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Untangling complex networks: risk minimization in financial markets through accessible spin glass ground states

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Abstract

Recurrent international financial crises inflict significant damage to societies and stress the need for mechanisms or strategies to control risk and tamper market uncertainties. Unfortunately, the complex network of market interactions often confounds rational approaches to optimize financial risks. Here we show that investors can overcome this complexity and globally minimize risk in portfolio models for any given expected return, provided the relative margin requirement remains below a critical, empirically measurable value. In practice, for markets with centrally regulated margin requirements, a rational stabilization strategy would be keeping margins small enough. This result follows from ground states of the random field spin glass Ising model that can be calculated exactly through convex optimization when relative spin coupling is limited by the norm of the network's Laplacian matrix. In that regime, this novel approach is robust to noise in empirical data and may be also broadly relevant to complex networks with frustrated interactions that are studied throughout scientific fields.

Keywords

networks; spin glass; ground states; econophysics; portfolio

1. Introduction

Large and abnormal fluctuations in financial markets can spread into other parts of the global economy with untoward and often incalculable effects—as observed to dramatic consequences in recent times. Therefore a key priority is to minimize risks and contain their propagation in spite of the tendencies of current financial markets to the contrary [1,2]. Important examples of such market places include exchanges where stocks, commodities, futures and other financial products can be bought and sold short by using leverage on margin accounts held by investors. A central financial decision problem in these markets is, for a given expected return r_P, to distribute the available capital among multiple assets, which comprise a portfolio P of size n, so to minimize the overall risk.

In portfolio selection models this goal can be mathematically formulated as finding the global minimum of a risk function [4-7], $R=1/2\sum_{i,k=1}^{n} C_{ik}p_i p_k - \sum_{i=1}^{n} p_i r_i - \gamma\sum_{i=1}^{n} p_i s_i$, where p_i is the positive or negative amount of capital invested in asset i, and s_i = sign (p_i) ∈ {−1, 1} are binary *spin* variables; r_i is the expected return of asset i such that

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$r_P = \sum_{i=1}^{n} r_i p_i$;$C_{ik}$ is the covariance between assets *i* and *k;* and γ is the margin account requirement which sets the fraction of capital that the investor must deposit in a margin account before buying or selling short assets. With the inverse C^{-1} of the covariance matrix *C* the minimum risk distribution $p = (p_1,\ldots, p_n)$ becomes $p_i = \sum_k C_{ik}^{-1} r_k + \gamma \sum_k C_{ik}^{-1} s_k$. It is known that finding the absolute risk minimum is computationally equivalent to the ground state problem of the random field Ising model [4,6]. This is evident after inserting *p* into the risk function while neglecting fixed terms that do not depend on spin variables which gives $R = -1/2 \sum_{i,k=1}^{n} \mathrm{J}_{ik} s_i s_k - \sum_{i=1}^{n} h_i s_i$, and where we introduced an interaction term $J_{ik} = \gamma C_{ik}^{-1}$ and a random local field $h = (h_1,\ldots, h_n)$ with $h_i = \sum_{k=1}^{n} C_{ik}^{-1} r_k$. Covariance among assets can be both positive and negative (see, for example, inset in Fig. 1 A), and globally minimizing risk means finding a ground state of the random field Ising model with random spin glass interactions, which in general belongs to the class of NP-complete decision problems [8,9] and for which efficient algorithms are not known. This computational intractability arises from the non-convexity of the cost function *R;* non-convex problems are much harder to solve computationally than convex optimization problems for which efficient algorithms do exist [10]. In the context of financial markets, the non-convexity of the spin glass model prevents equilibration into an optimum ground state and is viewed as an inherent source of risk [3,4].

2. Accessible ground states in the spin glass Ising model with random field

Here we demonstrate that ground states are efficiently accessible in the random field spin glass Ising model provided the margin requirement γ remains below a critical value, which we define as $\gamma_c = \|L\|^{-1} = \left[\max_i \left(\sum_{k=1}^{n} |L_{ik}|\right)\right]^{-1}$, where $L = D - C^{-1}$ is the network's Laplacian matrix, with $D = \mathrm{diag}\left(\sum_{i=1}^{n} \mathrm{C}_{ik}^{-1}\right)$ and $\|L\|$ is its maximum norm. This upper bound on the margin requirement ensures that there exists a related but convex risk function $R_c = 1/2 \sum_{i,k=1}^{n} J_{ik}(s_i - s_k)^2 + \sum_{i=1}^{n} (h_i - s_i)^2$, which in matrix form reads $R_c = (s - h)^{\mathrm{T}} (s - h) + \gamma s^T L\, s$. We note that in the special and simpler case with non-negative interactions $J_{ik} \geq 0$ similar objective functions have been studied in semi-supervised machine learning [11]. In the more challenging spin glass case, our prerequisite $\gamma < \gamma_c$ makes the Hessian matrix $H_c = 1 + \gamma L$ positive definite such that R_c remains convex with one global minimum even if the interaction is described by a random mix of positive and negative numbers. Let *s* denote the minimum configuration in R_c obtained after convex optimization, then *s* also depicts the ground state s^* of the spin glass Ising model with a random field because assuming the contrary, $R(s) > R(s^*)$, leads to a contradiction. To see this we choose a discrete path of single spin flips that leads from *s* to s^*. At the beginning $R_c(s)$ is a global minimum and nowhere on the path the cost in R_c can be lower. Concurrently, for any spin flip at site *i* the resulting change in R_c equals twice the risk change in *R, viz.* $\Delta R_{c,i} = R_c(-s_i) - R_c(s_i) = 2\Delta R_i \mathrm{with} \Delta R_i = 2 s_i \left(h_i + 1/2 \sum_{k=1}^{n} J_{ik} s_k\right)$, and so nowhere along the path–including its end–the risk in *R* can be lower than at the beginning. Therefore, in contradiction to the assumption, it is $R(s^*) \geq R(s)$ which proves that *s* is a global minimum of the Ising model. Note that this argument is valid only for the Ising model with non-zero external field *h;* without a random field the cost function R_c cannot be made convex by reducing the parameter γ.

This result directly implies that once $\gamma < \gamma_c$ is satisfied any feasible algorithm that converges to a local minimum of the random field Ising model will, due to the underlying convexity of

R_c, reach the absolute minimum in *R;* for example, this may be achieved by solving for all s_i the local stability condition through fixed points of the TAP (*Thouless-Anderson-Palmer*) equation [12], $s_i=\text{sign}\left[h_i+\sum_k J_{ik}s_k\right]$ for $i \in \{1,\ldots, n\}$. It also shows that the critical margin requirement separates two distinct regimes: a disordered regime for $\gamma > \gamma_c$ with an exponential number $\sim 2^n$ of equivalent local minima in the risk function [4,7], each one giving a different selection of the portfolio; and, for $\gamma < \gamma_c$, an ordered regime with only one distinguished minimum. Thus in the latter case the portfolio risk model significantly loses complexity and a computationally efficient, rational access to the optimum is opened.

3. Application to stock price data

To illustrate this general result with a numerical example we compared risk values from actual stock price data evaluated below and above the critical margin requirement. For the calculation of the covariance matrix C we used *end-of-day* (EOD) stock prices of $m = 395$ companies included in the *Standard and Poor's* 500 (S&P500) index over ten years, recorded from February 1999 to February 2009 in $t = 2511$ time points, which after centralization defined an $m\times t$ matrix M with zero mean. From this portfolios of any size $n \le m$ were randomly selected defined through a reduced $n\times t$ matrix M_n where only n rows out of m were retained. Given the selected portfolio's $n\times n$ covariance matrix $C=\left(M_n M_n^{\mathrm{T}}\right)/t$ and a random input distribution of the local field h, the portfolio risk was optimized efficiently by solving the TAP equation through iteration until a fixed point was reached. Relative risk is the lowest possible risk value (which was a negative number in our example) divided by the estimated risk after optimization. The lowest risk was found through exhaustive search in all spin states; this was computationally feasible due to our choice of a small portfolio size ($n = 16$) even though solving the TAP equation by iteration is computationally efficient for any network size n. Consistent with the theoretical prediction Fig. 1A shows that with margin requirements below γ_c the relative risk settled at its global minimum, *i.e.* at the spin glass Ising model ground state. The picture changes for $\gamma > \gamma_c$, where strong fluctuations significantly elevate the risk above the ground state; for instance, at $\gamma \gg \gamma_c$, the average relative risk from the TAP solutions leveled out at ~25% above the optimum.

The price data further allowed us to follow the critical margin requirement as a function of portfolio size n. Figure 1B shows that (a) γ_c is a decreasing function of n, indicating that in larger portfolios efficient risk minimization imposes stricter limitations on margins, and (b) γ_c vanishes for $q \equiv n / t \to 1$, which means that for a given number of observed prices t efficient risk minimization can only occur up to a maximum portfolio size $n = t$. This behavior can be understood by assuming that C is a random Wishart matrix where the bulk of eigenvalues follows a Marcenko-Pastur distribution [13] with a minimum eigenvalue $\lambda_{\min}=\sigma^2(1-\sqrt{q})^2$ and a maximum $\lambda_{\max}=\sigma^2(1+\sqrt{q})^2$; here, without loss of generality, $\sigma^2 \ge 1$ denotes the variance of the elements in M. Therefore—by duality—the spectral radius $\rho(C^{-1})$, defined as the largest eigenvalue of C^{-1}, equals $\lambda_{\min}^{-1}$. Since $(1-\sqrt{q})^{-2} \le \rho(C^{-1}) \le \|C^{-1}\| \le \|L\|+\|D\|$ it follows that in the limit $q \to 1$ the norm $\|L\|$ diverges and hence γ_c must vanish.

Random matrix theory can be further applied to analyze the influence of noise on the critical margin requirement. For that we normalized C by the variance to obtain the price correlation matrix $\tilde{C}$ with a small parameter $q = 395 / 2511 \approx 0.16$; we then filtered $\tilde{C}$ by setting all its eigenvalues below $\lambda_{\max}$ to zero because these are expected to contribute only uninformative noise [13,14]. Even though this procedure discarded all but the highest 23 eigenvalues (Fig. 1C) the resulting distribution of price correlations remained practically unaffected (compare inset in Fig. 1C with inset in Fig. 1A), which indicates that spin glass interactions from these data are genuine and not mere noise artifacts [14,15]. However, filtering significantly lifted the critical margin requirement estimates (Fig. 1D) which suggests that this number is systematically underestimated when computed from unfiltered correlations. In this filtered case, the critical margin requirement is expected to scale inversely proportional to *2n*, which directly follows from the definition of γ_c and from the fact that the underlying pairwise correlations form a totally connected graph. Indeed, Fig. 1D shows that this simple scaling, $\gamma_c \sim (2n)^{-1}$, is an accurate approximation to the critical margin requirement graph obtained from actual price data. Thus, in this example, noise filtering becomes a stabilizer that can extend the parameter range for efficient risk optimization.

4. Conclusions

Risk minimization in portfolio selection models with short selling is equivalent to the ground state problem of the random field Ising model with spin glass interactions. Because calculating its ground state is computationally hard in general, globally minimizing the risk has been regarded as unfeasible with a computationally efficient and rational approach [3,4]. Our result shows that, under realistic conditions, finding the ground state and thus efficient risk minimization is rationally possible. As a direct consequence in financial markets, this may provide an instrument for curbing volatility if financial products are traded below the critical margin requirement, and if investors and traders rationally optimize their portfolios. The second condition is both desirable and realistic in today's highly computerized markets, although it may have been less realistic in the past when computers were not widespread and therefore complex financial decisions were to a lesser degree rational. But the first condition seems to be in conflict with interests of traders and lenders who, in individual contracts, seek to reduce default risk by increasing margins. From a collective market perspective, however, higher margin requirements may have a destabilizing effect through higher transaction costs, which can drive traders from the market place; this may lead to a lower overall liquidity thus making the market more susceptible to volatility [16,17]. Hence, in financial markets where minimum margin requirements are regulated a reduction of risk by lowering margins is conceivable. Historically, the possibility of such a regulatory approach is indirectly supported by the fact that both the 1987 and the 1929 financial market crashes were accompanied by an increase in margin requirements which exacerbated liquidity problems and which might have contributed to rapid downfall [18,19]. Of practical relevance may be the observation that for portfolio sizes above $n \approx 10$ our estimates on the critical margin requirement from the recent American stock market fall below one (Fig. 1B), thus potentially imposing realistic upper limits on margins requirements.

The efficient access to an optimum is not restricted to portfolio risk models; in general, an efficient computation of a ground state is possible in any spin glass Ising model with a random field if the relative coupling strength between spins falls below the critical value. Further applications may follow in frustrated systems that routinely occur in artificial [20,21] and in biological [22,23] networks and where the goal is to find a ground state.

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References

1. International Monetary Fund. Global financial stability report. Washington DC: 2009.

2. Caporale GM, Pittis N, Spagnolo N. Volatility transmission and financial crises. J Econ Finance. 2006; 30:376–391.
3. Bouchaud JP. The (unfortunate) complexity of the economy. Physics World. 2009; 22:28–32.
4. Galluccio S, Bouchaud JP, Potters M. Rational decisions, random matrices and spin glasses. Physica A. 1998; 259:449.
5. Gabor A, Kondor I. Portfolios with nonlinear constraints and spin glasses. Physica A. 1999; 274:222–228.
6. Rosenow B, Gopikrishnan P, Plerou V, Stanley HE. Random magnets and correlations of stock price fluctuations. Physica A. 2002; 314:762–767.
7. Bongini L, Degli Esposti M, Giardina C, Schianchi A. Portfolio optimization with short-selling and spin-glass. Eur Phys J B. 2002; 27:263–272.
8. Bachas CP. Computer-intractability of the frustration model of a spin glass. J Phys A. 1984; 17:L709–L712.
9. Istrail, S. Statistical mechanics, three-dimensionality and NP-completeness. Proceedings of the 31st ACM Annual Symposium on the Theory of Computing. 87-96; 2000.
10. Rieger, H. Lecture Notes in Physics. Vol. 501. Springer Verlag; 1998. Frustrated systems: ground state properties via combinatorial optimization; p. 122-158.
11. Zhou, D.; Bousquet, O.; Weston, J.; Schölkopf, B. Advances in Neural Information Processing Systems. Vol. 16. MIT Press; 2004. Learning with local and global consistency; p. 321-328.
12. Thouless D, Anderson PW, Palmer RG. Solution of solvable model of a spin glass. Phil Mag. 1977; 35:593–601.
13. Laloux L, Cizeau P, Bouchaud JP, Potters M. Noise dressing of financial correlation matrices. Phys Rev Lett. 1999; 83:1467–1470.
14. Rosenow B, Plerou V, Gopikrishnan P, Stanley HE. Portfolio optimization and the random magnet problem. Europhys Lett. 2002; 59:500–506.
15. Pafka S, Kondor I. Noisy covariance matrices and portfolio optimization II. Physica A. 2002; 319:487–494.
16. France VG, Kodres L, Moser JT. A Review of Regulatory mechanisms to control the volatility of prices. Econ Perspect. 1994; 18:15–28.
17. Anshuman VR. Regulatory measures to curb stock price volatility. Econ Polit Week. 2003; 38:677–679.
18. Rappoport P, White EN. Was the crash of 1929 expected? Amer Econ Rev. 1994; 84:271–281.
19. Warshawsky MJ, Dietrich E. The adequacy and consistency of margin requirements in the markets for stocks and derivative products. Staff Studies from the Board of Governors of the Federal Reserve System (U.S.). 1989 158.
20. Hopfield JJ. Neural networks and physical systems with emergent collective computational abilities. Proc Natl Acad Sci USA. 1982; 79:2554–2558. [PubMed: 6953413]
21. Sourlas N. Spin-glass models as error-correcting codes. Nature. 1989; 339:695–697.
22. Goldstein RA, Luthey-Schulten ZA, Wolynes PG. Optimal protein-folding codes from spin-glass theory. Proc Natl Acad Sci USA. 1992; 89:4918–4922. [PubMed: 1594594]
23. Scheidman E, Berry MJ, Segev R, Bialek W. Weak pairwise correlations imply strongly correlated network states in a neural population. Nature. 2006; 440:1007–1012. [PubMed: 16625187]

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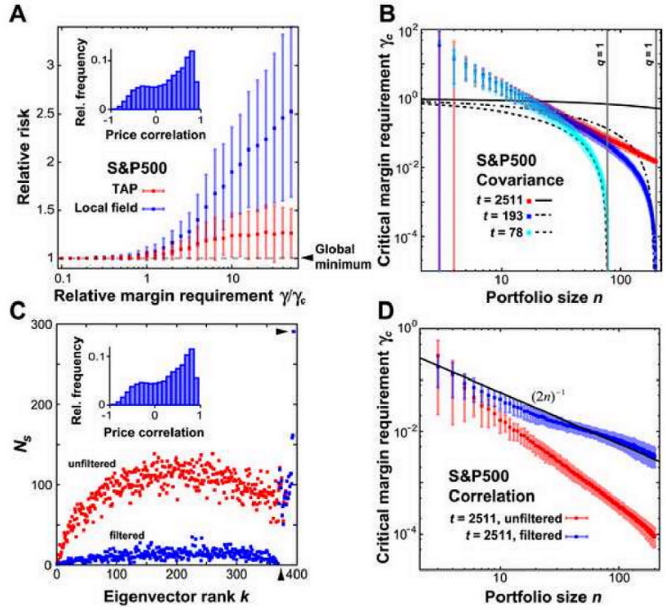

Figure 1.

(**A**) Portfolio risk can be globally and rationally minimized if the relative margin requirement satisfies $\gamma / y_c < 1$. In contrast, for $\gamma / \gamma_c > 1$ the estimated risk undergoes large fluctuations above the optimum. Red data points ("TAP") give the risk from solutions of the TAP equation for $n = 16$ with randomly selected assets from the S&P500 price data, and with a random field $h = (h_1,\ldots, h_n)$ with $|h_i| \leq 1$. Blue data points ("Local field") depict the risk obtained by taking the sign of local field *h*. Error bars represent standard deviations after 128 random trials. Inset shows the distribution of all price correlations between all pairs in the $m = 395$ assets taken from the S&P500 index. (**B**) Estimated critical margin requirement as a function of portfolio size $n \leq m$ and for three different choices of price samples, $t = \{2511, 193, 78\}$, where stock prices were selected every {1, 13, 32} days, respectively. Error bars represent standard deviations from 128 random selections in the S&P500 price data. Black solid and dashed graphs represent the function $\left(1 - \sqrt{n/t}\right)^2$. (**C**) The inverse partition ratio $N_s = \sum_{l=1}^{m} (u_l^k)^4$ for each normalized eigenvector u^k of the $m \times m$ correlation matrix $\tilde{C}$ ranked by its increasing eigenvalues [13]. Red dots represent the unfiltered correlation matrix which, up to a rank of $k = 372$, follow a semicircle distribution; blue dots represent the filtered correlation matrix after setting all eigenvalues with lower rank to zero, i.e. those in size smaller than λ_{max}. Inset shows the resulting histogram of pairwise price correlations after filtering. (**D**) Estimated critical margin requirement γ_c from the S&P500 correlation matrix $\tilde{C}$ before (red) and after (blue) eigenvalue filtering. Black solid line represents the graph $(2n)^{-1}$.