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# Sporadicity: Between periodic and chaotic dynamical behaviors

**(dynamical system/countable Markov chain/non-Gaussian fluctuation/algorithmic complexity)**

P. GASPARD* AND X.-J. WANG†

Université Libre de Bruxelles, Service de Chimie Physique II, Campus Plaine CP 231, Boulevard du Triomphe, B-1050 Brussels, Belgium

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**ABSTRACT** **We define the class of sporadic dynamical systems as the systems where the algorithmic complexity of Kolmogorov [Kolmogorov, A . N. (1983) *Russ. Math. Surv.* 38, 29–40] and Chaitin [Chaitin, G. J. (1987) *Algorithmic Information Theory* (Cambridge Univ. Press, Cambridge, U.K.)] as well as the logarithm of separation of initially nearby trajectories grow as $n^{\nu_0}(\log n)^{\nu_1}$ with $0 < \nu_0 < 1$ or $\nu_0 = 1$ and $\nu_1 < 0$ as time $n \to \infty$. These systems present a behavior intermediate between the multiperiodic ($\nu_0 = 0$, $\nu_1 = 1$) and the chaotic ones ($\nu_0 = 1$, $\nu_1 = 0$). We show that intermittent systems of Manneville [Manneville, P. (1980) *J. Phys. (Paris)* 41, 1235–1243] as well as some countable Markov chains may be sporadic and, furthermore, that the dynamical fluctuations of these systems may be of Lévy's type rather than Gaussian.**

## 1. Introduction

Exponential separation of trajectories of nearby initial conditions is a very general feature shared by a large variety of dissipative and conservative dynamical systems. This property has become in the last decades a theoretical cornerstone to understand irregular time evolutions in natural phenomena and in their models. Several quantities have been defined to characterize the exponential instability of a dynamical system $\Phi$ with an invariant measure $\mu$, such as the exponents of Lyapunov and Oseledec and the entropy per unit time of Kolmogorov and Sinai $h(\Phi, \mu)$. When the entropy is positive, the system is called chaotic and at least one Lyapunov exponent is then positive. The entropy is vanishing when the system is periodic or multiperiodic (1).

The connection between the exponential dynamical instability and the randomness of trajectories is provided by the algorithmic complexity of Kolmogorov (2) and Chaitin (3). After partitioning the phase space into cells $\{A_0, A_1, \ldots, A_{m-1}\}$, a given trajectory can be represented by a sequence of integers or symbols

$$S = s_0 s_1 s_2 \ldots s_n \ldots \quad [1.1]$$

if the position $x_n$ at time $n$ belongs to the cell $A_{s_n}$. The algorithmic complexity $K(S_n)$ of the string $S_n$ composed of the $n$ first symbols of $S$ is defined as the binary length of the shortest possible program $P$ able to reconstruct the string $S_n$ on a universal machine $A$; i.e.,

$$K(S_n) = \min_{S_n = A(P)} |P|, \quad [1.2]$$

where $|.|$ denotes the binary length. A periodic string can be constructed by specifying only the length $n$ of the string and the pattern of one period so that

$$K(S_n) \sim \log_2 n \text{ (periodic trajectory).} \quad [1.3]$$

However, if no regularity is observed in the string, as is the case for a random sequence, we have no possibility other than memorizing the whole string $S_n$, so that (3, 4)

$$K(S_n) \sim n \text{ (random trajectory).} \quad [1.4]$$

The following relation to the entropy per unit time shows that random trajectories prevail in chaotic systems,

$$\lim_{n\to\infty} \frac{1}{n} K(S_n) = h(\Phi, \mu) \quad [1.5]$$

for $\mu$-almost all trajectories (5).

The question arises whether intermediate dynamical behaviors could exist between multiperiodic and chaotic ones in the sense that the complexity $K(S_n)$ increases asymptotically as

$$n^{\nu_0}(\log n)^{\nu_1} \text{ with } 0 < \nu_0 < 1 \text{ or } \nu_0 = 1 \text{ and } \nu_1 < 0 \quad [1.6]$$

for almost all trajectories of initial condition in a given cell $A_i$ of the partition provided that $\mu(A_i) < \infty$. This latter condition is necessary because the invariant measure $\mu$ may be nonnormalizable. We call *strongly sporadic* such a dynamical system. If the asymptotic behavior (expression **1.6**) holds for the average complexity $E(K(S_n))$, we shall say that the system is *weakly sporadic*. In such systems, the dynamical instability would not be exponential anymore. So we could also use the logarithm of separation between nearby trajectories to define the sporadic systems when they are differentiable. In one-dimensional chaotic systems, the Lyapunov exponent $\lambda$ is given asymptotically by $\Lambda_n/n$ with

$$\Lambda_n = \sum_{i=1}^{n} \log_2 \left| \frac{d\Phi}{dx}(x_i) \right|. \quad [1.7]$$

In sporadic systems, $\lambda$ is vanishing and $\Lambda_n$ behaves as expression **1.6**. The exponent of sporadicity $\nu_0$ is then given asymptotically by $\ln \Lambda_n / \ln n$. We shall then speak of *stretched exponential instability*. The advantage of such a definition is that it can be generalized to differentiable dynamical systems in a phase space of dimension larger than one.

We shall show that the intermittent systems of Manneville (6)

$$x_{n+1} = \Phi(x_n) = x_n + c x_n^z \text{ (mod.1)} \qquad (z \geq 1) \quad [1.8]$$

are sporadic when $z \geq 2$. The invariant density of Eq. **1.8** behaves near the origin as $\rho(x) \simeq x^{1-z}$. It is not normalizable when $z \geq 2$, so that the invariant measure $\mu$ defines a probability for $z < 2$, but only a conditional probability for $z \geq 2$ as proposed by Mandelbrot in the context of continuous sto-

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*Present address: James Franck Institute, University of Chicago, 5640 Ellis Avenue, Chicago, IL 60637.
†Present address: Department of Physiology, University of California, Berkeley, CA 94720.

chastic processes (7). We consider the countable partition of the unit interval as constructed on Fig. 1. The measure of one of the subintervals $\{A_k\}$ and the transition probability between $A_0$ and $A_k$ behave as

$$\mu(A_k) \sim \frac{1}{(k+1)^{\frac{1}{z-1}}} \quad \textbf{[1.9a]}$$

and

$$Pr(A_0 \to A_k) = p_{0k} \sim \frac{1}{(k+1)^{\frac{z}{z-1}}}, \quad \textbf{[1.9b]}$$

respectively. Then we assume that the system is equivalent to the countable Markov chain defined by the transition matrix‡

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad \textbf{[1.10]}$$

A similar assumption was used with success by Ben-Mizrachi *et al.* (8) and Geisel *et al.* (9), where the Fourier spectral density $s(f)$ was shown to diverge in the infrared. Then, the autocorrelation function $C(n)$ and the variance of the dynamical fluctuations $\sigma_n^2$ obey power laws

$$C(n) \sim \frac{1}{n^{1-\gamma}} \qquad (n \to \infty), \quad \textbf{[1.11a]}$$

$$s(f) \sim \frac{1}{f^{\gamma}} \qquad (f \to 0), \quad \textbf{[1.11b]}$$

$$\sigma_n^2 \sim n^{1+\gamma} \qquad (n \to \infty), \quad \textbf{[1.11c]}$$

where $0 < \gamma \leq 1$. These results suggest that the fluctuations of intermittent systems may be non-Gaussian. This fact is important for our purpose here.

## 2. Non-Gaussian Dynamical Fluctuations

By using theory of recurrent events (10, 11), we construct the probability distribution of the random variable $N_n$, which is the number of passages by the cell $A_0$ during $n$ units of time. Then, the variance of $N_n$ gives $\sigma_n^2$, and $N_n/n$ is asymptotically the time average of the observable $I_{A_0}(x)$, which is the indicator of the cell $A_0$. The probability of a first passage at time $n$ by $A_0$ is given by the transition probability $p_{0n-1}$ and its distribution function is

$$F(t) \simeq 1 - At^{-\alpha}\,(t \to \infty) \text{ with } \alpha = \frac{1}{z-1}. \quad \textbf{[2.1]}$$

‡Such an equivalence is exact for the piecewise linear map of the unit interval, $x_{n+1} = \Phi(x_n)$, defined with

$$\Phi(x) = \begin{cases} \dfrac{\xi_{k-2} - \xi_{k-1}}{\xi_{k-1} - \xi_k}(x - \xi_k) + \xi_{k-1} & \xi_k \leq x < \xi_{k-1} \\ \dfrac{x-a}{1-a}, & a \leq x \leq 1 \end{cases}$$

with $\xi_k = \dfrac{a}{(k+1)^{\frac{1}{z-1}}}$ $k = 1, 2, 3, \ldots$.

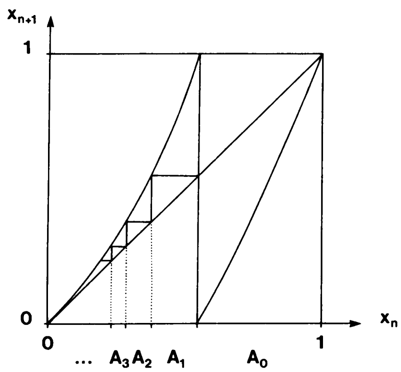


FIG. 1. Construction of the countable partition $\{A_0, A_1, A_2, \ldots\}$ of the unit interval for Manneville's intermittent systems (6) defined by Eq. **1.8**.

Let us denote the expectation and the variance of the recurrent time by $\tau$ and $\sigma^2$, respectively, when they exist. Let $G_\alpha(x)$ be the distribution function of Lévy's stable law of parameter $\alpha$ with $0 < \alpha < 2$, $\alpha \neq 1$ and of characteristic function

$$\psi_\alpha(s) = \exp\left\{-|s|^\alpha\Gamma(1-\alpha)\left[\cos\left(\frac{\pi\alpha}{2}\right) - i\,\mathrm{sgn}(s)\sin\left(\frac{\pi\alpha}{2}\right)\right]\right\}. \quad \textbf{[2.2]}$$

$G_2(x)$ is the standard Gauss distribution function. We can then prove the following results. $\alpha$ and $A$ are given by function **2.1**.

(*i*) If $1 \leq z < 3/2$, the fluctuations are Gaussian with

$$Pr\left\{N_n \geq \frac{n}{\tau} - x\frac{\sigma}{\tau^{3/2}}n^{1/2}\right\} \to_{n\to\infty} G_2(x) \quad \textbf{[2.3]}$$

and

$$E(N_n) \simeq n/\tau, \qquad Var(N_n) \simeq \sigma^2 n/\tau^3. \quad \textbf{[2.4]}$$

In the critical case $z = 3/2$, the fluctuations are still Gaussian but

$$E(N_n) \simeq n/\tau, \qquad Var(N_n) \sim n \ln n. \quad \textbf{[2.5]}$$

(*ii*) If $3/2 < z < 2$, the fluctuations are non-Gaussian with

$$Pr\left\{N_n \geq \frac{n}{\tau} - x\left(\frac{An}{\tau^{\alpha+1}}\right)^{1/\alpha}\right\} \to_{n\to\infty} G_\alpha(x) \quad (1 < \alpha < 2) \quad \textbf{[2.6]}$$

and

$$E(N_n) \simeq n/\tau, \qquad Var(N_n) \sim n^{\frac{3z-4}{z-1}}. \quad \textbf{[2.7]}$$

In the critical case $z = 2$, the fluctuations have a Cauchy-like distribution with

$$E(N_n) \simeq n/A \ln n, \qquad (\ln n/n)^2 Var(N_n) = o(1). \quad \textbf{[2.8]}$$

(*iii*) If $2 < z$, the fluctuations are non-Gaussian with

$$Pr\left\{N_n \geq \frac{n^\alpha}{Ax^\alpha}\right\} \to_{n\to\infty} G_\alpha(x) \qquad (0 < \alpha < 1) \quad \textbf{[2.9]}$$

and

$$E(N_n) \sim n^{\frac{1}{z-1}}, \qquad Var(N_n) \sim n^{\frac{2}{z-1}}. \qquad [2.10]$$

These probability distributions provide us with examples of violation of the Gaussian character of fluctuations in dynamical systems. Fig. 2 presents a numerical calculation of $Var(N_n)$ for different values of exponent $z$. Let us remark here that the fluctuations of observables other than $I_{A_0}(x)$ could have different probability distributions. For instance, Ben-Mizrachi *et al.* (8) considered the position $x$ rather than the indicator $I_{A_0}(x)$ and obtained a variance $\sigma_n^2$ as given by expression **1.11** from their spectral density, which is identical to ours when $z < 2$ but different when $2 \leq z$. This result suggests that the class of observables with the same fluctuations is larger in the first case, where we could speak of universality, than in the second case, where the fluctuations may depend on the observable.

### 3. Sporadicity and Stretched Exponential Instability

Now, we turn back to the problem of dynamical instability. The entropy per unit time and the Lyapunov exponent of the intermittent systems (Eq. **1.8**) are given by

$$\lambda(\Phi, \mu) = h(\Phi, \mu) = \frac{\int_0^1 \log_2\left|\frac{d\Phi}{dx}(x)\right|\rho(x)dx}{\int_0^1 \rho(x)dx}, \qquad [3.1]$$

where $\rho(x)$ is the invariant density. The numerator is finite for all $z \geq 1$, although the denominator is finite for $z < 2$ but infinite for $z \geq 2$. Thus the entropy is positive for $z < 2$ but vanishes as $h(\Phi, \mu) \sim (2 - z)$ near $z = 2$ and is zero for $z \geq 2$ (see Fig. 3). So intermittent systems are chaotic when $z < 2$ but neither chaotic nor periodic when $z \geq 2$.

The algorithmic complexity is then able to characterize the trajectories. Let us consider the symbolic sequence **1.1** produced by a given trajectory with $s_n = k$ if $x_n$ belongs to the cell $A_k$. An example compatible with the transition matrix **1.10** is

$$S = 0210543210107654321032100 43. \ldots \qquad [3.2]$$

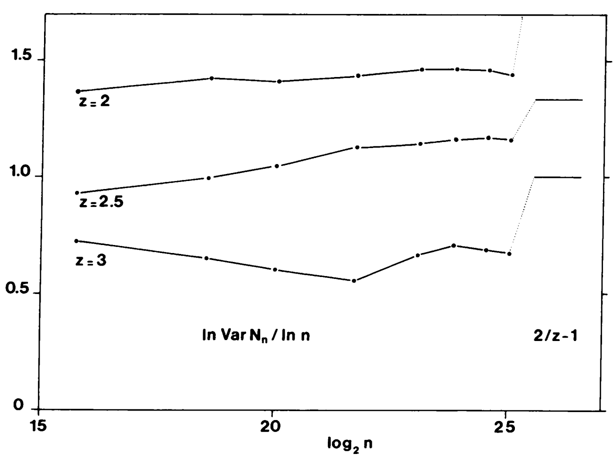


FIG. 2. Plot of ln $Var(N_n)$/ln $n$ versus $\log_2 n$ for $z$ = 2, 2.5, 3. The average was performed with 20 samplings by cutting a single trajectory of initial condition $x_0 = 0.4$ and time length of $20 \times 2^{25}$ into 20 pieces. At the right is plotted the exponent $2/(z - 1)$ predicted by formula **2.10**. Because of the trapping of trajectories near the origin, the statistics become very poor as exponent $z$ increases (about 10 events when $z = 3$). We observe, as expected, a decrease of the asymptotic value when exponent $z$ is increased, although a quantitative agreement cannot be concluded.

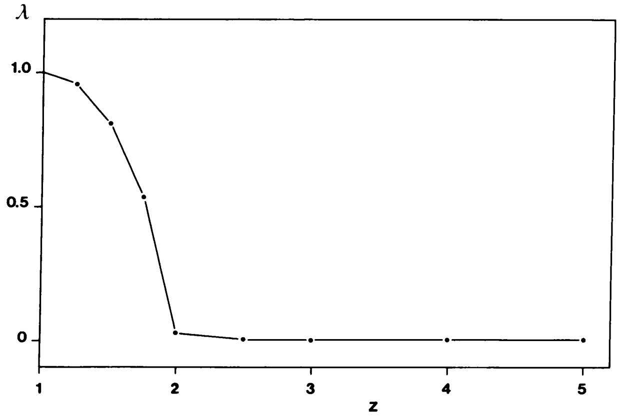


FIG. 3. Lyapunov exponent $\lambda$ calculated by $\Lambda_n/n$ at time $n = 2^{27}$ with a trajectory of initial condition $x_0 = 0.4$. As expected, $\lambda$ is vanishing when $z \geq 2$.

$S$ is compressible in the sense of Chaitin (4). Indeed, $S$ can be represented by the sequence of the recurrent times by $A_0$,

$$R = 02517304. \ldots \qquad [3.3]$$

$S$ is uniquely recovered from $R$. So we can associate to the string $S_n$ of the $n$ first symbols of $S$, the string

$$R_{N_n} = s_{k_1}s_{k_2}\ldots s_{k_{N_n}}, \qquad [3.4]$$

with $s_{k_i-1} = 0$ for all $i$. $R_{N_n}$ realizes a compression of $S_n$. $N_n$ is the number of recurrences by $A_0$ in $S_n$. The binary length of $R_{N_n}$ is an estimation of the complexity of $S_n$

$$K(S_n) = \sum_{i=1}^{N_n} \log_2 s_{k_i}. \qquad [3.5]$$

As a consequence, the average complexity behaves like

$$E(K(S_n)) \sim E(N_n). \qquad [3.6]$$

From formulae **2.4**, **2.5**, **2.7**, **2.8**, and **2.10**,

$$E(K(S_n)) \sim n \text{ if } 1 \leq z < 2, \qquad [3.7]$$

as expected because the system is then chaotic. However,

$$E(K(S_n)) \sim n/\ln n \text{ if } z = 2, \qquad [3.8]$$

and

$$E(K(S_n)) \sim n^{\frac{1}{z-1}} \text{ if } z > 2. \qquad [3.9]$$

Here the growth of complexity is slower than linear and the system is thus weakly sporadic. The exponent of sporadicity is

$$\alpha = \frac{1}{z-1} \text{ when } z > 2, \text{ and } 0 < \alpha < 1. \qquad [3.10]$$

Similarly, using the Markovian assumption, we can show that the intermittent systems have a stretched exponential instability when $z \geq 2$. Indeed, averaging expression (Eq. **1.7**) over trajectories of associated strings (Eq. **3.4**), we can write

$$E(\Lambda_n) \simeq \sum_{l=0}^{n-1} m_l\overline{\Lambda_l}, \qquad [3.11]$$

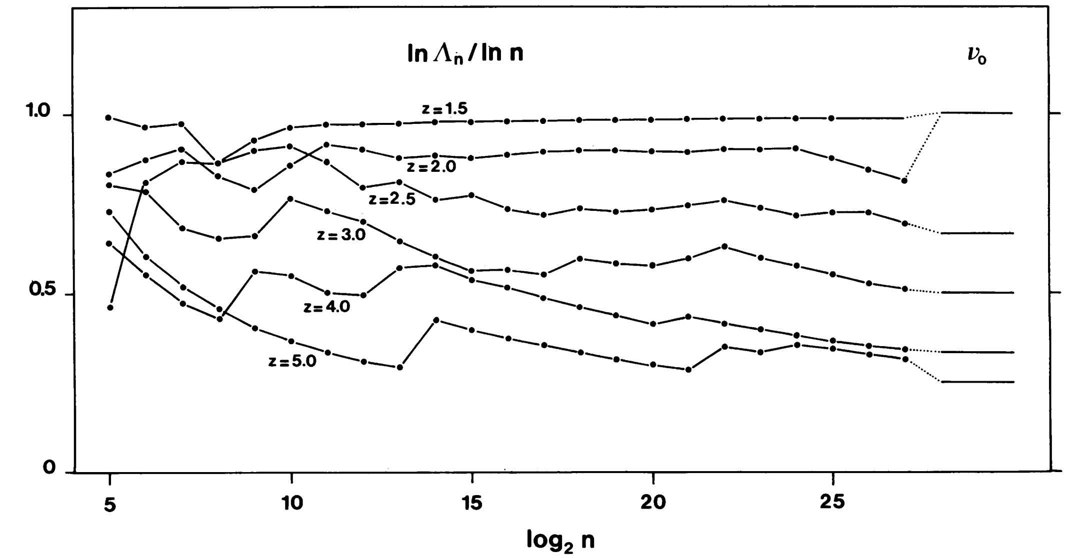


FIG. 4. Plot of $\ln \Lambda_n/\ln n$ versus $\log_2 n$ for different values of exponent $z$. The predicted exponent of sporadicity $\nu_0 = 1/(z-1)$ is plotted at the right. The initial condition of the trajectory is $x_0 = 0.4$. The effect of a trapping of the trajectory near the origin is a decrease of the curve, whereas a rise occurs when the trajectory is away from $x = 0$.

where $m_l$ is the average number of symbols $s_{k_i}$ equal to $l$ in $R_{N_n}$. We have

$$m_l \simeq E(N_n)p_{0l} \qquad [3.12]$$

and

$$\overline{\Lambda_l} = \int_{A_l} \sum_{i=0}^{l-1} \log_2\left|\frac{d\Phi}{dx}(x_i)\right|\rho(x_0)dx_0. \qquad [3.13]$$

An estimation of $\overline{\Lambda_l}$ yields

$$\overline{\Lambda_l} \sim l^{-\alpha}\log_2 l \ (l \to \infty). \qquad [3.14]$$

Combining these results, we obtain

$$E(\Lambda_n) \simeq CE(N_n), \qquad [3.15]$$

where $C$ is a positive constant bounded when $n \to \infty$. Consequently, $E(\Lambda_n)$ has the same asymptotic behavior as $E(K(S_n))$. When $z < 2$, the divergence of two nearby orbits is thus exponential. However, when $z \geq 2$, we are in the presence of a stretched exponential divergence of trajectories. Numerical calculations of $\Lambda_n$ are displayed on Fig. 4. In spite of the extremely slow convergence of $\ln \Lambda_n/\ln n$ due to non-Gaussian fluctuations when $z \geq 2$, we observe that the predicted exponents of sporadicity (Eq. **3.10**) are approached by above, suggesting a strong sporadicity.

## 4. Other Countable Markov Chains

Non-Gaussian fluctuations and sporadicity are also present in a particular countable chain on a tree defined by Meiss and Ott (12). The possible states of the tree are {Ø, 1, 2, 11, 12, 21, 22, . . .}. Each state is labeled by a string $s = \sigma_1\sigma_2 . . . \sigma_N$ with $\sigma_i = 1$ or 2. Following Meiss and Ott, $Ds$, $s1$, and $s2$ denote, respectively, the strings $\sigma_1\sigma_2 . . . \sigma_{N-1}$, $\sigma_1\sigma_2 . . . \sigma_N 1$, and $\sigma_1\sigma_2 . . . \sigma_N 2$. The allowed transitions in the Markov chain are $s \to Ds$, $s \to s$, $s \to s1$, $s \to s2$. The first one is not allowed if $s = Ø$ (see Fig. 5). The number of 1 and 2 in the string $s$ will be denoted by $|s|_1$ and $|s|_2$, respectively. The transition probabilities are then defined as follows (12), in terms of the parameters $\{P_0, \varepsilon_1, \varepsilon_2, w_1, w_2\}$ with $\varepsilon_1, \varepsilon_2 < 1$,

$$Pr(s1 \to s) = P_0\varepsilon_1^{|s|_1+1}\varepsilon_2^{|s|_2}, \qquad [4.1a]$$

$$Pr(s2 \to s) = P_0\varepsilon_1^{|s|_1}\varepsilon_2^{|s|_2+1}, \qquad [4.1b]$$

$$Pr(s \to s1) = P_0 w_1\varepsilon_1^{|s|_1}\varepsilon_2^{|s|_2}, \qquad [4.1c]$$

$$Pr(s \to s2) = P_0 w_2\varepsilon_1^{|s|_1}\varepsilon_2^{|s|_2}, \qquad [4.1d]$$

and they satisfy

$$\sum_{s'} Pr(s \to s') = 1. \qquad [4.2]$$

Generalization to trees with $M$ branchings at each state rather than 2 is straightforward. Meiss and Ott (12) proved that the probability of recurrence at time $t$ by the first state Ø of the tree behaves asymptotically like $t^{-(z+1)}$, where $z$ is de-

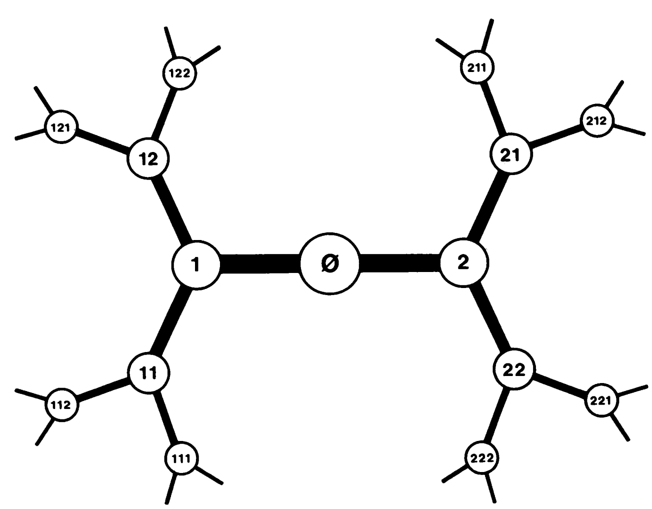


FIG. 5. Tree of states of the countable Markov chain defined by Meiss and Ott (12). The branching is here equal to $M = 2$.

Table 1. Classification of the different dynamical behaviors

| Lévy parameter | $0 < \alpha < 1$ | $1 < \alpha < 2$ | $\alpha = 2$ |
|---|---|---|---|
| Fluctuations | Lévy | Lévy | Gauss |
| Instability | Sporadic | Chaotic | Chaotic |
| In Manneville systems (6) for | $z > 2$ | $2 > z > 3/2$ | $3/2 > z > 1$ |
| In Meiss–Ott systems (12) for | $0 < z < 1$ | $1 < z < 2$ | $2 < z$ |

fined by

$$\sum_{j=1}^{M} w_j \varepsilon_j^{-z} = 1. \quad [4.3]$$

According to the theory of recurrent events (10, 11), the fluctuations in the number of recurrences by the state $\emptyset$ during a given time interval will have Lévy probability distributions like formulae **2.3**, **2.6**, and **2.9** with $\alpha = z$ if $z \leq 2$ and $\alpha = 2$ if $z > 2$ here.

The invariant measure is easily calculable and is given by

$$\mu(s) = \zeta\left(\frac{w_1}{\varepsilon_1}\right)^{|s|_1}\left(\frac{w_2}{\varepsilon_2}\right)^{|s|_2}. \quad [4.4]$$

It is normalizable and defines a probability at the condition that $z > 1$, which is equivalent to $(w_1/\varepsilon_1 + w_2/\varepsilon_2) < 1$. The normalizing constant is then

$$\zeta = 1 - \left(\frac{w_1}{\varepsilon_1} + \frac{w_2}{\varepsilon_2}\right). \quad [4.5]$$

When $z \leq 1$, the invariant measure is not normalizable and only conditional probability can be defined. We can show that the entropy per unit time of the Markov chain,

$$h(\Phi, \mu) = -\sum_{ss'} \frac{\mu(s)}{\Sigma_{s''}\mu(s'')} Pr(s \to s')\log_2 Pr(s \to s'), \quad [4.6]$$

is finite and positive when $z > 1$ but is zero when $z \leq 1$. As the Markov chain is not periodic, it is sporadic when $z \leq 1$, with $\alpha = z$ as the expected exponent of sporadicity.

The parallelism with the different dynamical behaviors of the intermittent systems is striking and suggests that the theoretical scheme summarized in Table 1 is very general. Let us remark that two-dimensional Hamiltonian mappings seem to correspond to the case $1 < z = \alpha < 2$ in the model of Meiss and Ott (12). The fluctuations would then be of Lévy's type but the system would still be chaotic. This example shows that non-Gaussian fluctuations and $1/f$ noise do not imply sporadicity, albeit sporadicity is always associated with such phenomena.

To conclude, sporadicity fills in a gap between multiperiodic and chaotic dynamical behaviors or equivalently between predictable and random patterns.

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