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RESEARCH ARTICLE

Mean square exponential and robust stability of stochastic discrete-time genetic regulatory networks with uncertainties

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Abstract This paper aims to analyze global robust exponential stability in the mean square sense of stochastic discrete-time genetic regulatory networks with stochastic delays and parameter uncertainties. Comparing to the previous research works, time-varying delays are assumed to be stochastic whose variation ranges and probability distributions of the time-varying delays are explored. Based on the stochastic analysis approach and some analysis techniques, several sufficient criteria for the global robust exponential stability in the mean square sense of the networks are derived. Moreover, two numerical examples are presented to show the effectiveness of the obtained results.

Keywords Discrete-time genetic regulatory networks · Exponential stability · Probability distribution · Linear matrix inequality · Stochastic delays

Introduction

The research of complex dynamical networks varies from biological and chemical oscillators to scientific collaboration networks as well as neurodynamics and biological neural networks (Dorogotsev and Mendes 2003; Becskei and Serrano 2000; Bolouri and Davidson 2002; Wang et al. 2008; Wang and Zhang 2007; Chaouiya 2007). As a special case, genetic regulatory networks (GRNs) consisting of DNA, RNA, proteins, small molecules and their mutual regulatory interactions, have become an important new area of research in the biological and biomedical sciences and received widely attention recently (Becskei and Serrano 2000; Bolouri and Davidson 2002; Weaver et al. 1999; De Jong 2002; Smolen et al. 2000). Several models have been developed to investigate the behaviors of the GRNs, for example, Boolean models (Weaver et al. 1999), the differential equation models (De Jong 2002; Smolen et al. 2000), the Petri net models (Chaouiya 2007) and discrete time piecewise affine model (Lima and Ugalde 2006; Coutinho et al. 2006). Among them, GRNs in the form of differential equation models have been well studied in He and Cao (2008), Ren and Cao (2008), Ribeiro et al. (2006) and Cao and Ren (2008).

It is revealed that time delay, which inevitably exists in GRNs due to slow biochemical reactions such as gene transcription, translation, diffusion, and translocation processes (see Hirata et al. 2002; Lewis 2003), is an important factor and should be considered. Various efforts have been paid in the past few years for the analysis of GRNs with time delay, see He and Cao (2008), Ren and Cao (2008), Chen and Aihara (2002a, b) and Li et al. (2006). In Chen and Aihara (2002), presented a model for GRNs with constant delay and analyzed nonlinear properties of the model in terms of local stability and bifurcation. Subsequently, they explained periodic oscillations which are mainly generated by nonlinearly negative and positive feedback loops in gene regulatory systems, and explored effects of time delay on stability region of the oscillations (see Chen and Aihara 2002). In Li et al. (2006), a nonlinear model for GRNs with SUM regulatory

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functions was presented and some sufficient conditions for the stability of the GRNs involving time varying delays and stochastic perturbations were derived by using the Lyapunov method and the Lur'e system approach. He and Cao (2008) investigated global asymptotic stability of GRNs with distributed delay. In Ren and Cao (2008), by using the Lyapunov method and linear matrix inequality (LMI) approach, sufficient conditions were proposed to ensure robust asymptotic stability of GRNs with time-varying delays and parameter uncertainties. On the other hand, due to small numbers of transcriptional factors and other key signaling proteins, considerable experimental evidences show that noise plays a very important role in gene regulation (Tian et al. 2007; Jonathan and Erin 2005). In addition, gene expression involves a series of molecular events in cells, which are often subject to significant intrinsic fluctuations and extrinsic disturbances, thus being best viewed as a stochastic process (Jonathan and Erin 2005; Michael et al. 2002; Sun et al. 2009). So the stochastic differential equation model has recently been developed to describe the molecular fluctuation in gene networks (Lestas et al. 2008, Li et al. 2007).

It is worth noting that most references for delayed GRNs were only concerned with the case of deterministic time delay(s). But in many real systems, such as the networked control systems, the network-induced delay often appears as some probabilistic properties and its probability distribution can be measured by the statistical method (Yue et al. 2009). On the other hand, it is shown in Ribeiro et al. (2006) that time delays in some GRNs are often existent in a stochastic fashion. And their probabilistic characteristics can also be obtained by statistical methods. Hence, it is necessary to consider stochastic delay effects in GRNs. In addition, as pointed out in Lima and Ugalde (2006), Coutinho et al. (2006) and Cao and Ren (2008), some GRN models are discrete-time dynamical systems which can be viewed as an extension of discrete-time delay systems and are more important than their continuous-time counterpart in a sense. These kinds of discrete-time models are directly inspired by the systems of differential equations mentioned above, though they do not correspond to a time discretization of the differential equations but rather to a natural discrete-time version of them. Hence, it is clear that theoretical analysis of stability of discrete-time GRNs is an important and necessary step. However, to the best of the author's knowledge, little attention has been paid to this issue, especially investigation on stability of discrete-time GRNs with stochastic delay when considering the information of both variation range and probability distribution of the time delay.

In this paper, we aim to solve the problem of global robust exponential stability in the mean square sense (GRES-MSE) of discrete-time GRNs with parameter uncertainties and stochastic disturbances. The parameter uncertainties are assumed to be norm-bounded and the stochastic disturbances are described in terms of a Brownian motion. By using two stochastic variables which satisfy Bernoulli random binary distribution, we construct a new model of discrete-time GRNs with stochastic time-varying delays. Then some sufficient conditions for GRES-MSE of the stochastic discrete-time GRNs with uncertainties are exploited. It should be noted that the solvability of the derived conditions depends on not only the size of the delay but also the probability of the delay appearing in some intervals. Numerical examples are presented to show the effectiveness and applicability of the proposed results.

Notations Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices. I is the identity matrix of the appropriate dimensions. $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate. diag$(\cdot)$ denotes a diagonal matrix. The superscript "T" represents the matrix transposition. The notation $X > 0$ (respectively, $X \geq 0$) for $X \in \mathbb{R}^{n\times n}$ means that the matrix X is positive definite (respectively, positive semidefinite). $\mathbb{E}\{\cdot\}$ stands for the expectation. $[a, b]$ denotes a set involving all integers between a and b. $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalue of the real symmetric matrix P. In symmetric block matrices, the symbol "*" is used as an ellipsis for terms induced by symmetry. $\mathbb{Z}_{\geq 0}$ denotes the set including zero and positive integers. $\emptyset$ denotes the empty set. $(\Omega, \mathscr{F}, \mathscr{P})$ is a probability space, where Ω is the sample space, $\mathscr{F}$ is the ϱ -algebra of subsets of the sample space and $\mathscr{P}$ is the probability measure on $\mathscr{F}$.

Model description and preliminaries

Consider a discrete-time GRN with variable delays containing of n mRNAs and n proteins can be formulated by the following difference equation

$$\begin{cases} M_i(k+1) = e^{-a_i h} M_i(k) + \phi_i(h) \\ \qquad\qquad \times \left[\sum_{j=1}^{n} b_{ij} f_j(P_j(k - d(k))) + W_i \right], \\ P_i(k+1) = e^{-c_i h} P_i(k) + \varphi_i(h)[d_i M_i(k - \tau(k))], \\ \qquad\qquad i = 1, 2, \ldots, n. \end{cases} \tag{1}$$

This mathematical model is taken from Cao and Ren (2008) with slack variation on time delays, where $M_i(k) \in \mathbb{R}$ and $P_i(k) \in \mathbb{R}$ are the concentrations of mRNA and protein of the ith gene; h is a fixed positive real number denoting a uniform discretionary step size; $a_i > 0$ and $c_i > 0$ are the degradation rates of mRNA and protein, respectively; d_i is the translation rate; $d(k) > 0$ and $\tau(k) > 0$ denote random time delays for mRNAs and Proteins, respectively; $W_i = \sum_{j \in \mathbb{I}_i} v_{ij}$, where v_{ij} is the bounded constant and denotes the dimensionless transcriptional rate of transcription factor j to i, and $\mathbb{I}_i$ is the set of all the j genes; $\phi_i(h) = \frac{1-e^{-a_i h}}{a_i}$ and $\varphi_i(h) = \frac{1-e^{-c_i h}}{c_i}$. Obviously, $\phi_i(h) > 0$, $\varphi_i(h) > 0$. The coupling coefficient b_{ij} $(i, j = 1, 2, ..., n)$ is defined as follows:

$$b_{ij} = \begin{cases} v_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i, \\ 0 & \text{if there is no link from node } j \text{ to } i, \\ -v_{ij} & \text{if transcription factor } j \text{ is a repressor of gene } i. \end{cases} \tag{2}$$

In addition, the nonlinear function $f(\cdot) \in \mathbb{R}^n$ represents the feedback regulation of the protein on the transcription. It is a monotonic function in Hill form, that is, $f_j(s) = \frac{s^{h_j}}{1+s^{h_j}}$ $(j = 1, 2, ..., n)$, where h_j is the Hill coefficient.

Let us rewrite system (1) into the following compact matrix form

$$\begin{cases} M(k+1) = AM(k) + Bf(P(k-d(k))) + V, \\ P(k+1) = CP(k) + DM(k-\tau(k)), \end{cases} \tag{3}$$

where

$$\begin{aligned}
&M(k) = [M_1(k), M_2(k), \ldots, M_n(k)]^{\mathrm{T}}, \\
&P(k) = [P_1(k), P_2(k), \ldots, P_n(k)]^{\mathrm{T}}, \\
&f(P(k-d(k))) = [f_1(P_1(k-d(k))), f_2(P_2(k-d(k))), \\
&\qquad \ldots, f_n(P_n(k-d(k)))]^{\mathrm{T}}, \\
&V = [\phi_1(h)W_1, \phi_2(h)W_2, \ldots, \phi_n(h)W_n]^{\mathrm{T}}, \\
&A = \mathrm{diag}(e^{-a_1 h}, e^{-a_2 h}, \ldots, e^{-a_n h}), \\
&C = \mathrm{diag}(e^{-c_1 h}, e^{-c_2 h}, \ldots, e^{-c_n h}), \\
&D = \mathrm{diag}(\varphi_1(h)d_1, \varphi_2(h)d_2, \ldots, \varphi_n(h)d_n), \\
&B = \begin{cases} \phi_i(h)b_{ii} & i = j, \\ b_{ij} & i \neq j. \end{cases}
\end{aligned}$$

Let $[M^{*\mathrm{T}}, P^{*\mathrm{T}}]^{\mathrm{T}} = [M_1^*, \ldots, M_n^*, P_1^*, \ldots, P_n^*]$ be an equilibrium point of system (3). Then it satisfies

$$\begin{cases} M^* = AM^* + Bf(P^*) + V, \\ P^* = CP^* + DM^*. \end{cases} \tag{4}$$

For convenience, let us shift an intended equilibrium point $(M^{*\mathrm{T}}, P^{*\mathrm{T}})^{\mathrm{T}}$ of system (3) to the origin through the transformations $x(k) = M(k) - M^*$, $y(k) = P(k) - P^*$. Then system (3) can be transformed into

$$\begin{cases} x(k+1) = Ax(k) + Bg(y(k-d(k))), \\ y(k+1) = Cy(k) + Dx(k-\tau(k)), \end{cases} \tag{5}$$

where $g(y(k)) = f(y(k) + P^*) - f(P^*)$.

As mentioned before, little study has been performed on GRNs when considering unavoidable uncertainties or external perturbations, but in the applications and designs of networks, such as genetic networks and neural networks, there are often some unavoidable uncertainties such as modeling errors, external perturbations, and parameter fluctuations, which may cause the networks to be unstable. Hence, it is essential to take into account parameter uncertainties and stochastic disturbance additionally as studied in Ren and Cao (2008) and Li et al. (2007). A general GRN model containing these influences can be described as follows

$$\begin{cases} x(k+1) = (A + \Delta A(k))x(k) + (B + \Delta B(k))g(y(k-d(k))) \\ \qquad + \sigma(k, x(k), y(k))w(k), \\ y(k+1) = (C + \Delta C(k))y(k) + (D + \Delta D(k))x(k-\tau(k)), \end{cases} \tag{6}$$

where $\Delta A(k)$, $\Delta B(k)$, $\Delta C(k)$ and $\Delta D(k)$ denote the parameter uncertainties satisfying the following condition

$$\begin{aligned} &[\,\Delta A(k) \quad \Delta B(k) \quad \Delta C(k) \quad \Delta D(k)\,] \\ &\quad = HF(k)[\,E_1 \quad E_2 \quad E_3 \quad E_4\,], \end{aligned}$$

where H, E_1, E_2, E_3 and E_4 are constant matrices of appropriate dimensions, and $F(k)$ is an unknown time-varying matrix satisfying $F^{\mathrm{T}}(k)F(k) \leq I$. $\sigma(k, x(k), y(k)) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ represents a noise intensity function vector; $w(k)$ is a scalar Wiener process on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with $\mathbb{E}\{w(k)\} = 0$, $\mathbb{E}\{w^2(k)\} = 1$, $\mathbb{E}\{w(i)w(j)\} = 0$ $(i \neq j)$.

Assumption 1 For $i \in \{1, 2, ..., n\}$, each function $g_i(\cdot)$ is continuous and bounded, and satisfies that

$$l_i \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq L_i, \ \forall s_1, s_2 \in \mathbb{R} \ (s_1 \neq s_2), \ g_i(0) = 0,$$

where l_i and L_i are known constants.

Remark 1 The constants l_i and L_i here are allowed to be positive, negative, or zero, which makes this assumption on function $g(\cdot)$ less conservative than those stated in He and Cao (2008), Ren and Cao (2008), Chen and Aihara (2002a, b) and Li et al. (2006).

Assumption 2 Suppose that

$$\begin{aligned} &\sigma^{\mathrm{T}}(k, x(k), y(k))\sigma(k, x(k), y(k)) \leq x^{\mathrm{T}}(k)H_1 x(k) \\ &\quad + y^{\mathrm{T}}(k)H_2 y(k), \end{aligned}$$

where $H_1 > 0$ and $H_2 > 0$ are two known matrices.

Assumption 3 Suppose that the time-varying delays $d(k)$ and $\tau(k)$ are bounded with $0 < d_m \leq d(k) \leq d_M$, $0 < \tau_m \leq \tau(k) \leq \tau_M$, and their probability distributions can be observed.

Remark 2 In what follows, in order to transform system (6) with random delays $d(k)$ and $\tau(k)$ into an equivalent system which dependents on distributed sequences, similar analysis as exploited in Yue et al. [2008, 2009] can also be carried out for the random delays. Suppose that $d(k)$ takes values in $[d_m, d_0]$ or $(d_0, d_M]$ and $\mathrm{Prob}\{d(k) \in [d_m, d_0]\} = \alpha_0$, where

d_0, d_m, d_M are integers satisfying $d_m \le d_0 < d_M$, and $0 \le \alpha_0 \le 1$. Similarly, $\tau(k)$ takes values in $[\tau_m, \tau_0]$ or $(\tau_0, \tau_M]$ and $\mathrm{Prob}\{\tau(k) \in [\tau_m, \tau_0]\} = \beta_0$, where τ_0, τ_m, τ_M are integers satisfying $\tau_m \le \tau_0 < \tau_M$, and $0 \le \beta_0 \le 1$.

Define four sets $\mathcal{A}_1 = \{k|d(k) \in [d_m, d_0]\}$, $\mathcal{A}_2 = \{k|d(k) \in (d_0, d_M]\}$, $\mathcal{B}_1 = \{k|\tau(k) \in [\tau_m, \tau_0]\}$ and $\mathcal{B}_2 = \{k|\tau(k) \in (\tau_0, \tau_M]\}$. Obviously, $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathbb{Z}_{\ge 0}$, $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathbb{Z}_{\ge 0}$, $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Furthermore, define four mapping functions

$$d_1(k) = \begin{cases} d(k), & k \in \mathcal{A}_1 \\ d_m, & k \in \mathcal{A}_2 \end{cases}, \quad d_2(k) = \begin{cases} d(k), & k \in \mathcal{A}_2 \\ d_0, & k \in \mathcal{A}_1 \end{cases}$$

and

$$\tau_1(k) = \begin{cases} \tau(k), & k \in \mathcal{B}_1 \\ \tau_m, & k \in \mathcal{B}_2 \end{cases}, \quad \tau_2(k) = \begin{cases} \tau(k), & k \in \mathcal{B}_2 \\ \tau_0, & k \in \mathcal{B}_1 \end{cases}$$

Then one can define two stochastic variables $\alpha(k)$ and $\beta(k)$ which are Bernoulli distributed white sequences taking the values of 0 and 1 with

$$\begin{cases} \mathrm{Prob}\{\alpha(k) = 1\} = \mathbb{E}\{\alpha(k)\} = \alpha_0, \\ \mathrm{Prob}\{\alpha(k) = 0\} = 1 - \mathbb{E}\{\alpha(k)\} = 1 - \alpha_0, \\ \mathrm{Prob}\{\beta(k) = 1\} = \mathbb{E}\{\beta(k)\} = \beta_0, \\ \mathrm{Prob}\{\beta(k) = 0\} = 1 - \mathbb{E}\{\beta(k)\} = 1 - \beta_0. \end{cases}$$

Therefore, system (6) can be equivalently rewritten as

$$\begin{cases} x(k+1) = & (A + \Delta A(k))x(k) + \alpha(k)(B + \Delta B(k)) \\ & g(y(k - d_1(k))) \\ & +(1 - \alpha(k))(B + \Delta B(k))g(y(k - d_2(k))) \\ & +\sigma(k, x(k), y(k))w(k), \\ y(k+1) = & (C + \Delta C(k))y(k) \\ & +\beta(k)(D + \Delta D(k))x(k - \tau_1(k)) \\ & +(1 - \beta(k))(D + \Delta D(k))x(k - \tau_2(k)). \end{cases} \tag{7}$$

For brevity of the following analysis, denote $x(k)$, $y(k)$, $\alpha(k)$, $1 - \alpha(k)$, $\beta(k)$, $1 - \beta(k)$, $w(k)$, $\tau_1(k)$, $\tau_2(k)$, $x(k - \tau_1(k))$, $x(k - \tau_2(k))$, $d_1(k)$, $d_2(k)$, $g(y(k - d_1(k)))$, $g(y(k - d_2(k)))$, $\Delta A(k)$, $\Delta B(k)$, $\Delta C(k)$ and $\Delta D(k)$ by x_k, y_k, α_k, $\bar{\alpha}_k$, β_k, $\bar{\beta}_k$, w_k, τ_1^k, τ_2^k, $x_{\tau,1}$, $x_{\tau,2}$, d_1^k, d_2^k, $g(y_{d,1})$, $g(y_{d,2})$, ΔA_k, ΔB_k, ΔC_k and ΔD_k, respectively.

Then system (7) can be rewritten as

$$\begin{cases} x_{k+1} = (A + \Delta A_k)x_k + \alpha_k(B + \Delta B_k)g(y_{d,1}) \\ +\bar{\alpha}_k(B + \Delta B_k)g(y_{d,2}) + \sigma(k, x_k, y_k)w_k, \\ y_{k+1} = (C + \Delta C_k)y_k + \beta_k(D + \Delta \\ D_k)x_{\tau,1} + \bar{\beta}_k(D + \Delta D_k)x_{\tau,2}. \end{cases} \tag{8}$$

Remark 3 It should be pointed out that, up to now, most existing literatures concentrate on the stability of continuous-time GRNs, but few attempts are devoted to the problem of stability of discrete-time GRNs with stochastic delays. Although the introduction of binary stochastic variables has been presented in the Wang et al. (2006, 2004) and then developed in Yue et al. (2008, 2009), the stability problem for GRNs with stochastic delays still remains challenging.

Definition 1 The origin of system (8) is said to be globally robustly exponentially stable in the mean square sense with $w_k = 0$, if there exist constants $\gamma > 0$ and $0 < \sigma < 1$ such that every solution of system (8) for all parameter uncertainties (that is, ΔA_k, ΔB_k, ΔC_k and ΔD_k) satisfies

$$\mathbb{E}\{\|x_k\|^2 + \|y_k\|^2\} \le \gamma\sigma^k \mathbb{E}\left\{\max_{-\tau_M \le i \le 0} \|x_i\|^2 + \max_{-d_M \le i \le 0} \|y_i\|^2\right\}.$$

Lemma 1 (Schur complement) (Mahmoud and Shi 2003) Given constant matrices Ω_1, Ω_2, Ω_3, where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\begin{pmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{pmatrix} < 0.$$

Lemma 2 For any vectors a, $b \in R^n$, the inequality

$$\pm 2a^T b \le a^T Y a + b^T Y^{-1} b$$

holds, in which Y is any matrix with $Y > 0$.

Proof Since $Y > 0$, we have

$$a^T Y a \pm 2a^T b + b^T Y^{-1} b = (Y^{1/2}a \pm Y^{-1/2}b)^T (Y^{1/2}a \pm Y^{-1/2}b) \ge 0.$$

This completes the proof. □

Main results

In this section, we shall derive sufficient conditions for mean square exponential robust stability of stochastic discrete-time GRNs with random delays. The main results will be stated in two parts.

Case I GRNs without parameter uncertainties.

Firstly, consider the following stochastic GRNs without parameter uncertainties

$$\begin{cases} x_{k+1} = Ax_k + \alpha_k Bg(y_{d,1}) + \bar{\alpha}_k Bg(y_{d,2}) + \sigma(k, x_k, y_k)w_k, \\ y_{k+1} = Cy_k + \beta_k Dx_{\tau,1} + \bar{\beta}_k Dx_{\tau,2}. \end{cases} \tag{9}$$

For simplifying the following representation, denote

$$\hat{L} = \mathrm{diag}(l_1L_1, l_2L_2, \ldots, l_nL_n),$$
$$\check{L} = \mathrm{diag}\left(-\frac{l_1 + L_1}{2}, -\frac{l_2 + L_2}{2}, \ldots, -\frac{l_n + L_n}{2}\right),$$
$$M^T = \begin{bmatrix} M_1^T & 0 & M_2^T & 0 \end{bmatrix},$$
$$N^T = \begin{bmatrix} N_1^T & 0 & 0 & N_2^T \end{bmatrix},$$
$$S^T = \begin{bmatrix} S_1^T & 0 & S_2^T & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$Z^T = \begin{bmatrix} Z_1^T & 0 & 0 & Z_2^T & 0 & 0 & 0 \end{bmatrix},$$

where M_i, N_i, S_i and Z_i ($i = 1, 2$) are any matrices with appropriate dimensions.

Theorem 1 The origin of system (9) is said to be globally exponentially stable in the mean square sense if there exist three positive diagonal matrices $\Lambda_1, \Lambda_2, \Lambda_3$, positive definite matrices $P_1, P_2, Q_1, Q_2, Q_3, Q_4, R_1, R_2, R_3, R_4, K_1, K_2$, any matrices $M_1, M_2, N_1, N_2, S_1, S_2, Z_1, Z_2$ of appropriate dimensions and a positive scalar $\mu_* > 0$ such that the following LMIs

$$P_1 < \mu_* I, \tag{10}$$

$$\Sigma_1 = \begin{bmatrix} \Omega_1 & \tau_0 M & \tau_M N \\ * & -\tau_0 R_1 & 0 \\ * & * & -\tau_M R_2 \end{bmatrix} < 0, \tag{11}$$

$$\Sigma_2 = \begin{bmatrix} \Omega_2 & d_0 S & d_M Z \\ * & -d_0 R_3 & 0 \\ * & * & -d_M R_4 \end{bmatrix} < 0, \tag{12}$$

hold, where

$$\Omega_1 = \begin{bmatrix} \Upsilon_1 & (A-I)^{\mathrm{T}} K_1 & M_2^{\mathrm{T}} - M_1 & N_2^{\mathrm{T}} - N_1 \\ * & \Upsilon_2 & 0 & 0 \\ * & * & \Upsilon_3 & 0 \\ * & * & * & \Upsilon_4 \end{bmatrix},$$

$$\Omega_2 = \begin{bmatrix} \Phi_1 & (C-I)^{\mathrm{T}} K_2 & S_2^{\mathrm{T}} - S_1 & Z_2^{\mathrm{T}} - Z_1 & 0 & 0 & -\Lambda_1 \check{L} \\ * & \Phi_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Phi_3 & 0 & -\Lambda_2 \check{L} & 0 & 0 \\ * & * & * & \Phi_4 & 0 & -\Lambda_3 \check{L} & 0 \\ * & * & * & * & \Phi_5 & 0 & 0 \\ * & * & * & * & * & \Phi_6 & 0 \\ * & * & * & * & * & * & \Phi_7, \end{bmatrix}$$

with

$$\Upsilon_1 = 2A^{\mathrm{T}} P_1 A - P_1 + (\tau_0 - \tau_m + 1) Q_1 + (\tau_M - \tau_0) Q_2 + \mu_* H_1 + M_1 + M_1^{\mathrm{T}} + N_1 + N_1^{\mathrm{T}},$$
$$\Upsilon_2 = \tau_0 R_1 + \tau_M R_2 - K_1,$$
$$\Upsilon_3 = \beta_0 D^{\mathrm{T}} (2P_2 + K_2) D - Q_1 - M_2 - M_2^{\mathrm{T}},$$
$$\Upsilon_4 = \bar{\beta}_0 D^{\mathrm{T}} (2P_2 + K_2) D - Q_2 - N_2 - N_2^{\mathrm{T}},$$
$$\Phi_1 = 2C^{\mathrm{T}} P_2 C - P_2 + \mu_* H_2 + S_1 + S_1^{\mathrm{T}} + Z_1 + Z_1^{\mathrm{T}} - \Lambda_1 \hat{L},$$
$$\Phi_2 = d_0 R_3 + d_M R_4 - K_2,$$
$$\Phi_3 = -S_2 - S_2^{\mathrm{T}} - \Lambda_2 \hat{L},$$
$$\Phi_4 = -Z_2 - Z_2^{\mathrm{T}} - \Lambda_3 \hat{L},$$
$$\Phi_5 = \alpha_0 B^{\mathrm{T}} (2P_1 + K_1) B - Q_3 - \Lambda_2,$$
$$\Phi_6 = \bar{\alpha}_0 B^{\mathrm{T}} (2P_1 + K_1) B - Q_4 - \Lambda_3,$$
$$\Phi_7 = (d_0 - d_m + 1) Q_3 + (d_M - d_0) Q_4 - \Lambda_1.$$

Proof See Appendix A. □

Case II GRNs with parameter uncertainties. In this part, we consider the stochastic GRN (8) with parameter uncertainties.

Theorem 2 The origin of system (8) is said to be globally robustly exponentially stable in the mean square sense if there exist three positive diagonal matrices $\Lambda_1, \Lambda_2, \Lambda_3$, positive definite matrices P_1, P_2, Q_1, Q_2, Q_3, Q_4, R_1, R_2, R_3, R_4, K_1, K_2, any matrices M_1, M_2, N_1, N_2, S_1, S_2, Z_1, Z_2 of appropriate dimensions and scalars $\mu_* > 0$, $k_i > 0$ ($i = 1, ..., 6$) such that the following LMIs

$$P_1 < \mu_* I, \tag{13}$$

$$\Sigma_1^* = \begin{bmatrix} \Omega_1^* & \tau_0 M^* & \tau_M N^* \\ * & -\tau_0 R_1 & 0 \\ * & * & -\tau_M R_2 \end{bmatrix} < 0, \tag{14}$$

$$\Sigma_2^* = \begin{bmatrix} \Omega_2^* & d_0 S^* & d_M Z^* \\ * & -d_0 R_3 & 0 \\ * & * & -d_M R_4 \end{bmatrix} < 0, \tag{15}$$

hold, where

$$\Omega_1^* = \begin{bmatrix} \Upsilon_1^* & (A-I)^{\mathrm{T}} K_1 & M_2^{\mathrm{T}} - M_1 & N_2^{\mathrm{T}} - N_1 & 2A^{\mathrm{T}} P_1 H & 0 & 0 \\ * & \Upsilon_2^* & 0 & 0 & K_1 H & 0 & 0 \\ * & * & \Upsilon_3^* & 0 & 0 & \Theta_1 & 0 \\ * & * & * & \Upsilon_4^* & 0 & 0 & \Theta_2 \\ * & * & * & * & \Upsilon_5^* & 0 & 0 \\ * & * & * & * & * & \Upsilon_6^* & 0 \\ * & * & * & * & * & 0 & \Upsilon_7^* \end{bmatrix},$$

$$\Omega_2^* = \begin{bmatrix} \Phi_1^* & (C-I)^{\mathrm{T}} K_2 & S_2^{\mathrm{T}} - S_1 & Z_2^{\mathrm{T}} - Z_1 & 0 & 0 & -\Lambda_1 \check{L} & 2C^{\mathrm{T}} P_2 H & 0 & 0 \\ * & \Phi_2^* & 0 & 0 & 0 & 0 & 0 & K_2 H & 0 & 0 \\ * & * & \Phi_3^* & 0 & -\Lambda_2 \check{L} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_4^* & 0 & -\Lambda_3 \check{L} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Phi_5^* & 0 & 0 & 0 & \Theta_3 & 0 \\ * & * & * & * & * & \Phi_6^* & 0 & 0 & 0 & \Theta_4 \\ * & * & * & * & * & * & \Phi_7^* & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Phi_8^* & 0 & 0 \\ * & * & * & * & * & * & * & * & \Phi_9^* & 0 \\ * & * & * & * & * & * & * & * & * & \Phi_{10}^*, \end{bmatrix}$$

$$M^{*\mathrm{T}} = \begin{bmatrix} M_1^{\mathrm{T}} & 0 & M_2^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$N^{*\mathrm{T}} = \begin{bmatrix} N_1^{\mathrm{T}} & 0 & 0 & N_2^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix},$$
$$S^{*\mathrm{T}} = \begin{bmatrix} S_1^{\mathrm{T}} & 0 & S_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$Z^{*\mathrm{T}} = \begin{bmatrix} Z_1^{\mathrm{T}} & 0 & 0 & Z_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with

$$\Upsilon_1^* = 2A^{\mathrm{T}} P_1 A - P_1 + (\tau_0 - \tau_m + 1) Q_1 + (\tau_M - \tau_0) Q_2 + \mu_* H_1 + M_1 + M_1^{\mathrm{T}} + N_1 + N_1^{\mathrm{T}} + k_1 E_1^{\mathrm{T}} E_1,$$
$$\Upsilon_2^* = \tau_0 R_1 + \tau_M R_2 - K_1,$$
$$\Upsilon_3^* = \beta_0 D^{\mathrm{T}} (2P_2 + K_2) D - Q_1 - M_2 - M_2^{\mathrm{T}} + k_2 E_4^{\mathrm{T}} E_4,$$
$$\Upsilon_4^* = \bar{\beta}_0 D^{\mathrm{T}} (2P_2 + K_2) D - Q_2 - N_2 - N_2^{\mathrm{T}} + k_3 E_4^{\mathrm{T}} E_4,$$
$$\Upsilon_5^* = 2H^{\mathrm{T}} P_1 H - k_1 I,$$
$$\Upsilon_6^* = \beta_0 H^{\mathrm{T}} (2P_2 + K_2) H - k_2 I,$$
$$\Upsilon_7^* = \bar{\beta}_0 H^{\mathrm{T}} (2P_2 + K_2) H - k_3 I,$$
$$\Theta_1 = \beta_0 D^{\mathrm{T}} (2P_2 + K_2) H,$$
$$\Theta_2 = \bar{\beta}_0 D^{\mathrm{T}} (2P_2 + K_2) H,$$

$$\begin{aligned}
\Phi_1^* &= 2C^{\mathrm{T}}P_2C - P_2 + \mu_* H_2 + S_1 + S_1^{\mathrm{T}} + Z_1 \\
&\quad + Z_1^{\mathrm{T}} - \Lambda_1 \hat{L} + k_4 E_3^{\mathrm{T}} E_3, \\
\Phi_2^* &= d_0 R_3 + d_M R_4 - K_2, \\
\Phi_3^* &= -S_2 - S_2^{\mathrm{T}} - \Lambda_2 \hat{L}, \\
\Phi_4^* &= -Z_2 - Z_2^{\mathrm{T}} - \Lambda_3 \hat{L}, \\
\Phi_5^* &= \alpha_0 B^{\mathrm{T}}(2P_1 + K_1)B - Q_3 - \Lambda_2 + k_5 E_2^{\mathrm{T}} E_2, \\
\Phi_6^* &= \bar{\alpha}_0 B^{\mathrm{T}}(2P_1 + K_1)B - Q_4 - \Lambda_3 + k_6 E_2^{\mathrm{T}} E_2, \\
\Phi_7^* &= (d_0 - d_m + 1)Q_3 + (d_M - d_0)Q_4 - \Lambda_1, \\
\Phi_8^* &= 2H^{\mathrm{T}}P_2H - k_4 I, \\
\Phi_9^* &= \alpha_0 H^{\mathrm{T}}(2P_1 + K_1)H - k_5 I, \\
\Phi_{10}^* &= \bar{\alpha}_0 H^{\mathrm{T}}(2P_1 + K_1)H - k_6 I, \\
\Theta_3 &= \alpha_0 B^{\mathrm{T}}(2P_1 + K_1)H, \\
\Theta_4 &= \bar{\alpha}_0 B^{\mathrm{T}}(2P_1 + K_1)H.
\end{aligned}$$

Proof See Appendix B. □

Examples

In this section, two numerical examples are presented to illustrate the applicability and effectiveness of our results.

Example 1 Consider a two-node GRN (9) with the following parameters:

$$A = \mathrm{diag}(0.4, 0.2), C = \mathrm{diag}(0.2, 0.1), D = \mathrm{diag}(0.2, 0.3),$$
$$B = \begin{bmatrix} -0.1 & -0.6 \\ 0 & 0.3 \end{bmatrix}, H_1 = H_2 = \mathrm{diag}(0.2, 0.2),$$
$$g(s) = \tanh(0.4s).$$

Here, suppose $\tau_m = 1, \tau_0 = 3, d_m = 1, d_0 = 3$. It can be calculated that $\hat{L} = \mathrm{diag}(0, 0)$, $\check{L} = \mathrm{diag}(-0.2, -0.2)$. By setting $\alpha_0 = 0.8, \beta_0 = 0.6, \tau_M = 7, d_M = 10$ in Theorem 1 and using Matlab LMI toolbox, a set of one feasible solutions of LMIs (10)- (12) can be obtained as follows:

$$P_1 = \begin{bmatrix} 3.7304 & 0.4187 \\ 0.4187 & 5.2740 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.1671 & 0.4672 \\ 0.4672 & 4.7820 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.1895 & 0.0419 \\ 0.0419 & 0.5639 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 0.1336 & 0.0297 \\ 0.0297 & 0.3833 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.3658 & 0.1620 \\ 0.1620 & 2.8601 \end{bmatrix},$$
$$Q_4 = \begin{bmatrix} 0.2218 & 0.0129 \\ 0.0129 & 0.7810 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 0.0346 & 0.0085 \\ 0.0085 & 0.0305 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.0125 & 0.0030 \\ 0.0030 & 0.0113 \end{bmatrix},$$
$$R_3 = \begin{bmatrix} 0.0810 & 0.0143 \\ 0.0143 & 0.0488 \end{bmatrix},$$

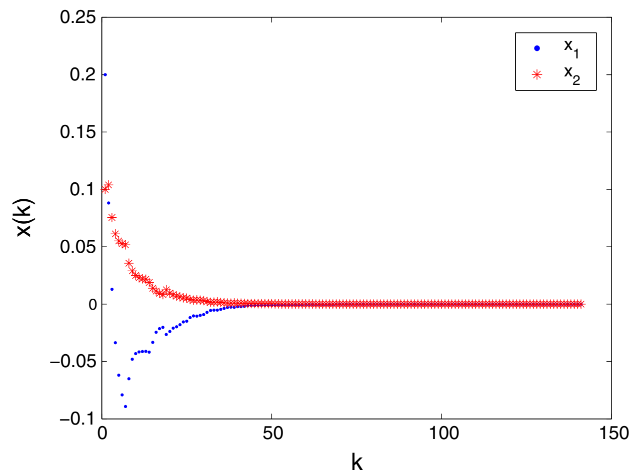

Fig. 1 Transient responses of state variables x_k in system (9)

$$R_4 = \begin{bmatrix} 0.0200 & 0.0031 \\ 0.0031 & 0.0091 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 4.2566 & 0 \\ 0 & 20.5599 \end{bmatrix},$$
$$\Lambda_2 = \begin{bmatrix} 0.4128 & 0 \\ 0 & 0.2350 \end{bmatrix},$$
$$\Lambda_3 = \begin{bmatrix} 0.0701 & 0 \\ 0 & 0.0352 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0.4175 & 0.1239 \\ 0.1239 & 0.4279 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 0.8877 & 0.1396 \\ 0.1396 & 0.4917 \end{bmatrix}, \quad \mu_* = 5.4855.$$

Therefore, all the conditions in Theorem 1 are satisfied, which indicates that the origin of system (9) with stochastic delays and disturbances is globally exponentially stable in the mean square sense. Computer simulations for transient responses of state variables x_k and y_k in system (9) are depicted in Figs. 1 and 2, respectively.

Example 2 Consider another five-node GRN (8) with the following parameters:

$$A = \mathrm{diag}(0.3, 0.2, 0.3, 0.3, 0.2),$$
$$C = \mathrm{diag}(0.2, 0.3, 0.4, 0.2, 0.2),$$
$$D = \mathrm{diag}(0.2, 0.2, 0.1, 0.3, 0.2),$$
$$H = \mathrm{diag}(0.2, 0.2, 0.2, 0.2, 0.2),$$
$$F(s) = \mathrm{diag}(\sin(s), \cos(s), -\sin(2s), \cos(2s), \sin(s)),$$
$$E_1 = E_2 = E_3 = E_4 = \mathrm{diag}(0.3, 0.3, 0.3, 0.3, 0.3),$$
$$B = \begin{bmatrix} -0.1 & -0.6 & 0 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0 & 0 \\ 0.1 & -0.2 & 0.1 & 0 & 0 \\ 0.2 & -0.1 & 0 & -0.3 & 0 \\ 0 & 0 & 0 & 0 & -0.5 \end{bmatrix},$$
$$H_1 = H_2 = \mathrm{diag}(0.16, 0.16, 0.16, 0.16, 0.16),$$
$$g_i(s) = \tanh(0.5s), i = 1, \ldots, 5.$$

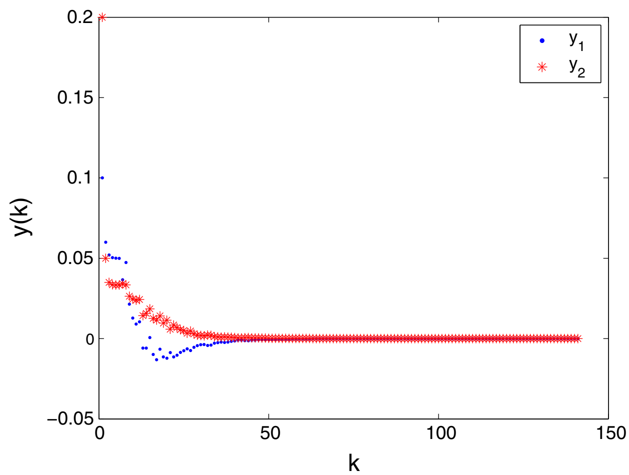

Fig. 2 Transient responses of state variables y_k in system (9)

It is easy to obtain that $\hat{L} = \mathrm{diag}(0,0,0,0,0)$, $\check{L} = \mathrm{diag}(-0.25,-0.25,-0.25,-0.25,-0.25)$. Set $\tau_m = 1$, $\tau_0 = 4$, $\tau_M = 8$, $d_m = 1$, $d_0 = 3$, $d_M = 10$, $\alpha_0 = 0.6$, $\beta_0 = 0.7$. By applying Theorem 2, one can verify by Matlab LMI toolbox that feasible solutions of the LMIs (13–15) exist. Therefore, for all parameter uncertainties and stochastic perturbations, the origin of system (8) is said to be globally robustly exponentially stable in the mean square sense. Computer simulations for transient responses of state variables x_k and y_k in system (8) are shown in Figs. 3 and 4, respectively.

Conclusions

The problem of global robust exponential stability of stochastic discrete-time GRNs with parameter uncertainties and random delays has been studied. By constructing a proper Lyapunov-Krasovskii functional and adopting a new modelling method, two delay-distribution-dependent conditions are derived. Different from the existing GRN models, the probability distributions of the time delays have been translated into the networks' parameter matrices. Two numerical examples and their simulations have been given to illustrate the effectiveness and applicability of the obtained results.

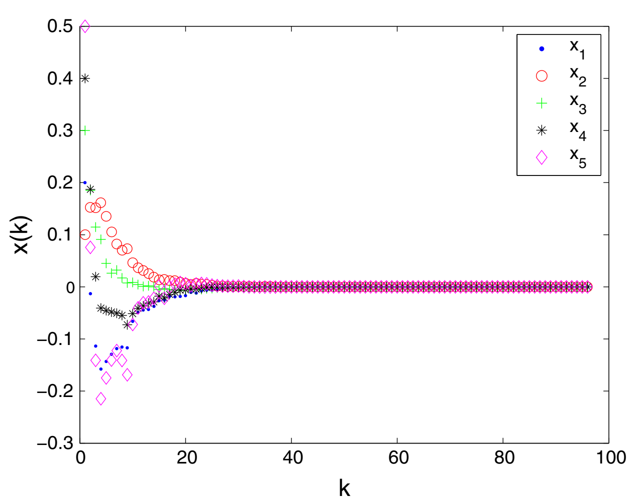

Fig. 3 Transient responses of state variables x_k in system (8)

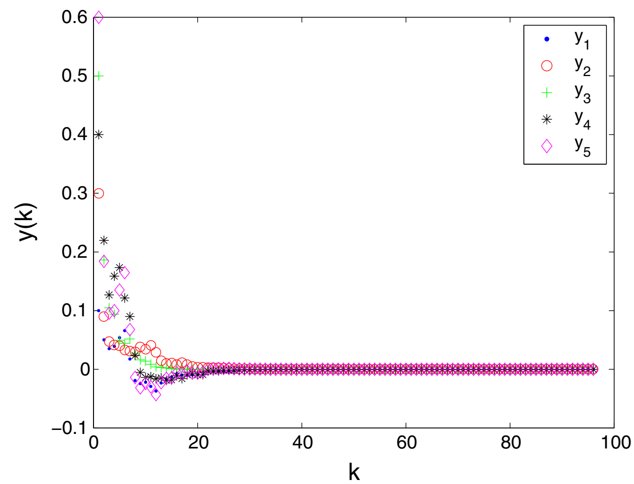

Fig. 4 Transient responses of state variables y_k in system (8)

Appendix A: Proof of Theorem 1

Consider the following Lyapunov-Krasovskii functional candidate:

$$V_k = V_{k,1} + V_{k,2} + V_{k,3} + V_{k,4} + V_{k,5} \tag{16}$$

where

$$
\begin{aligned}
V_{k,1} &= x_k^{\mathrm{T}} P_1 x_k + y_k^{\mathrm{T}} P_2 y_k,\\
V_{k,2} &= \sum_{i=k-\tau_1^k}^{k-1} x_i^{\mathrm{T}} Q_1 x_i + \sum_{j=-\tau_0+1}^{-\tau_m} \sum_{i=k+j}^{k-1} x_i^{\mathrm{T}} Q_1 x_i\\
&\quad + \sum_{i=k-\tau_2^k}^{k-1} x_i^{\mathrm{T}} Q_2 x_i + \sum_{j=-\tau_M+1}^{-\tau_0-1} \sum_{i=k+j}^{k-1} x_i^{\mathrm{T}} Q_2 x_i,\\
V_{k,3} &= \sum_{i=k-d_1^k}^{k-1} g^{\mathrm{T}}(y_i) Q_3 g(y_i) + \sum_{j=-d_0+1}^{-d_m} \sum_{i=k+j}^{k-1} g^{\mathrm{T}}(y_i) Q_3 g(y_i)\\
&\quad + \sum_{i=k-d_2^k}^{k-1} g^{\mathrm{T}}(y_i) Q_4 g(y_i) + \sum_{j=-d_M+1}^{-d_0-1} \sum_{i=k+j}^{k-1} g^{\mathrm{T}}(y_i) Q_4 g(y_i),\\
V_{k,4} &= \sum_{j=1}^{\tau_0} \sum_{i=k-j}^{k-1} \eta_i^{\mathrm{T}} R_1 \eta_i + \sum_{j=1}^{\tau_M} \sum_{i=k-j}^{k-1} \eta_i^{\mathrm{T}} R_2 \eta_i, \quad \eta_k = x_{k+1} - x_k,\\
V_{k,5} &= \sum_{j=1}^{d_0} \sum_{i=k-j}^{k-1} \delta_i^{\mathrm{T}} R_3 \delta_i + \sum_{j=1}^{d_M} \sum_{i=k-j}^{k-1} \delta_i^{\mathrm{T}} R_4 \delta_i, \quad \delta_k = y_{k+1} - y_k.
\end{aligned}
$$

Calculating the difference of V_k along the solution of (9) and taking its mathematical expectation yield

$$\begin{aligned}\mathbb{E}\{\Delta V_{k,1}\} &= \mathbb{E}\{\mathbb{E}\{V_{k+1,1}\} - V_{k,1}\} \\ &= \mathbb{E}\{x_k^{\mathrm{T}}(A^{\mathrm{T}}P_1A - P_1)x_k + 2\alpha_0 x_k^{\mathrm{T}} \\ &\quad A^{\mathrm{T}}P_1Bg(y_{d,1}) + 2\bar{\alpha}_0 x_k^{\mathrm{T}}A^{\mathrm{T}}P_1Bg(y_{d,2}) \\ &\quad + \alpha_0 g^{\mathrm{T}}(y_{d,1})B^{\mathrm{T}}P_1Bg(y_{d,1}) \\ &\quad + \bar{\alpha}_0 g^{\mathrm{T}}(y_{d,2})B^{\mathrm{T}}P_1Bg(y_{d,2}) \\ &\quad + \sigma^{\mathrm{T}}(k,x_k,y_k)P_1\sigma(k,x_k,y_k) \\ &\quad + y_k^{\mathrm{T}}(C^{\mathrm{T}}P_2C - P_2)y_k + 2\beta_0 y_k^{\mathrm{T}}C^{\mathrm{T}}P_2Dx_{\tau,1} \\ &\quad + 2\bar{\beta}_0 y_k^{\mathrm{T}}C^{\mathrm{T}}P_2Dx_{\tau,2} \\ &\quad + \beta_0 x_{\tau,1}^{\mathrm{T}}D^{\mathrm{T}}P_2Dx_{\tau,1} + \bar{\beta}_0 x_{\tau,2}^{\mathrm{T}}D^{\mathrm{T}}P_2Dx_{\tau,2}\}, \quad (17)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{k,2}\} &= \mathbb{E}\{\mathbb{E}\{V_{k+1,2}\} - V_{k,2}\} \\ &\le \mathbb{E}\{(\tau_0 - \tau_m + 1)x_k^{\mathrm{T}}Q_1x_k \\ &\quad - x_{\tau,1}^{\mathrm{T}}Q_1x_{\tau,1} + (\tau_M - \tau_0)x_k^{\mathrm{T}}Q_2x_k - x_{\tau,2}^{\mathrm{T}}Q_2x_{\tau,2}\}, \quad (18)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{k,3}\} &= \mathbb{E}\{\mathbb{E}\{V_{k+1,3}\} - V_{k,3}\} \\ &\le \mathbb{E}\{(d_0 - d_m + 1)g^{\mathrm{T}}(y_k)Q_3g(y_k) \\ &\quad - g^{\mathrm{T}}(y_{d,1})Q_3g(y_{d,1}) \quad (19) \\ &\quad + (d_M - d_0)g^{\mathrm{T}}(y_k)Q_4g(y_k) \\ &\quad - g^{\mathrm{T}}(y_{d,2})Q_4g(y_{d,2})\},\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{k,4}\} &= \mathbb{E}\{\mathbb{E}\{V_{k+1,4}\} - V_{k,4}\} \\ &= \mathbb{E}\left\{\eta_k^{\mathrm{T}}(\tau_0R_1 + \tau_MR_2)\eta_k - \sum_{j=k-\tau_0}^{k-1}\eta_i^{\mathrm{T}}R_1\eta_i - \sum_{j=k-\tau_M}^{k-1}\eta_i^{\mathrm{T}}R_2\eta_i\right\}, \quad (20)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{k,5}\} &= \mathbb{E}\{\mathbb{E}\{V_{k+1,5}\} - V_{k,5}\} \\ &= \mathbb{E}\left\{\delta_k^{\mathrm{T}}(d_0R_3 + d_MR_4)\delta_k - \sum_{j=k-d_0}^{k-1}\delta_i^{\mathrm{T}}R_3\delta_i - \sum_{j=k-d_M}^{k-1}\delta_i^{\mathrm{T}}R_4\delta_i\right\}. \quad (21)\end{aligned}$$

Considering Assumption 2 and (10), it can be easily obtained

$$\begin{aligned}&\mathbb{E}\{\sigma^{\mathrm{T}}(k,x_k,y_k)P_1\sigma(k,x_k,y_k)\} \\ &\quad \le \mathbb{E}\{\lambda_{\max}(P_1)\sigma^{\mathrm{T}}(k,x_k,y_k)\sigma(k,x_k,y_k)\} \quad (22) \\ &\quad \le \mathbb{E}\{\mu_* x_k^{\mathrm{T}}H_1x_k + \mu_* y_k^{\mathrm{T}}H_2y_k\}.\end{aligned}$$

Making use of Lemma 2, we can derive

$$\begin{aligned}2\alpha_0 x_k^{\mathrm{T}}A^{\mathrm{T}}P_1Bg(y_{d,1}) &\le \alpha_0 x_k^{\mathrm{T}}A^{\mathrm{T}}P_1Ax_k \\ &\quad + \alpha_0 g^{\mathrm{T}}(y_{d,1})B^{\mathrm{T}}P_1Bg(y_{d,1}), \quad (23)\end{aligned}$$

$$\begin{aligned}2\bar{\alpha}_0 x_k^{\mathrm{T}}A^{\mathrm{T}}P_1Bg(y_{d,2}) &\le \bar{\alpha}_0 x_k^{\mathrm{T}}A^{\mathrm{T}}P_1Ax_k \\ &\quad + \bar{\alpha}_0 g^{\mathrm{T}}(y_{d,2})B^{\mathrm{T}}P_1Bg(y_{d,2}), \quad (24)\end{aligned}$$

$$2\beta_0 y_k^{\mathrm{T}}C^{\mathrm{T}}P_2Dx_{\tau,1} \le \beta_0 y_k^{\mathrm{T}}C^{\mathrm{T}}P_2Cy_k + \beta_0 x_{\tau,1}^{\mathrm{T}}D^{\mathrm{T}}P_2Dx_{\tau,1}, \quad (25)$$

$$2\bar{\beta}_0 y_k^{\mathrm{T}}C^{\mathrm{T}}P_2Dx_{\tau,2} \le \bar{\beta}_0 y_k^{\mathrm{T}}C^{\mathrm{T}}P_2Cy_k + \bar{\beta}_0 x_{\tau,2}^{\mathrm{T}}D^{\mathrm{T}}P_2Dx_{\tau,2}. \quad (26)$$

Obviously, the following zero equations hold.

$$2\xi_1^{\mathrm{T}}(k)M\left[x_k - x_{\tau,1} - \sum_{i=k-\tau_1^k}^{k-1}(x_{i+1} - x_i)\right] = 0, \quad (27)$$

$$2\xi_1^{\mathrm{T}}(k)N\left[x_k - x_{\tau,2} - \sum_{i=k-\tau_2^k}^{k-1}(x_{i+1} - x_i)\right] = 0, \quad (28)$$

$$2\xi_2^{\mathrm{T}}(k)S\left[y_k - y_{d,1} - \sum_{i=k-d_1^k}^{k-1}(y_{i+1} - y_i)\right] = 0, \quad (29)$$

$$2\xi_2^{\mathrm{T}}(k)Z\left[y_k - y_{d,2} - \sum_{i=k-d_2^k}^{k-1}(y_{i+1} - y_i)\right] = 0. \quad (30)$$

where

$$\xi_1^{\mathrm{T}}(k) = \left[x_k^{\mathrm{T}} \quad \eta_k^{\mathrm{T}} \quad x_{\tau,1}^{\mathrm{T}} \quad x_{\tau,2}^{\mathrm{T}}\right],$$

$$\xi_2^{\mathrm{T}}(k) = \left[y_k^{\mathrm{T}} \quad \delta_k^{\mathrm{T}} \quad y_{d,1}^{\mathrm{T}} \quad y_{d,2}^{\mathrm{T}} \quad g^{\mathrm{T}}(y_{d,1}) \quad g^{\mathrm{T}}(y_{d,2}) \quad g^{\mathrm{T}}(y_k)\right].$$

By applying Lemma 2, we have

$$\begin{aligned}-2\xi_1^{\mathrm{T}}(k)M\sum_{i=k-\tau_1^k}^{k-1}(x_{i+1} - x_i) &\le \tau_0\xi_1^{\mathrm{T}}(k)MR_1^{-1}M^{\mathrm{T}}\xi_1(k) \\ &\quad + \sum_{i=k-\tau_1^k}^{k-1}\eta_i^{\mathrm{T}}R_1\eta_i, \quad (31)\end{aligned}$$

$$\begin{aligned}-2\xi_1^{\mathrm{T}}(k)N\sum_{i=k-\tau_2^k}^{k-1}(x_{i+1} - x_i) &\le \tau_M\xi_1^{\mathrm{T}}(k)NR_2^{-1}N^{\mathrm{T}}\xi_1(k) \\ &\quad + \sum_{i=k-\tau_2^k}^{k-1}\eta_i^{\mathrm{T}}R_2\eta_i, \quad (32)\end{aligned}$$

$$\begin{aligned}-2\xi_2^{\mathrm{T}}(k)S\sum_{i=k-d_1^k}^{k-1}(y_{i+1} - y_i) &\le d_0\xi_2^{\mathrm{T}}(k)SR_3^{-1}S^{\mathrm{T}}\xi_2(k) \\ &\quad + \sum_{i=k-d_1^k}^{k-1}\delta_i^{\mathrm{T}}R_3\delta_i, \quad (33)\end{aligned}$$

$$\begin{aligned}-2\xi_2^{\mathrm{T}}(k)Z\sum_{i=k-d_2^k}^{k-1}(y_{i+1} - y_i) &\le d_M\xi_2^{\mathrm{T}}(k)ZR_4^{-1}Z^{\mathrm{T}}\xi_2(k) \\ &\quad + \sum_{i=k-d_2^k}^{k-1}\delta_i^{\mathrm{T}}R_4\delta_i. \quad (34)\end{aligned}$$

For positive definite matrices K_1 and K_2, it follows from the definition of η_k and δ_k that

$$\begin{aligned}0 &= 2\eta_k^{\mathrm{T}}K_1(x_{k+1} - x_k - \eta_k) \\ &= 2\eta_k^{\mathrm{T}}K_1[(A - I)x_k + \alpha_kBg(y_{d,1}) \\ &\quad + \bar{\alpha}_kBg(y_{d,2}) + \sigma(k,x_k,y_k)w_k - \eta_k], \quad (35)\end{aligned}$$

$$
\begin{aligned}
0 &= 2\delta_k^{\mathrm{T}} K_2(x_{k+1} - x_k - \delta_k) \\
&= 2\delta_k^{\mathrm{T}} K_2[(C - I)y_k + \beta_k D x_{\tau,1} + \bar{\beta}_k D x_{\tau,2} - \delta_k].
\end{aligned} \tag{36}
$$

Taking the expectations on both side of (35) and (36), and employing Lemma 2 yield

$$
\begin{aligned}
0 &= \mathbb{E}\{2\eta_k^{\mathrm{T}} K_1[(A - I)x_k + \alpha_0 B g(y_{d,1}) + \bar{\alpha}_0 B g(y_{d,2}) - \eta_k]\} \\
&= \mathbb{E}\{2\eta_k^{\mathrm{T}} K_1(A - I)x_k + 2\alpha_0 \eta_k^{\mathrm{T}} K_1 B g(y_{d,1}) \\
&\quad + 2\bar{\alpha}_0 \eta_k^{\mathrm{T}} K_1 B g(y_{d,2}) - 2\eta_k^{\mathrm{T}} K_1 \eta_k\} \\
&\le \mathbb{E}\{2\eta_k^{\mathrm{T}} K_1(A - I)x_k + \alpha_0 \eta_k^{\mathrm{T}} K_1 \eta_k \\
&\quad + \alpha_0 g^{\mathrm{T}}(y_{d,1}) B^{\mathrm{T}} K_1 B g(y_{d,1}) \\
&\quad + \bar{\alpha}_0 \eta_k^{\mathrm{T}} K_1 \eta_k + \bar{\alpha}_0 g^{\mathrm{T}}(y_{d,2}) B^{\mathrm{T}} K_1 B g(y_{d,2}) \\
&\quad - 2\eta_k^{\mathrm{T}} K_1 \eta_k\} \\
&= \mathbb{E}\{2\eta_k^{\mathrm{T}} K_1(A - I)x_k - \eta_k^{\mathrm{T}} K_1 \eta_k + \alpha_0 g^{\mathrm{T}}(y_{d,1}) B^{\mathrm{T}} K_1 B g(y_{d,1}) \\
&\quad + \bar{\alpha}_0 g^{\mathrm{T}}(y_{d,2}) B^{\mathrm{T}} K_1 B g(y_{d,2})\},
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
0 =& \mathbb{E}\{2\delta_k^{\mathrm{T}} K_2[(C - I)y_k + \beta_0 D x_{\tau,1} + \bar{\beta}_0 D x_{\tau,2} - \delta_k]\} \\
=& \mathbb{E}\{2\delta_k^{\mathrm{T}} K_2(C - I)y_k + 2\beta_0 \delta_k^{\mathrm{T}} K_2 D x_{\tau,1} \\
& + 2\bar{\beta}_0 \delta_k^{\mathrm{T}} K_2 D x_{\tau,2} - 2\delta_k^{\mathrm{T}} K_2 \delta_k\} \\
\le& \mathbb{E}\{2\delta_k^{\mathrm{T}} K_2(C - I)y_k + \beta_0 \delta_k^{\mathrm{T}} K_2 \delta_k + \beta_0 x_{\tau,1}^{\mathrm{T}} D^{\mathrm{T}} K_2 D x_{\tau,1} \\
& + \bar{\beta}_0 \delta_k^{\mathrm{T}} K_2 \delta_k + \bar{\beta}_0 x_{\tau,2}^{\mathrm{T}} D^{\mathrm{T}} K_2 D x_{\tau,2} - 2\delta_k^{\mathrm{T}} K_2 \delta_k\} \\
=& \mathbb{E}\{2\delta_k^{\mathrm{T}} K_2(C - I)y_k - \delta_k^{\mathrm{T}} K_2 \delta_k \\
& + \beta_0 x_{\tau,1}^{\mathrm{T}} D^{\mathrm{T}} K_2 D x_{\tau,1} + \bar{\beta}_0 x_{\tau,2}^{\mathrm{T}} D^{\mathrm{T}} K_2 D x_{\tau,2}\}.
\end{aligned} \tag{38}
$$

In view of Assumption 1, we can conclude that

$$
[g_i(y_i(k)) - l_i y_i(k)][g_i(y_i(k)) - L_i y_i(k)] \le 0, \tag{39}
$$

$$
\begin{aligned}
&[g_i(y_i(k - d_1(k))) - l_i y_i(k - d_1(k))][g_i(y_i(k - d_1(k))) \\
&\quad - L_i y_i(k - d_1(k))] \le 0, (40)
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&[g_i(y_i(k - d_2(k))) - l_i y_i(k - d_2(k))][g_i(y_i(k - d_2(k))) \\
&\quad - L_i y_i(k - d_2(k))] \le 0. (41)
\end{aligned} \tag{41}
$$

It can be deduced from (39) that there exists a diagonal matrix $\Lambda_1 = \mathrm{diag}\{\lambda_{1,1}, \ldots, \lambda_{1,n}\} > 0$ such that

$$
\begin{aligned}
&\sum_{i=1}^{n} \lambda_{1,i} \begin{bmatrix} y_k \\ g(y_k) \end{bmatrix}^{\mathrm{T}} l_i \begin{bmatrix} l_i L_i e_i e_i^{\mathrm{T}} & -\frac{l_i + L_i}{2} e_i e_i^{\mathrm{T}} \\ -\frac{l_i + L_i}{2} e_i e_i^{\mathrm{T}} & e_i e_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} y_k \\ g(y_k) \end{bmatrix} \\
&\quad = \begin{bmatrix} y_k \\ g(y_k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Lambda_1 \hat{L} & \Lambda_1 \check{L} \\ \Lambda_1 \check{L} & \Lambda_1 \end{bmatrix} \begin{bmatrix} y_k \\ g(y_k) \end{bmatrix} \le 0,
\end{aligned} \tag{42}
$$

where e_i denotes a column vector having "1" element on its ith row and zeros elsewhere. Similarly, by means of (40) and (41), there exist diagonal matrices Λ_2 and Λ_3 such that

$$
\begin{bmatrix} y_{d,1} \\ g(y_{d,1}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Lambda_2 \hat{L} & \Lambda_2 \check{L} \\ \Lambda_2 \check{L} & \Lambda_2 \end{bmatrix} \begin{bmatrix} y_{d,1} \\ g(y_{d,1}) \end{bmatrix} \le 0 \tag{43}
$$

and

$$
\begin{bmatrix} y_{d,2} \\ g(y_{d,2}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Lambda_3 \hat{L} & \Lambda_3 \check{L} \\ \Lambda_3 \check{L} & \Lambda_3 \end{bmatrix} \begin{bmatrix} y_{d,2} \\ g(y_{d,2}) \end{bmatrix} \le 0, \tag{44}
$$

respectively.

Therefore, we have

$$
\begin{aligned}
\mathbb{E}\{\Delta V_k\} \le \mathbb{E}\Bigg\{ & x_k^{\mathrm{T}}(2A^{\mathrm{T}} P_1 A - P_1 + (\tau_0 - \tau_m + 1)Q_1 \\
& + (\tau_M - \tau_0)Q_2 + \mu_* H_1)x_k \\
& + 2\eta_k^{\mathrm{T}} K_1(A - I)x_k + \eta_k^{\mathrm{T}}(\tau_0 R_1 + \tau_M R_2 - K_1)\eta_k \\
& + x_{\tau,1}^{\mathrm{T}}[\beta_0 D^{\mathrm{T}}(2P_2 + K_2)D - Q_1]x_{\tau,1} \\
& + x_{\tau,2}^{\mathrm{T}}[\bar{\beta}_0 D^{\mathrm{T}}(2P_2 + K_2)D - Q_2]x_{\tau,2} \\
& + y_k^{\mathrm{T}}(2C^{\mathrm{T}} P_2 C - P_2 + \mu_* H_2)y_k \\
& + 2\delta_k^{\mathrm{T}} K_2(C - I)y_k + \delta_k^{\mathrm{T}}(d_0 R_3 + d_M R_4 - K_2)\delta_k \\
& + g^{\mathrm{T}}(y_k)[(d_0 - d_m + 1)Q_3 + (d_M - d_0)Q_4]g(y_k) \\
& + g^{\mathrm{T}}(y_{d,1})[\alpha_0 B^{\mathrm{T}}(2P_1 + K_1)B - Q_3]g(y_{d,1}) \\
& + g^{\mathrm{T}}(y_{d,2})[\bar{\alpha}_0 B^{\mathrm{T}}(2P_1 + K_1)B - Q_4]g(y_{d,2}) \\
& + 2\xi_1^{\mathrm{T}}(k)M(x_k - x_{\tau,1}) + 2\xi_1^{\mathrm{T}}(k)N(x_k - x_{\tau,2}) \\
& + 2\xi_2^{\mathrm{T}}(k)S(y_k - y_{d,1}) + 2\xi_2^{\mathrm{T}}(k)Z(y_k - y_{d,2}) \\
& + \xi_1^{\mathrm{T}}(k)(\tau_0 M R_1^{-1} M^{\mathrm{T}} + \tau_M N R_2^{-1} N^{\mathrm{T}})\xi_1(k) \\
& + \xi_2^{\mathrm{T}}(k)(d_0 S R_3^{-1} S^{\mathrm{T}} + d_M Z R_4^{-1} Z^{\mathrm{T}})\xi_2(k) \\
& - \begin{bmatrix} y_k \\ g(y_k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Lambda_1 \hat{L} & \Lambda_1 \check{L} \\ \Lambda_1 \check{L} & \Lambda_1 \end{bmatrix} \begin{bmatrix} y_k \\ g(y_k) \end{bmatrix} \\
& - \begin{bmatrix} y_{d,1} \\ g(y_{d,1}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Lambda_2 \hat{L} & \Lambda_2 \check{L} \\ \Lambda_2 \check{L} & \Lambda_2 \end{bmatrix} \begin{bmatrix} y_{d,1} \\ g(y_{d,1}) \end{bmatrix} \\
& - \begin{bmatrix} y_{d,2} \\ g(y_{d,2}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Lambda_3 \hat{L} & \Lambda_3 \check{L} \\ \Lambda_3 \check{L} & \Lambda_3 \end{bmatrix} \begin{bmatrix} y_{d,2} \\ g(y_{d,2}) \end{bmatrix} \Bigg\} \\
= \mathbb{E}\{ & \xi_1^{\mathrm{T}}(k)\left[\Omega_1 + \tau_0 M R_1^{-1} M^{\mathrm{T}} + \tau_M N R_2^{-1} N^{\mathrm{T}}\right]\xi_1(k) \\
& + \xi_2^{\mathrm{T}}(k)\left[\Omega_2 + d_0 S R_3^{-1} S^{\mathrm{T}} + d_M Z R_4^{-1} Z^{\mathrm{T}}\right]\xi_2(k)\},
\end{aligned} \tag{45}
$$

where

$$
\begin{aligned}
\xi_1^{\mathrm{T}}(k) &= \begin{bmatrix} x_k^{\mathrm{T}} & \eta_k^{\mathrm{T}} & x_{\tau,1}^{\mathrm{T}} & x_{\tau,2}^{\mathrm{T}} \end{bmatrix}, \\
\xi_2^{\mathrm{T}}(k) &= \begin{bmatrix} y_k^{\mathrm{T}} & \delta_k^{\mathrm{T}} & y_{d,1}^{\mathrm{T}} & y_{d,2}^{\mathrm{T}} & g^{\mathrm{T}}(y_{d,1}) & g^{\mathrm{T}}(y_{d,2}) & g^{\mathrm{T}}(y_k) \end{bmatrix}.
\end{aligned}
$$

According to the well-known Schur complement (see, Lemma 1), one can get

$$\begin{aligned}\mathbb{E}\{\Delta V_k\} &\leq \mathbb{E}\{\xi_1^{\mathrm{T}}(k)\Sigma_1\xi_1(k) + \xi_2^{\mathrm{T}}(k)\Sigma_2\xi_2(k)\} \\ &\leq \lambda_{\max}(\Sigma_1)\mathbb{E}\{\|x_k\|^2 + \|\eta_k\|^2 + \|x_{\tau,1}\|^2 + \|x_{\tau,2}\|^2\} \\ &\quad + \lambda_{\max}(\Sigma_2)\mathbb{E}\{\|y_k\|^2 + \|\delta_k\|^2 + \|y_{d,1}\|^2 + \|y_{d,2}\|^2 \\ &\quad + \|g(y_{d,1})\|^2 + \|g(y_{d,2})\|^2 + \|g(y_k)\|^2\}.\end{aligned}$$

In view of the conditions $\Sigma_1 < 0$ and $\Sigma_2 < 0$ in Theorem 1, it follows that

$$\mathbb{E}\{\Delta V_k\} \leq \lambda_{\max}(\Sigma_1)\mathbb{E}\{\|x_k\|^2\} + \lambda_{\max}(\Sigma_2)\mathbb{E}\{\|y_k\|^2\} \leq 0, \tag{46}$$

which implies that the origin of system (9) is globally asymptotically stable in the mean square sense.

Now, we are in a position to proceed with the global exponential stability analysis of the system (9). It follows from Assumption 1 that

$$\begin{aligned}&\|g(y_k)\| \leq L_{\max}\|y_k\|, \\ &\|g(y_{d,1})\| \leq L_{\max}\|y_{d,1}\|, \\ &\|g(y_{d,2})\| \leq L_{\max}\|y_{d,2}\|,\end{aligned}$$

where $L_{\max} = \max\{|l_1|, ..., |l_n|, |L_1|, ..., |L_n|\}$. Based upon expression of V_k, one can get

$$\begin{aligned}\mathbb{E}\{V_k\} \leq{}& \rho_1\mathbb{E}\{\|x_k\|^2\} + \rho_2\mathbb{E}\{\|y_k\|^2\} + \rho_3\sum_{i=k-\tau_M}^{k-1}\mathbb{E}\{\|x_i\|^2\} \\ &+ \rho_4\sum_{i=k-d_M}^{k-1}\mathbb{E}\{\|y_i\|^2\} + \rho_5\sum_{i=k-d_M}^{k-1}\mathbb{E}\{\|y_{d,1}\|^2\} \\ &+ \rho_6\sum_{i=k-d_M}^{k-1}\mathbb{E}\{\|y_{d,2}\|^2\} + \rho_7\sum_{i=k-\tau_M}^{k-1}\mathbb{E}\{\|x_{\tau,1}\|^2\} \\ &+ \rho_8\sum_{i=k-\tau_M}^{k-1}\mathbb{E}\{\|x_{\tau,2}\|^2\},\end{aligned} \tag{47}$$

where

$$\begin{aligned}\rho_1 &= \lambda_{\max}(P_1), \rho_2 = \lambda_{\max}(P_2), \\ \rho_3 &= (\tau_0 - \tau_m + 1)\lambda_{\max}(Q_1) + (\tau_M - \tau_0)\lambda_{\max}(Q_2) \\ &\quad + [\tau_0\lambda_{\max}(R_1) + \tau_M\lambda_{\max}(R_2)]\|A - I\|, \\ \rho_4 &= [(d_0 - d_m + 1)\lambda_{\max}(Q_3) + (d_M - d_0)\lambda_{\max}(Q_4)]L_{\max} \\ &\quad + [d_0\lambda_{\max}(R_3) + d_M\lambda_{\max}(R_4)]\|C - I\|, \\ \rho_5 &= [\tau_0\lambda_{\max}(R_1) + \tau_M\lambda_{\max}(R_2)]\alpha_0\|B\|L_{\max}, \\ \rho_6 &= [\tau_0\lambda_{\max}(R_1) + \tau_M\lambda_{\max}(R_2)]\bar{\alpha}_0\|B\|L_{\max}, \\ \rho_7 &= [d_0\lambda_{\max}(R_3) + d_M\lambda_{\max}(R_4)]\beta_0\|D\|, \\ \rho_8 &= [d_0\lambda_{\max}(R_3) + d_M\lambda_{\max}(R_4)]\bar{\beta}_0\|D\|.\end{aligned}$$

For any scalar $\mu > 1$, the above inequality (47), together with (46), implies that

$$\begin{aligned}\mu^{k+1}&\mathbb{E}\{V_{k+1}\} - \mu^k\mathbb{E}\{V_k\} = \mu^{k+1}\mathbb{E}\{\Delta V_k\} + \mu^k(\mu-1)\mathbb{E}\{V_k\} \\ &\leq \psi_1(\mu)\mu^k\mathbb{E}\{\|x_k\|^2\} + \psi_2(\mu)\mu^k\mathbb{E}\{\|y_k\|^2\} \\ &\quad + \psi_3(\mu)\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_i\|^2\} \\ &\quad + \psi_4(\mu)\sum_{i=k-d_M}^{k-1}\mu^k\mathbb{E}\{\|y_i\|^2\} + \psi_5(\mu)\sum_{i=k-d_M}^{k-1}\mu^k\mathbb{E}\{\|y_{d,1}\|^2\} \\ &\quad + \psi_6(\mu)\sum_{i=k-d_M}^{k-1}\mu^k\mathbb{E}\{\|y_{d,2}\|^2\} + \psi_7(\mu)\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_{\tau,1}\|^2\} \\ &\quad + \psi_8(\mu)\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_{\tau,2}\|^2\},\end{aligned} \tag{48}$$

where $\psi_i(\mu) = \mu\lambda_{\max}(\Sigma_i) + (\mu - 1)\rho_i, i = 1, 2$ and $\psi_j(\mu) = (\mu - 1)\rho_j, j = 3, ..., 8$.

Furthermore, for any integer $N \geq 1$, summing up both sides of (48) from 0 to $N - 1$ with respect to k, yields

$$\begin{aligned}\mu^N&\mathbb{E}\{V_N\} - \mathbb{E}\{V_0\} \leq \\ &+ \psi_3(\mu)\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_i\|^2\} \\ &+ \psi_4(\mu)\sum_{k=0}^{N-1}\sum_{i=k-d_M}^{k-1}\mu^k\mathbb{E}\{\|y_i\|^2\} \\ &+ \psi_5(\mu)\sum_{k=0}^{N-1}\sum_{i=k-d_M}^{k-1}\mu^k\mathbb{E}\{\|y_{i-d_1^i}\|^2\} \\ &+ \psi_6(\mu)\sum_{k=0}^{N-1}\sum_{i=k-d_M}^{k-1}\mu^k\mathbb{E}\{\|y_{i-d_2^i}\|^2\} \\ &+ \psi_7(\mu)\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_{i-\tau_1^i}\|^2\} \\ &+ \psi_8(\mu)\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_{i-\tau_2^i}\|^2\}.\end{aligned} \tag{49}$$

Note that for $\tau_M \geq 1$. It follows that

$$\begin{aligned}&\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\mu^k\mathbb{E}\{\|x_i\|^2\} \\ &\leq \left(\sum_{i=-\tau_M}^{-1}\sum_{k=0}^{i+\tau_M} + \sum_{i=0}^{N-1-\tau_M}\sum_{k=i+1}^{i+\tau_M} + \sum_{i=N-\tau_M}^{N-1}\sum_{k=i+1}^{N-1}\right)\mu^k\mathbb{E}\{\|x_i\|^2\} \\ &\leq \tau_M\sum_{i=-\tau_M}^{-1}\mu^{i+\tau_M}\mathbb{E}\{\|x_i\|^2\} + \tau_M\sum_{i=0}^{N-1-\tau_M}\mu^{i+\tau_M}\mathbb{E}\{\|x_i\|^2\} \\ &\quad + \tau_M\sum_{i=N-1-\tau_M}^{N-1}\mu^{i+\tau_M}\mathbb{E}\{\|x_i\|^2\} \\ &\leq \tau_M\mu^{\tau_M}\max_{-\tau_M \leq i \leq 0}\mathbb{E}\{\|x_i\|^2\} + \tau_M\mu^{\tau_M}\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|x_i\|^2\}.\end{aligned} \tag{50}$$

Hence, Eq. (49) can be written as

$$\mu^N\mathbb{E}\{V_N\} \le \mathbb{E}\{V_0\} + [\psi_3(\mu)+\psi_7(\mu)+\psi_8(\mu)]\tau_M\mu^{\tau_M} \max_{-\tau_M\le i\le 0}\mathbb{E}\{\|x_i\|^2\} + [\psi_1(\mu)+\tau_M\mu^{\tau_M}\psi_3(\mu)]\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|x_i\|^2\} + \tau_M\mu^{\tau_M}\psi_7(\mu)\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|x_{i-\tau_1^i}\|^2\} + \tau_M\mu^{\tau_M}\psi_8(\mu)\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|x_{i-\tau_2^i}\|^2\} + [\psi_4(\mu)+\psi_5(\mu)+\psi_6(\mu)]d_M\mu^{d_M} \max_{-d_M\le i\le 0}\mathbb{E}\{\|y_i\|^2\} + [\psi_2(\mu)+d_M\mu^{d_M}\psi_4(\mu)]\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|y_i\|^2\} + d_M\mu^{d_M}\psi_5(\mu)\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|y_{i-d_1^i}\|^2\} + d_M\mu^{d_M}\psi_6(\mu)\sum_{i=0}^{N-1}\mu^i\mathbb{E}\{\|y_{i-d_2^i}\|^2\}. \quad (51)$$

Let $\sigma_1 = \max\{\rho_3, \rho_7, \rho_8\}$, $\zeta_1(\mu) = (\mu-1)\sigma_1$, $\sigma_2 = \max\{\rho_4, \rho_5, \rho_6\}$ and $\zeta_2(\mu) = (\mu-1)\sigma_2$. It follows from (51) that

$$\mu^N\mathbb{E}\{V_N\} \le \mathbb{E}\{V_0\} + 3\zeta_1(\mu)\tau_M\mu^{\tau_M}\max_{-\tau_M\le i\le 0}\mathbb{E}\{\|x_i\|^2\} + [\psi_1(\mu)+\tau_M\mu^{\tau_M}\zeta_1(\mu)] \times\sum_{i=0}^{N-1}\mu^i\left(\mathbb{E}\{\|x_i\|^2\}+\mathbb{E}\{\|x_{\tau,1}\|^2\}+\mathbb{E}\{\|x_{\tau,2}\|^2\}\right) + 3\zeta_2(\mu)d_M\mu^{d_M}\max_{-d_M\le i\le 0}\mathbb{E}\{\|y_i\|^2\} + [\psi_2(\mu)+d_M\mu^{d_M}\zeta_2(\mu)] \times\sum_{i=0}^{N-1}\mu^i\left(\mathbb{E}\{\|y_i\|^2\}+\mathbb{E}\{\|y_{d,1}\|^2\}+\mathbb{E}\{\|y_{d,2}\|^2\}\right). \quad (52)$$

Define $\Phi(\mu) = \max\{\psi_1(\mu)+\tau_M\mu^{\tau_M}\zeta_1(\mu), \psi_2(\mu)+d_M\mu^{d_M}\zeta_2(\mu)\}$. Note that it can be verified that there exists a scalar $\theta > 1$ such that $\Phi(\theta) = 0$. Therefore, for such a scalar θ, we have

$$\theta^N\mathbb{E}\{V_N\} \le \mathbb{E}\{V_0\} + 3\zeta_1(\theta)\tau_M\theta^{\tau_M}\max_{-\tau_M\le i\le 0}\mathbb{E}\{\|x_i\|^2\} + 3\zeta_2(\theta)d_M\theta^{d_M}\max_{-d_M\le i\le 0}\mathbb{E}\{\|y_i\|^2\}. \quad (53)$$

Meanwhile, one can derive from (47) that

$$\mathbb{E}\{V_0\} \le (\rho_1+3\sigma_1\tau_M)\max_{-\tau_M\le i\le 0}\mathbb{E}\{\|x_i\|^2\} + (\rho_2+3\sigma_2 d_M)\max_{-d_M\le i\le 0}\mathbb{E}\{\|y_i\|^2\}. \quad (54)$$

Substituting (54) into (53) yields

$$\theta^N\mathbb{E}\{V_N\} \le (\rho_1+3\sigma_1\tau_M+3\zeta_1(\theta)\tau_M\theta^{\tau_M})\max_{-\tau_M\le i\le 0}\mathbb{E}\{\|x_i\|^2\} + (\rho_2+3\sigma_2 d_M+3\zeta_2(\theta)d_M\theta^{d_M})\max_{-d_M\le i\le 0}\mathbb{E}\{\|y_i\|^2\}. \quad (55)$$

On the other hand, from (16), it is easy to obtain

$$\mathbb{E}\{V_N\} \ge \lambda_{\min}(P_1)\mathbb{E}\{\|x_N\|^2\} + \lambda_{\min}(P_2)\mathbb{E}\{\|y_N\|^2\} \ge \lambda_m\mathbb{E}\{\|x_N\|^2+\|y_N\|^2\}. \quad (56)$$

where $\lambda_m = \min\{\lambda_{\min}(P_1), \lambda_{\min}(P_2)\}$.

Combining (55) and (56), one can get

$$\theta^N\lambda_m\mathbb{E}\{\|x_N\|^2+\|y_N\|^2\} \le [\rho_1+3\sigma_1\tau_M+3\tau_M\theta^{\tau_M}\zeta_1(\theta)]\max_{-\tau_M\le i\le 0}\mathbb{E}\{\|x_i\|^2\} + \left[\rho_2+3\sigma_2 d_M+3d_M\theta^{d_M}\zeta_2(\theta)\right]\max_{-d_M\le i\le 0}\mathbb{E}\{\|y_i\|^2\}, \quad (57)$$

yielding

$$\mathbb{E}\{\|x_N\|^2 + \|y_N\|^2\} \le \rho_*\left(\frac{1}{\theta}\right)^N\mathbb{E}\left\{\max_{-\tau_M\le i\le 0}\|x_i\|^2 + \max_{-d_M\le i\le 0}\|y_i\|^2\right\}, \quad (58)$$

where $\rho_* = \max\{\rho_1+3\sigma_1\tau_M+3\zeta_1(\theta)\tau_M\theta^{\tau_M}, \rho_2+3\sigma_2 d_M +3\zeta_2(\theta)d_M\theta^{d_M}\}/\lambda_m$.

Since N is an any positive integer, it can be concluded from Definition 1 that the origin of system (9) is globally exponentially stable in the mean square. This completes the proof of the theorem. □

Appendix B: Proof of Theorem 2

Consider the same Lyapunov–Krasovskii functional as that in the proof of Theorem 1. Then replace A, B, C and D in Theorem 1 by $A + HF(k)E_1$, $B + HF(k)E_2$, $C + HF(k)E_3$ and $D + HF(k)E_4$, respectively.

Since $F^{\mathrm{T}}(k)F(k) \le I$, it follows that

$$\begin{aligned}
&k_1x_k^{\mathrm{T}}E_1^{\mathrm{T}}E_1x_k - k_1(F(k)E_1x_k)^{\mathrm{T}}(F(k)E_1x_k) \ge 0,\\
&k_2x_{\tau,1}^{\mathrm{T}}E_4^{\mathrm{T}}E_4x_{\tau,1} - k_2(F(k)E_4x_{\tau,1})^{\mathrm{T}}(F(k)E_4x_{\tau,1}) \ge 0,\\
&k_3x_{\tau,2}^{\mathrm{T}}E_4^{\mathrm{T}}E_4x_{\tau,2} - k_3(F(k)E_4x_{\tau,2})^{\mathrm{T}}(F(k)E_4x_{\tau,2}) \ge 0,\\
&k_4y_k^{\mathrm{T}}E_3^{\mathrm{T}}E_3y_k - k_4(F(k)E_3y_k)^{\mathrm{T}}(F(k)E_3y_k) \ge 0,\\
&k_5g(y_{d,1})^{\mathrm{T}}E_2^{\mathrm{T}}E_2g(y_{d,1}) - k_5(F(k)E_2g(y_{d,1}))^{\mathrm{T}}\\
&\times(F(k)E_2g(y_{d,1})) \ge 0,\\
&k_6g(y_{d,2})^{\mathrm{T}}E_2^{\mathrm{T}}E_2g(y_{d,2}) - k_6(F(k)E_2g(y_{d,2}))^{\mathrm{T}}\\
&\times(F(k)E_2g(y_{d,2})) \ge 0.
\end{aligned} \quad (59)$$

Calculating $\mathbb{E}\{\Delta V_k\}$ together with the above inequalities, we obtain

$$\mathbb{E}\{\Delta V_k\} \le \mathbb{E}\Big\{\tilde{\xi}_1^{\mathrm{T}}(k)\big[\Omega_1^* + \tau_0 M^* R_1^{-1} M^{*\mathrm{T}} + \tau_M N^* R_2^{-1} N^{*\mathrm{T}}\big]\tilde{\xi}_1(k) + \tilde{\xi}_2^{\mathrm{T}}(k)\big[\Omega_2^* + d_0 S^* R_3^{-1} S^{*\mathrm{T}} + d_M Z^* R_4^{-1} Z^{*\mathrm{T}}\big]\tilde{\xi}_2(k)\Big\}.$$

where

$$\tilde{\xi}_1^{\mathrm{T}}(k) = \Big[x_k^{\mathrm{T}} \quad \eta_k^{\mathrm{T}} \quad x_{\tau,1}^{\mathrm{T}} \quad x_{\tau,2}^{\mathrm{T}} \quad (F(k)E_1 x_k)^{\mathrm{T}} \quad (F(k)E_4 x_{\tau,1})^{\mathrm{T}} \quad (F(k)E_4 x_{\tau,2})^{\mathrm{T}}\Big],$$

$$\tilde{\xi}_2^{\mathrm{T}}(k) = \Big[y_k^{\mathrm{T}} \quad \delta_k^{\mathrm{T}} \quad y_{d,1}^{\mathrm{T}} \quad y_{d,2}^{\mathrm{T}} \quad g^{\mathrm{T}}(y_{d,1}) \quad g^{\mathrm{T}}(y_{d,2}) \quad g^{\mathrm{T}}(y_k) \quad (F(k)E_3 y_k)^{\mathrm{T}} \quad (F(k)E_2 g(y_{d,1}))^{\mathrm{T}} \quad (F(k)E_2 g(y_{d,2}))^{\mathrm{T}}\Big],$$

The remaining proof for global robust exponential stability is similar to those in the proof of Theorem 1. For the sake of simplicity, we omit it here. □

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