gamma, R, X, \gamma_0) - \mu$$

and this equals

$$\frac{RY}{\pi_0(X)} - \frac{R - \pi_0(X)}{\pi_0(X)} \left\{ m(X, \beta^*) + c^{*\top} \frac{\pi_{\gamma,0}(X)}{1 - \pi_0(X)} \right\} - \mu \tag{15}$$

for arbitrary  $(\beta^*, c^*)$ . Identifying the expression in braces in (15) as a function of  $(\beta^*, c^*)$  with  $m(X, \beta^*)$  in (7) and (8), by analogy to (7) and (8),  $(\beta_{\text{opt}}^{**}, c_{\text{opt}}^{**})$  solving

$$E \left[ \frac{1 - \pi_0(X)}{\pi_0(X)} \left\{ m_0(X) - m(X, \beta^*) - c^{*\top} \frac{\pi_{\gamma,0}(X)}{1 - \pi_0(X)} \right\} \left\{ \begin{matrix} m_\beta(X, \beta^*) \\ \frac{\pi_{\gamma,0}(X)}{1 - \pi_0(X)} \end{matrix} \right\} \right] = 0$$

minimize the variance of (15). This yields  $c_{\text{opt}}^{**} = \Gamma_0^{\top}(\beta_{\text{opt}}^{**})\Sigma_{\gamma\gamma,0}^{-1}$ , so that (15) with  $(\beta_{\text{opt}}^{**}, c_{\text{opt}}^{**})$  substituted has the same form as (14), and hence  $\beta_{\text{opt}}^{**}$  minimizes the variance of (13). Thus, an estimator for  $\mu$  of the form (4), with the smallest asymptotic variance when  $\pi(X, \gamma)$  is correctly specified but  $m(X, \beta)$  may not be, is achieved by using  $\hat{\beta}$  converging in probability to  $\beta_{\text{opt}}^{**}$ . By analogy to (12), we propose estimating  $\beta$  by solving jointly in  $(\beta, c)$

$$\sum_{i=1}^n \left[ \frac{R_i}{\pi(X_i, \hat{\gamma})} \frac{1 - \pi(X_i, \hat{\gamma})}{\pi(X_i, \hat{\gamma})} \left\{ \begin{matrix} m_\beta(X_i, \beta^*) \\ \frac{\pi_\gamma(X_i, \hat{\gamma})}{1 - \pi(X_i, \hat{\gamma})} \end{matrix} \right\} \left\{ Y_i - m(X_i, \beta) - c^\top \frac{\pi_\gamma(X_i, \hat{\gamma})}{1 - \pi(X_i, \hat{\gamma})} \right\} \right] = 0. \tag{16}$$

By an argument entirely similar to that following (12), when the propensity model is correct but  $m(X, \beta)$  may or may not be,  $\hat{\beta}_4$ , say, solving (16) converges in probability to  $\beta_{\text{opt}}^{**}$ . When  $m(X, \beta)$  is correct but  $\pi(X, \gamma)$  is not, assuming that  $\hat{\gamma}$  converges in probability to some  $\gamma^*$ , the quantity to which the left-hand side of (16) converges in probability equals zero when  $(\beta, c) = (\beta_0, 0)$ . Thus, taking  $\hat{\beta} = \hat{\beta}_4$  in (4) yields an estimator for  $\mu$  that is (i) doubly robust and (ii) achieves minimum asymptotic variance when the propensity model is correct. In the sequel, we denote this estimator by  $\hat{\mu}_{\text{PROJ}}$  and denote the usual doubly robust estimator taking  $\hat{\beta} = \hat{\beta}_1$ , the ordinary least-squares estimator for  $\beta$  solving (9), by  $\hat{\mu}_{\text{USUAL}}$ .

Tan (2006) proposed a doubly robust estimator for  $\mu$  that is closely related to  $\hat{\mu}_{\text{PROJ}}$ . In the present context, Tan’s estimator is equivalent to modelling  $E(Y | X)$  by  $m(X, \beta)$  and estimating  $\beta$  by ordinary or iteratively reweighted least squares ( $\hat{\beta}_1$ ); replacing  $m(X, \beta)$  in (4) and (16) by  $\tilde{m}(X, \tilde{\beta}) = \alpha_0 + \alpha_1 m(X, \beta)$ ,  $\tilde{\beta} = (\alpha_0, \alpha_1, \beta^\top)^\top$ ; holding  $\beta$  fixed at  $\hat{\beta}_1$  and solving (16) in  $(\alpha_0, \alpha_1, c)$ , where  $m_\beta(X, \beta)$  is replaced by  $\{1, m(X, \hat{\beta}_1)\}^\top$ ; and substituting the resulting estimates for  $(\alpha_0, \alpha_1)$  and  $\hat{\beta}_1$  for  $\tilde{\beta}$  in (4). Denote this estimator by  $\hat{\mu}_{\text{TAN}}$ . If, in constructing  $\hat{\mu}_{\text{PROJ}}$ , we similarly replace  $m(X, \beta)$  by  $\tilde{m}(X, \tilde{\beta})$  in (4) and (16), but estimate all elements of  $\tilde{\beta}$  simultaneously by solving (16) with  $m_\beta(X, \beta)$  replaced by  $\partial/\partial\tilde{\beta}\{\tilde{m}(X, \tilde{\beta})\}$ , then, by the same reasoning as above, the resulting estimator for  $\mu$  will have asymptotic variance at least as small as that of  $\hat{\mu}_{\text{TAN}}$  when the propensity score is correct, as this estimator for  $\beta$  will converge in probability to the optimal value minimizing this variance, while  $\hat{\beta}_1$  used by Tan will not. If  $m(X, \beta)$  is correctly specified but  $\pi(X, \gamma)$  is not, because the estimator for  $\tilde{\beta}$  obtained by either method converges in probability to  $(0, 1, \beta_0^\top)^\top$ , both  $\hat{\mu}_{\text{TAN}}$  and this version of  $\hat{\mu}_{\text{PROJ}}$  are doubly robust; this would also hold if the true form of  $E(Y | X)$  were  $\alpha_0 + \alpha_1 m(X, \beta)$  for  $(\alpha_0, \alpha_1) \neq (0, 1)$ . Thus, although these versions of  $\hat{\mu}_{\text{PROJ}}$  and  $\hat{\mu}_{\text{TAN}}$  are doubly robust, the former is at least as efficient as the latter.

All of the estimators  $\hat{\mu}_{\text{USUAL}}$ ,  $\hat{\mu}_{\text{PROJ}}$  and  $\hat{\mu}_{\text{TAN}}$  involve solving jointly a set of M-estimating equations (Stefanski & Boos, 2002); for example,  $\hat{\mu}_{\text{USUAL}}$  is found by solving the usual score equation for  $\gamma$ , the ordinary least-squares equation (9) and the estimating equation implied by (4). Thus, the asymptotic variance of the estimator for  $\mu$  can be approximated by the usual empirical sandwich technique; see Stefanski & Boos (2002). The resulting estimator for variance

will be consistent for the true sampling variance even if one or both of the propensity or outcome regression models is incorrectly specified.

#### 4. ENHANCED PROPENSITY SCORE MODEL

Doubly robust estimators such as  $\hat{\mu}_{\text{PROJ}}$  that also achieve minimum variance when the propensity model is correct but the outcome regression model may not be should lead to improved performance over  $\hat{\mu}_{\text{USUAL}}$  under these conditions. However, the problem of large weights  $1/\pi(X_i, \hat{\gamma})$  can also affect performance; as illustrated by [Kang & Schafer \(2007\)](#), if both models are even mildly misspecified, then  $\hat{\mu}_{\text{USUAL}}$  may be severely biased due to a few very large weights. If the propensity model in particular is slightly misspecified,  $\pi(X_i, \hat{\gamma})$  can be erroneously close to zero for some  $i$ . We consider an approach to address this issue.

If the propensity score model is correct, we expect that  $\sum_{i=1}^n R_i/\pi(X_i, \hat{\gamma}) \approx n$ . When the estimated propensities for some observations are close to zero, this quantity can be very different from  $n$ . We thus consider propensity models and estimators that impose the restriction that this quantity be equal to  $n$ ; if the chosen model is misspecified, this restriction will drive estimated propensities away from zero. We thus propose an enhanced propensity score model, given by

$$\text{pr}(R = 1 \mid X) = \pi(X, \delta, \gamma) = 1 - \frac{\exp(\delta + \tilde{X}^T \gamma)}{1 + \exp(\tilde{X}^T \gamma)}, \quad (17)$$

where  $\delta$  is a scalar parameter. If  $\delta = 0$ , (17) reduces to a usual logistic regression model; otherwise,  $\delta$  is an enhancement imposing the constraint  $\sum_{i=1}^n R_i/\pi(X_i, \hat{\delta}, \hat{\gamma}) = n$ . This follows because the score for  $\delta$  is  $n - \sum_{i=1}^n R_i/\pi(X_i, \delta, \gamma)$ , so that if maximum likelihood is used to estimate  $(\delta, \gamma^T)^T$ , the constraint is satisfied automatically. Because  $\pi(X, \delta, \gamma)$  can take values outside  $(0, 1)$ , we impose  $0 < \pi(X, \delta, \gamma) < 1$  and implement maximum likelihood subject to this restriction, which can be carried out with standard optimization packages.

From a semiparametric theory perspective, it may be shown that use of the enhanced model should lead to an increase in efficiency in estimation of  $\mu$  by any of the methods in § 3 relative to using the logistic regression model with  $\gamma$  alone as long as (17) contains  $\pi_0(X)$ . This follows because the influence functions for these estimators when (17) is used involve an additional term relative to those for the same estimators using the model with  $\delta = 0$ . Those with the additional term have smaller variance; see [Tsiatis \(2006, chapter 9\)](#).

#### 5. SIMULATION STUDIES

We carried out several simulation studies to assess performance of the proposed methods under two scenarios. For both scenarios, for each of  $n = 200$  and  $1000$ , we considered the four possible combinations of correct and misspecified outcome regression and propensity score models. For each scenario/setting combination, 1000 Monte Carlo datasets were generated, and the estimators  $\hat{\mu}_{\text{OR}}$ ,  $\hat{\mu}_{\text{USUAL}}$ ,  $\hat{\mu}_{\text{TAN}}$  and  $\hat{\mu}_{\text{PROJ}}$  were calculated for each, where  $\hat{\mu}_{\text{PROJ}}$  was constructed using  $\tilde{m}(X, \tilde{\beta})$  as described in § 3. We also constructed the estimators  $\hat{\mu}_{\text{USUAL}}^{\text{en}}$  and  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$ , which are the indicated estimators with the enhanced propensity model (17), replacing the usual logistic propensity model described below and fitted by constrained maximum likelihood. For each estimator, sandwich standard errors and nominal 95% Wald confidence intervals for  $\mu$  were calculated. To calculate  $\hat{\mu}_{\text{USUAL}}^{\text{en}}$  and  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$ , we used the SAS IML optimizer nlpqn ([SAS Institute, 2006](#)) to fit the enhanced propensity model.

We duplicated the scenarios in Kang & Schafer (2007) and Tan (2007), which were designed so that, when misspecified, the assumed outcome regression and propensity score models were nonetheless nearly correct; our choice of these scenarios allows consideration of the proposed methods in a familiar context that was designed to highlight differences among estimators. Kang & Schafer found that, under their scenario,  $\hat{\mu}_{\text{USUAL}}$  exhibited severe bias when both models were misspecified but nearly correct, while  $\hat{\mu}_{\text{OR}}$  was not as severely affected, leading the authors to contend that ‘two wrong models are not necessarily better than one.’ Tan modified Kang & Schafer’s scenario slightly and showed that versions of  $\hat{\mu}_{\text{TAN}}$  offered improvement over  $\hat{\mu}_{\text{USUAL}}$ . For the Kang & Schafer scenario, for each  $i$  ( $i = 1, \dots, n$ ),  $Z_i = (Z_{i1}, Z_{i2}, Z_{i3}, Z_{i4})^T$  was generated as standard multivariate normal, and the elements of  $X_i = (X_{i1}, X_{i2}, X_{i3}, X_{i4})^T$  were defined as  $X_{i1} = \exp(Z_{i1}/2)$ ,  $X_{i2} = Z_{i2}/\{1 + \exp(Z_{i1})\} + 10$ ,  $X_{i3} = (Z_{i1}Z_{i3}/25 + 0.6)^3$  and  $X_{i4} = (Z_{i2} + Z_{i4} + 20)^2$ , so that  $Z_i$  may be expressed in terms of  $X_i$ . For each  $i$ ,  $Y_i = m_0(X_i) + \epsilon_i$  for  $\epsilon_i$  standard normal and  $m_0(X_i) = 210 + 27.4Z_{i1} + 13.7Z_{i2} + 13.7Z_{i3} + 13.7Z_{i4}$ , and  $R_i$  was generated as Bernoulli with true propensity  $\pi_0(X_i) = \text{expit}(-Z_{i1} + 0.5Z_{i2} - 0.25Z_{i3} - 0.1Z_{i4})$ , where  $\text{expit}(u) = e^u/(1 + e^u)$ . Correctly specified outcome regression and propensity models were thus achieved when an additive linear regression of  $Y_i$  on  $Z_i$  and a logistic regression with linear predictor additive in the  $Z_i$  for  $R_i$ , respectively, were fitted; nearly correctly specified models involved fitting these models with  $X_i$  replacing  $Z_i$ ; see Kang & Schafer (2007). The Tan scenario was identical to that of Kang & Schafer, except that  $X_{i4} = (Z_{i3} + Z_{i4} + 20)^2$ . The true value of the mean is  $\mu = 210$ .

Results for the Kang & Schafer and Tan scenarios are in Tables 1 and 2, respectively. When both models are correct, all estimators perform similarly, and all of the doubly robust estimators show negligible Monte Carlo bias when at least one of the models is correctly specified, as expected. Moreover,  $\hat{\mu}_{\text{PROJ}}$  and  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$  for the most part exhibit efficiencies no worse or better than those of  $\hat{\mu}_{\text{OR}}$  and the other doubly robust estimators on the basis of the root mean square error and the median absolute error, and in particular dominate the others when the outcome regression model is misspecified but the propensity model is correct, consistent with the basis of their construction. When both models are incorrectly specified,  $\hat{\mu}_{\text{USUAL}}$  shows nonnegligible bias, as observed by Kang & Schafer (2007) and Tan (2007); however, the use of the enhanced propensity model in  $\hat{\mu}_{\text{USUAL}}^{\text{en}}$  eliminates this behaviour. The proposed estimators  $\hat{\mu}_{\text{PROJ}}$  and  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$  exhibit the best performance in terms of bias and efficiency when both models are misspecified; in the Appendix, we sketch a heuristic argument suggesting that this behaviour is not unexpected. Overall,  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$  shows the best performance across the range of settings in both scenarios.

Confidence intervals based on sandwich standard errors based on the doubly robust estimators for the most part attain nominal coverage except when both models are misspecified in the Kang & Schafer scenario; those for  $\hat{\mu}_{\text{PROJ}}$  and  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$  perform consistently well except in this case. Not unexpectedly, when the outcome regression model is misspecified, confidence intervals based on  $\hat{\mu}_{\text{OR}}$  can suffer from undercoverage.

## 6. DISCUSSION

Our work complements that of (Tan, 2006, 2007) and Robins et al. (2007), who also demonstrated that it is possible to identify doubly robust estimators that do not suffer the drawbacks demonstrated by Kang & Schafer (2007) under model misspecification. We have focused our development on estimation of a single treatment mean in order to demonstrate the approach to developing optimal, doubly robust estimators in § 3 in an accessible context; however, the results are relevant to more complex estimands. In the case where a difference of treatment means is of interest, if one restricts attention to outcome regression models linear in a vector of known



Table 1. Simulation results based on 1000 Monte Carlo replications for the Kang & Schafer scenario. Smallest, median, second largest, and largest standard errors for table entries: BIAS (0.04, 0.08, 0.39, 5.58); AVESE (0.0008, 0.004, 0.58, 6.64); COV (0.006, 0.007, 0.015, 0.015)

	BIAS	RMSE	MAE	MCS	AVESE	COV	BIAS	RMSE	MAE	MCS	AVESE	COV
<i>n</i> = 200												
OR correct, PS correct						OR correct, PS incorrect						
$\hat{\mu}_{OR}$	-0.06	2.51	1.66	2.51	2.56	0.96	-0.06	2.51	1.66	2.51	2.56	0.96
$\hat{\mu}_{USUAL}$	-0.06	2.51	1.66	2.51	2.56	0.95	-0.05	2.53	1.70	2.53	2.57	0.95
$\hat{\mu}_{PROJ}$	-0.07	2.51	1.69	2.51	2.56	0.95	-0.06	2.50	1.68	2.50	2.56	0.96
$\hat{\mu}_{TAN}$	-0.05	2.51	1.68	2.51	2.58	0.95	-0.05	2.51	1.67	2.51	2.51	0.96
$\hat{\mu}_{USUAL}^{en}$	-0.06	2.51	1.66	2.51	2.56	0.96	-0.06	2.51	1.67	2.51	2.62	0.96
$\hat{\mu}_{PROJ}^{en}$	-0.06	2.51	1.68	2.51	2.58	0.95	-0.06	2.51	1.70	2.51	2.63	0.96
OR incorrect, PS correct						OR incorrect, PS incorrect						
$\hat{\mu}_{OR}$	-0.55	3.29	2.14	3.24	3.24	0.93	-0.55	3.29	2.14	3.24	3.24	0.93
$\hat{\mu}_{USUAL}$	0.36	3.53	2.33	3.51	3.22	0.94	-5.19	13.26	3.62	12.20	6.54	0.92
$\hat{\mu}_{PROJ}$	-0.06	2.57	1.72	2.57	2.60	0.95	-0.39	3.58	2.00	3.55	3.28	0.93
$\hat{\mu}_{TAN}$	0.16	2.88	1.96	2.88	2.81	0.95	-1.77	3.52	2.36	3.05	3.04	0.90
$\hat{\mu}_{USUAL}^{en}$	0.54	3.26	2.27	3.22	3.37	0.94	-1.53	3.51	2.29	3.16	5.48	0.91
$\hat{\mu}_{PROJ}^{en}$	-0.04	2.57	1.70	2.57	2.85	0.96	-0.31	3.48	1.89	3.47	3.63	0.94
<i>n</i> = 1000												
OR correct, PS correct						OR correct, PS incorrect						
$\hat{\mu}_{OR}$	-0.03	1.13	0.73	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
$\hat{\mu}_{USUAL}$	-0.03	1.13	0.73	1.13	1.15	0.95	0.01	1.72	0.74	1.72	1.28	0.95
$\hat{\mu}_{PROJ}$	-0.03	1.13	0.72	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
$\hat{\mu}_{TAN}$	-0.03	1.13	0.73	1.12	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
$\hat{\mu}_{USUAL}^{en}$	-0.03	1.13	0.73	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
$\hat{\mu}_{PROJ}^{en}$	-0.03	1.13	0.72	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
OR incorrect, PS correct						OR incorrect, PS incorrect						
$\hat{\mu}_{OR}$	-0.78	1.68	1.18	1.49	1.48	0.91	-0.78	1.68	1.18	1.49	1.48	0.91
$\hat{\mu}_{USUAL}$	0.12	1.64	1.09	1.64	1.54	0.93	-18.05	177.45	5.25	176.53	16.60	0.61
$\hat{\mu}_{PROJ}$	0.01	1.14	0.73	1.14	1.16	0.95	-1.25	1.78	1.35	1.27	1.24	0.83
$\hat{\mu}_{TAN}$	0.04	1.27	0.85	1.27	1.26	0.95	-1.69	2.24	1.80	1.47	1.43	0.76
$\hat{\mu}_{USUAL}^{en}$	0.24	1.55	1.02	1.53	1.42	0.92	-2.00	2.44	2.08	1.41	1.39	0.69
$\hat{\mu}_{PROJ}^{en}$	0.02	1.14	0.73	1.14	1.16	0.95	-0.96	1.58	1.16	1.25	1.24	0.88

BIAS, Monte Carlo bias; RMSE, root mean square error; MAE, median of absolute errors; MCS, Monte Carlo standard deviation; AVESE, average of sandwich standard errors; COV, Monte Carlo coverage of 95% Wald confidence intervals; OR, outcome regression; PS, propensity score.

functions  $g(X)$  for both treatments, then taking the difference of the optimal, doubly robust estimators proposed here will lead to an optimal, doubly robust estimator for the mean difference; see Tan (2006, p. 1623). However, this need not hold in general, for example, if the posited outcome regression models are nonlinear in their parameters. In this case, it is possible to adapt the approach here to derive directly an optimal, doubly robust estimator for the difference; a sketch of the argument is available at <http://www.stat.ncsu.edu/~davidian>. The proposed methods may also be adapted to the case of estimation of the parameter in a regression model, where an estimator based on the full data may be derived as the solution to an M-estimating equation; we are currently developing such methods in the case of monotonely coarsened longitudinal data and will report the results elsewhere.

Table 2. *Simulation results based on 1000 Monte Carlo replications for the Tan scenario. Entries are as in Table 1. The Tan and Kang & Schafer scenarios are distributionally identical in the OR correct and PS correct cases. Smallest, median, second largest, and largest standard errors for table entries: BIAS (0.04, 0.08, 0.76, 5.66); AVESE (0.0008, 0.004, 1.59, 9.78); COV (0.006, 0.007, 0.010, 0.015)*

	BIAS	RMSE	MAE	MCS D	AVESE	COV	BIAS	RMSE	MAE	MCS D	AVESE	COV
<i>n</i> = 200												
OR correct, PS correct						OR correct, PS incorrect						
$\hat{\mu}_{OR}$	-0.06	2.51	1.66	2.51	2.56	0.96	-0.06	2.51	1.66	2.51	2.56	0.96
$\hat{\mu}_{USUAL}$	-0.06	2.51	1.66	2.51	2.56	0.95	-0.04	2.55	1.70	2.55	2.59	0.95
$\hat{\mu}_{PROJ}$	-0.07	2.51	1.69	2.51	2.56	0.95	-0.06	2.51	1.69	2.51	2.56	0.95
$\hat{\mu}_{TAN}$	-0.05	2.51	1.68	2.51	2.58	0.95	-0.05	2.50	1.65	2.50	2.56	0.96
$\hat{\mu}_{USUAL}^{en}$	-0.06	2.51	1.66	2.51	2.56	0.96	-0.06	2.51	1.67	2.51	2.57	0.95
$\hat{\mu}_{PROJ}^{en}$	-0.06	2.51	1.68	2.51	2.58	0.95	-0.06	2.51	1.68	2.51	2.62	0.95
OR incorrect, PS correct						OR incorrect, PS incorrect						
$\hat{\mu}_{OR}$	2.64	4.10	3.02	3.14	3.08	0.88	2.64	4.10	3.02	3.14	3.08	0.88
$\hat{\mu}_{USUAL}$	0.74	3.80	2.44	3.72	3.30	0.93	-2.76	24.18	2.76	24.02	7.71	0.95
$\hat{\mu}_{PROJ}$	0.56	2.70	1.76	2.64	2.67	0.95	0.51	2.91	1.90	2.87	2.80	0.95
$\hat{\mu}_{TAN}$	0.64	2.79	1.88	2.72	2.72	0.96	0.94	2.99	1.91	2.84	2.84	0.95
$\hat{\mu}_{USUAL}^{en}$	1.37	3.42	2.33	3.13	3.22	0.91	1.36	3.28	2.18	2.99	3.65	0.93
$\hat{\mu}_{PROJ}^{en}$	0.52	2.69	1.75	2.64	3.14	0.95	0.48	2.86	1.86	2.82	3.02	0.95
<i>n</i> = 1000												
OR correct, PS correct						OR correct, PS incorrect						
$\hat{\mu}_{OR}$	-0.03	1.13	0.73	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
$\hat{\mu}_{USUAL}$	-0.03	1.13	0.73	1.13	1.15	0.95	0.03	2.07	0.74	2.07	1.32	0.95
$\hat{\mu}_{PROJ}$	-0.03	1.13	0.72	1.13	1.15	0.95	-0.03	1.13	0.72	1.13	1.15	0.95
$\hat{\mu}_{TAN}$	-0.03	1.12	0.73	1.12	1.15	0.95	-0.03	1.12	0.73	1.12	1.15	0.95
$\hat{\mu}_{USUAL}^{en}$	-0.03	1.13	0.73	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
$\hat{\mu}_{PROJ}^{en}$	-0.03	1.13	0.72	1.13	1.15	0.95	-0.03	1.13	0.73	1.13	1.15	0.95
OR incorrect, PS correct						OR incorrect, PS incorrect						
$\hat{\mu}_{OR}$	2.31	2.72	2.32	1.43	1.41	0.63	2.31	2.72	2.32	1.43	1.41	0.63
$\hat{\mu}_{USUAL}$	0.18	1.84	1.14	1.84	1.64	0.93	-17.89	179.88	2.65	178.98	22.60	0.94
$\hat{\mu}_{PROJ}$	0.19	1.18	0.76	1.16	1.17	0.95	0.13	1.25	0.78	1.25	1.20	0.94
$\hat{\mu}_{TAN}$	0.22	1.23	0.79	1.21	1.21	0.94	0.89	1.61	1.00	1.35	1.31	0.91
$\hat{\mu}_{USUAL}^{en}$	0.53	1.56	1.03	1.47	1.27	0.89	0.73	1.47	0.97	1.28	1.27	0.91
$\hat{\mu}_{PROJ}^{en}$	0.18	1.17	0.74	1.16	1.17	0.94	0.22	1.23	0.78	1.21	1.19	0.94

BIAS, Monte Carlo bias; RMSE, root mean square error; MAE, median of absolute errors; MCS D, Monte Carlo standard deviation; AVESE, average of sandwich standard errors; COV, Monte Carlo coverage of 95% Wald confidence intervals; OR, outcome regression; PS, propensity score.

Like the stabilized weights discussed by [Robins et al. \(2000\)](#), the enhanced propensity score model proposed in § 4 is an effort to avoid weighting that is too disparate across individuals, leading to instability of the estimator for the mean. In the simple context of estimating a single mean, taking a stabilized weights approach is not possible; accordingly, the proposed enhanced model provides an effective alternative. Other methods, such as truncating or smoothing estimated propensities, may also yield improved performance.

It is worth noting that, when the outcome regression model is correct but the propensity model is not, attempting to improve efficiency would be fruitless. Here, the optimal estimator is  $\hat{\mu}_{OR}$ , and the propensity score plays no role; see [Tsiatis & Davidian \(2007, p. 573\)](#).

Detailed formulæ for the asymptotic variances of the estimators in this paper are available at <http://www.stat.ncsu.edu/~davidian>.

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## APPENDIX

*Performance under misspecification*

We argue heuristically that  $\hat{\mu}_{\text{PROJ}}$  and  $\hat{\mu}_{\text{PROJ}}^{\text{en}}$  may perform well under misspecification of both the propensity and outcome regression models. Suppose that  $m(X, \beta)$  in (5) may be misspecified and that  $\pi(X)$  in (5) is misspecified as  $\pi_n(X) = \pi_0(X) + \theta_n s(X)$ , where  $\lim_{n \rightarrow \infty} n^{1/2} \theta_n = \tau$ . If we substitute  $\hat{\beta}_3$  solving (12) in (5), because  $\pi_n(X)$  converges to  $\pi_0(X)$ ,  $\hat{\beta}_3$  still converges in probability to  $\beta_{\text{opt}}^*$ . Thus, the resulting estimator for  $\mu$ ,  $\hat{\mu}_n$ , say, would be asymptotically equivalent to (6) with  $\pi_n(X)$  replacing  $\pi_0(X)$  and  $\beta^* = \beta_{\text{opt}}^*$ . Expanding this expression about  $\pi_0(X)$  shows that  $n^{1/2}(\hat{\mu}_n - \mu)$  converges in distribution to a normal random variable with mean  $-\tau E[s(X)\{Y - m(X, \beta_{\text{opt}}^*)/\pi_0(X)\}]$ , so  $\hat{\mu}_n$  exhibits an asymptotic bias. Because of (8),  $E[\{1 - \pi_0(X)\}\pi_0^{-1}(X)\{Y - m(X, \beta_{\text{opt}}^*)\}c^\top m_\beta(X, \beta_{\text{opt}}^*)] = 0$  for any constant vector  $c$ . It follows that, letting  $q_0(X) = \{[1 - \pi_0(X)]/\pi_0(X)\}^{1/2}$ , the asymptotic bias may be written as  $-\tau E\{(s(X)/[\pi_0(X)\{1 - \pi_0(X)\}]^{1/2} - q_0(X)c^\top m_\beta(X, \beta_{\text{opt}}^*))q_0(X)\{Y - m(X, \beta_{\text{opt}}^*)\}$ , the absolute value of which, by the Cauchy–Schwarz inequality, is bounded by

$$\tau [\inf_c E\{(s(X)/[\pi_0(X)\{1 - \pi_0(X)\}]^{1/2} - q_0(X)c^\top m_\beta(X, \beta_{\text{opt}}^*))^2\}]^{1/2} \quad (\text{A1})$$

$$\times (E[\{q_0(X)\}^2\{Y - m(X, \beta_{\text{opt}}^*)\}^2])^{1/2}. \quad (\text{A2})$$

If we were to use in (5) another estimator  $\tilde{\beta}$ , which converges in probability to some  $\beta^{**}$ , by a similar argument, the resulting estimator  $\tilde{\mu}_n$  would have associated asymptotic bias whose absolute value is bounded by  $\tau [E\{(s(X)/[\pi_0(X)\{1 - \pi_0(X)\}]^{1/2})^2\}]^{1/2} \times (E[\{q_0(X)\}^2\{Y - m(X, \beta^{**})\}^2])^{1/2}$ . The first term in this expression must be at least as large as (A1), because (A1) is the projection of  $s(X)/[\pi_0(X)\{1 - \pi_0(X)\}]^{1/2}$  onto the linear space spanned by  $q_0(X)m_\beta(X, \beta_{\text{opt}}^*)$ , while the second must be greater than or equal to (A2) by the definition of  $\beta_{\text{opt}}^*$ . Thus, the bound on asymptotic bias of  $\tilde{\mu}_n$  is greater than that for  $\hat{\mu}_n$ ; moreover, the asymptotic variance of  $\tilde{\mu}_n$  is of course greater than that of  $\hat{\mu}_n$  by construction. Although the first result does not guarantee that  $\hat{\mu}_n$  will show smaller bias, it does suggest that smaller bias may obtain in many circumstances, particularly if  $s(X)/[\pi_0(X)\{1 - \pi_0(X)\}]^{1/2}$  may be well approximated by a linear combination of  $q_0(X)m_\beta(X, \beta_{\text{opt}}^*)$ .

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