$1/\rho$, and under the conditions of Theorem 3 the first member of (4) is also $1/\rho$.

Theorem 2 and later results including the corollaries have application to the dual problem, approximation by functions of type $R_{\mu n}(z)$ considered in reference 3.

The methods of the present note for the study of convergence and degree of convergence on subsets of E_ρ apply when the number of free poles of the approximating rational functions is finite, and by dualization apply when the number of their free zeros is finite; they do not apply when the number of both free poles and zeros is infinite.

* This research was sponsored (in part) by the U.S. Air Force Office of Scientific Research.

[1] Walsh, J. L., *Math. Ann.*, **155**, 252–264 (1964).

[2] Walsh, J. L., "The convergence of sequences of rational functions of best approximation, II," in preparation.

[3] Walsh, J. L., these PROCEEDINGS, **50**, 791–794 (1963).

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*RANDOM POLYGONS DETERMINED BY RANDOM LINES IN A PLANE**

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Introduction.—Consider a system of straight lines in different directions, distributed at random homogeneously throughout a plane (see Fig. 1). This paper presents the main results of a study concerned primarily with the statistical properties of the aggregates of polygons into which such systems divide the plane. This problem was first considered by Goudsmit,[2] who was able to derive a number of properties. The present study utilizes more powerful and more general methods, and consequently yields many additional results. In order to give a brief and simplified description of the theory comprehensible to nonspecialists interested in applications, a number of mathematical points must needs be glossed over or even ignored; the following treatment should therefore only be regarded as heuristic. Parentheses will often signify heuristic ideas. Only a knowledge of elementary probability theory is presupposed. The precise specifications of the system are followed by the sequence of main results. The second and concluding part of this paper, which will appear in the November issue of these PROCEEDINGS, contains a short account of methods and of an application.

The Line System $\mathfrak{L}$.—By way of notation, the equation of any line in the (x,y) plane may be written as

$$p = x \cos \theta + y \sin \theta \qquad (-\infty < p < \infty, 0 \leqslant \theta < \pi),$$

where p is the signed (i.e., positive or negative) length of the perpendicular, or its *distance*, from the origin O and θ is its *orientation*—the angle this perpendicular makes with Ox. There is thus a one-to-one correspondence between lines in the (x,y) plane and points in the strip $0 \leqslant \theta < \pi$ of the (p,θ) plane.

The distances $\ldots \leqslant p_{-2} \leqslant p_{-1} \leqslant p_0 \leqslant p_1 \leqslant p_2 \leqslant \ldots$ of the lines from O constitute the coordinates of the events of a Poisson (or purely random) process on a one-dimen-

sional axis, of constant density τ. (Thus, the number of events in any interval of length L has a Poisson distribution with mean τL, the mean interevent interval length being τ^{-1}. For definiteness p_0 may be taken, for example, as the p_i with least modulus.) The corresponding orientations θ_i are mutually independent, having a common uniform probability distribution in the interval $0 \leqslant \theta < \pi$. The sequence $\{(p_i, \theta_i)\}$ ($i = 0, \pm 1, \pm 2, \ldots$) represents a system of random lines, denoted by $\mathcal{L}$, in the plane. Equivalently, the (p_i,θ_i) are the points of a two-dimensional Poisson process of density τ/π in the strip $0 \leqslant \theta < \pi$ of the (p,θ) plane.

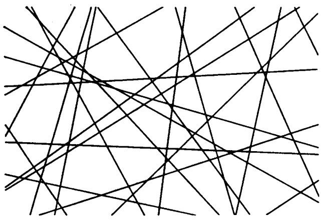

FIG. 1.—Random lines. A realization of $\mathcal{L}$ within a rectangle.

One possible realization of $\mathcal{L}$ is that in which there are no lines at all! The probability of this is, of course, zero. It is on account of such "irregular" realizations, of total probability zero, that many of the following assertions must be qualified by "with probability 1" (often omitted, however, for brevity). Note that the points at which the perpendiculars from O meet the lines of $\mathcal{L}$ are not uniformly distributed in the plane, but constitute a Poisson point process of density $\tau/\pi R$, where R measures radial distance from O. With probability 1, no pair of lines of $\mathcal{L}$ has the same orientation, ensuring that every pair intersects in a unique point. Furthermore, with probability 1, no triple of lines of $\mathcal{L}$ intersects in a common point. The aggregate of points, or *vertices*, of intersection of pairs of lines of $\mathcal{L}$ is denoted by $\mathcal{V}$. One of the principal reasons for employing this particular random construction is the basic property

THEOREM 1. *$\mathcal{L}$ is homogeneous.*

That is, from a probability point of view, the lines of $\mathcal{L}$ have a uniform density, of which τ is a measure, throughout the plane. The actual position of the origin in the plane is of no significance, and it is immaterial from which origin $\mathcal{L}$ is randomly generated. Theorem 1 is the foundation of a fruitful body of theory: without homogeneity, ergodic theory would be inapplicable and consequently most of what follows would be meaningless. A considerable additional advantage of the construction is that many of the "independence properties" of the linear Poisson process carry over to $\mathcal{L}$. On account of the common uniform orientation distribution, $\mathcal{L}$ is, again in a probability sense, *isotropic*. On the average, there is a length τ of lines of $\mathcal{L}$, and τ^2/π vertices of $\mathcal{V}$, per unit area; the corresponding mean values of these quantities contained in an arbitrary set of area A are given on multiplying by A. An alternative random construction of $\mathcal{L}$ is furnished by

THEOREM 2. *The points of intersection of an arbitrary line l with $\mathcal{L}$ constitute a Poisson process of density $2\tau/\pi$, the associated angles of intersection being mutually independent with common probability density $^1/_2 \sin \theta$ ($0 \leqslant \theta < \pi$).*

Note that the angle distribution is not uniform as one might at first suppose, but it is symmetrical about $\theta = \pi/2$. Note also that this theorem holds for the intersection of a *line of* $\mathcal{L}$ with the remaining lines of $\mathcal{L}$. There is some overlap between Theorem 2 and

THEOREM 3. *The number M of lines of $\mathcal{L}$ intersecting an arbitrary convex figure in the plane, of perimeter S, has a Poisson distribution with mean value $\tau S/\pi$. Furthermore, given that $M = m$, the m lines are independently and identically distributed.*

Such an individual intersecting line is called a *random secant* of the figure, for a precise specification of which, and an account of the beautiful theory which ensues, the reader is referred to Deltheil.[1] An example of a convex figure is a line segment of length L, which has (*sic*) perimeter $2L$. It is in Theorem 3, an excellent characterization of $\mathcal{L}$ as well as a useful tool, that the deep underlying connection with integral geometry is perhaps most evident. For instance,[4] the "invariant measure" of the set of straight lines intersecting a convex figure is (up to a constant factor) S.

The Polygon Aggregate $\mathcal{P}$.—The present investigations have been primarily concerned with the probability distributions of the *aggregate* $\mathcal{P}$ *of random convex polygons* into which the plane is partitioned by the lines of $\mathcal{L}$, in the sense in which *each polygon of* $\mathcal{P}$ *is given "equal weight."* More specifically, with the class of distributions of certain "descriptions" of a convex polygon, the basic ones are N, the number of sides (or vertices); S, the perimeter, $= \pi$ times its mean orthogonal projection onto a random uniformly oriented line; A, the area; and D, the in-circle diameter (the in-circle, the largest circle contained in a convex polygon, is in general tangential to three of its sides). It is a matter of ergodic theory to demonstrate[3] the *existence* of these distributions, which are not to be confused with the corresponding class of well-defined distributions relating to the unique polygon P_0 containing the origin O. For, on account of homogeneity, O is like a "random point in the plane," and so the larger the area of a polygon of $\mathcal{P}$, the more likely it is to contain O. (Suffice it to say,[3] corresponding pairs of joint probability density functions, in which one of the descriptions is A, differ essentially by a factor A, *vide* the sketch derivation of the second-order moments below.) Denoting "mean value" and "probability" by E and P, respectively, the principal results relating to $\mathcal{P}$ are:

THEOREM 4. *The distribution of* D *is negative exponential, with* $E[D] = \tau^{-1}$.

Theorem 4 generalizes to two dimensions the negative exponential distribution of the interval lengths in a linear Poisson process. The behavior of this distribution for small values yields information about how close triples of lines of $\mathcal{L}$ come to intersecting in a common point; incidentally, this consideration verifies that, with probability 1, there are no such points. It is also the distribution of the diameter of the largest circle, center O, contained in P_0: but this circle is, with probability 1, smaller than the in-circle of P_0! This apparent paradox is dispelled by a previous remark.

THEOREM 5. *The distribution of* $2\tau S/\pi$ *for the class of* k*-sided polygons of* $\mathcal{P}$ *is* χ^2 *on* $2(k-2)$ *degrees of freedom* ($k = 3,4,5,\ldots$).

Thus, in particular, the perimeter distribution for the triangular polygons is negative exponential, with mean value π/τ. A corollary is that the mean length of a side for the class of k-sided polygons is $(k-2)\pi/k\tau$, which $\uparrow \pi/\tau$ as $k \to \infty$. Information about the distributions of N, S, and A for small values is given by

THEOREM 6. $P[N = 3] = P$[*a "random" polygon of* $\mathcal{P}$ *is a triangle*] $= 2 - \pi^2/6 = 0.3551$. *For* $S \ll \tau^{-1}$ *and* $A \ll \tau^{-2}$, *the probability densities of* S *and* A *are* $(12 - \pi^2)\tau/6\pi + 0(\tau^2 S)$ *and* $c\tau A^{-1/2} + 0(\tau^2)$, *respectively, where*
$$c = \frac{1}{3\pi}\int_0^\pi \int_0^{\pi-\phi} [2\sin\phi \sin\psi \sin(\phi+\psi)]^{1/2}\, d\psi\, d\phi.$$

Now follow the principal mean values:

$$E[N] = 4 \qquad E[S] = 2\pi/\tau \qquad E[A] = \pi/\tau^2 \tag{1}$$

$$\left.\begin{array}{lll} E[N^2] = (\pi^2 + 24)/2 & & \\ E[SN] = \pi(\pi^2 + 8)/2\tau & E[S^2] = \pi^2(\pi^2 + 4)/2\tau^2 & \\ E[AN] = \pi^3/2\tau^2 & E[AS] = \pi^4/2\tau^3 & E[A^2] = \pi^4/2\tau^4 \end{array}\right\} \quad (2)$$

The values of $E[N]$, $E[A]$, and $E[A^2]$ are essentially those given by Goudsmit,[2] who normalized by taking $E[A] = 1$. (1) and (2) together yield the interesting variance-covariance matrix

$$\begin{array}{cccc} & N & S & A \\ \begin{array}{c} N \\ S \\ A \end{array} & \multicolumn{3}{l}{\left[\begin{array}{lll} (\pi^2 - 8)/2 & \pi(\pi^2 - 8)/2\tau & \pi(\pi^2 - 8)/2\tau^2 \\ \pi(\pi^2 - 8)/2\tau & \pi^2(\pi^2 - 4)/2\tau^2 & \pi^2(\pi^2 - 4)/2\tau^3 \\ \pi(\pi^2 - 8)/2\tau^2 & \pi^2(\pi^2 - 4)/2\tau^3 & \pi^2(\pi^2 - 2)/2\tau^4 \end{array}\right]} \end{array} \quad (3)$$

of N, S, and A.

$$E[NA^2] = \pi^4(8\pi^2 - 21)/21\tau^4 \qquad E[SA^2] = 8\pi^7/21\tau^5 \qquad E[A^3] = 4\pi^7/7\tau^6. \quad (4)$$

$$E[SA^{m-1}] = 2\tau E[A^m]/m \qquad (m = 1,2,\ldots). \quad (5)$$

D. G. Kendall obtained the value of $E[A^3]$, while P. I. Richards obtained (5), both in unpublished papers. Combining these two results (with $m = 3$) yields $E[SA^2]$, which the author utilized to find $E[NA^2]$. The reader may find it rewarding to "check" the values of $P[N = 3]$, $E[N]$, and $E[N^2]$ by covering a sheet of paper with 20 or so "random" straight lines, and computing the obvious estimates.

Just as for $\mathcal{P}$, ergodic theory establishes probability distributions for $\mathcal{V}$ and the aggregates $\mathcal{J}_k$ and $\mathcal{G}_k$ defined below. For the sake of completeness, the next theorem gives together all the main orientation distributions.

THEOREM 7. (i) *The orientation α of a "random" line of $\mathcal{L}$ has probability density $1/\pi$ $(0 \leqslant \alpha < \pi)$.* (ii) *The orientations α, β of the two lines of $\mathcal{L}$ through a "random" vertex of $\mathcal{V}$ have joint probability density $|\sin(\beta - \alpha)|/2\pi$ $(0 \leqslant \alpha < \pi, 0 \leqslant \beta < \pi)$.* (iii)(a) *The orientations α, β, γ $(0 \leqslant \alpha < \pi, 0 \leqslant \beta < \pi, 0 \leqslant \gamma < \pi)$ of the three lines of $\mathcal{L}$ tangential to the in-circle of a "random" polygon of $\mathcal{P}$ have joint probability density*

$$2\cos\frac{\alpha^*}{2}\cos\frac{\beta^*}{2}\cos\frac{\gamma^*}{2}\Big/3\pi^2,$$

where α^*, β^*, and γ^* are the interior angles of the triangle so formed. (b) *The corresponding density for the lines forming the sides of a "random" triangle of $\mathcal{P}$ is*

$$4\sin\frac{\alpha^*}{2}\sin\frac{\beta^*}{2}\sin\frac{\gamma^*}{2}\Big/\pi(12 - \pi^2).$$

It is left as an exercise for the reader to express α^*, β^*, and γ^* in terms of α, β, and γ. Observe in (*ii*) that, although individually both α and β have uniform distributions, they are not independent. The same remark applies to both parts of (*iii*). From (*ii*) may be derived the distribution of the intersection angle at a "random" vertex, which as might be expected is the distribution appearing in Theorem 2; it is also a simple matter to derive in the two cases of (*iii*) the joint and individual distributions of α^* and $\beta^*(\gamma^* = \pi - \alpha^* - \beta^*)$. Theorems 4 and 7(*iii*)(*a*) combined are the basis of a random construction of a "random" polygon of $\mathcal{P}$, from which it may be concluded that those polygons with small D, S, or A tend to be triangles, whereas those with large D, S, or A tend to have many sides, and so be "rounded" in appearance.

Consider now the class of line segments joining pairs of vertices of $\mathcal{V}$. Such a segment is a member of the aggregate $\mathcal{S}_k$ if it is part of a line of $\mathcal{L}$ and contains exactly k other vertices of $\mathcal{V}(k = 0,1,\ldots)$; and a member of the aggregate $\mathcal{J}_k$ if it is *not* part of a line of $\mathcal{L}$ and its interior is crossed by exactly k lines of $\mathcal{L}(k = 0,1,\ldots)$. For example, $\mathcal{S}_0$ and $\mathcal{J}_0$ are, respectively, the aggregates of sides and diagonals of the polygons of $\mathcal{P}$. Let L denote segment length.

THEOREM 8. *The distribution of* $4\tau L/\pi$ *for the segments of* (i) $\mathcal{S}_k$ *is* χ^2 *on* $2(k + 1)$ *degrees of freedom;* (ii) $\mathcal{J}_k$ *is* χ^2 *on* $2(k + 2)$ *degrees of freedom.*

Part (*i*) is fairly immediate (cf. Theorem 2), unlike (*ii*).

Thick Lines.—Suppose each line is given width w (with a width $w/2$ on either side).

THEOREM 9. *The new polygon aggregate comprising the interstices between these thick lines has precisely the same probability distributions as* $\mathcal{P}$, *so that, for example, Theorems* 4, 5, 6, 7(iii), *and* (1)–(5) *carry over unchanged. Moreover, this property continues to hold in the progressively more general situations:* (i) *the widths of the lines are random, being mutually independent with a common probability distribution; and* (ii) *as in* (i), *but the distribution of the line's width depends on the line's orientation, e.g., the case where the "North-South" and "East-West" lines tend to be thick and thin, respectively.*

This theorem is plausible, since by giving the lines width the larger polygons are reduced in size, the smaller ones disappear, the general effect in a given large area being a loss only of the number of polygons, not of size generally. If $\bar{w}(\theta)$ is the mean thickness of lines with orientation θ, then $\bar{w} = \int_0^\pi \bar{w}(\theta)\, d\theta/\pi$ is the over-all mean line thickness. Theorem 3 remains true if $\tau(\bar{w} + S/\pi)$ is substituted for $\tau S/\pi$. Remembering that homogeneity is preserved, the "fraction of the plane" not covered by any thick line $= P[O \text{ is not covered}]$ which, putting $S = 0$ into the generalization of Theorem 3, $= e^{-\tau\bar{w}}$.

The Anisotropic Case.—If the condition of isotropy is dropped, the common orientation distribution being altered from a uniform to a quite general one, with distribution function $H(\theta)$ $(0 \leqslant \theta < \pi)$, say, then the above theory continues to hold in varying degrees. If $H(\theta)$ has any finite jumps δ_j, corresponding to orientations $\theta^{(j)}$ having positive probability, then a "proportion" δ_j of the lines of $\mathcal{L}$ will have orientation $\theta^{(j)}$, and it is no longer true that every pair of lines intersects in a unique point. In view of this, the in-circle is no longer necessarily unique, although the value of D is. In order to generalize Theorem 7(*iii*)(*a*) it is sufficient in the ambiguous cases to define the in-circle to be a random one (probability $^1/_2$) of the two extreme largest inscribed circles. However, for all choices of $H(\theta)$, still no triple of lines of $\mathcal{L}$ intersects in a common point (with probability 1). Homogeneity is preserved (Theorem 1). Define

$$\xi(\phi) = \int_0^\pi |\sin(\psi - \phi)|\, dH(\psi),$$

$$\zeta = \int_0^\pi \xi(\phi)\, dH(\phi).$$

and $$\eta = \int_0^\pi [\xi(\phi)]^{-2}\, d\phi.$$

The average length of line and number of vertices per unit area are, respectively, τ(as before) and $\zeta\tau^2/2$. ζ is maximal for $H(\theta) = \theta/\pi$, so, for a fixed line density, the maximum vertex density occurs in the isotropic case. If in Theorem 2 the orientation of l is ϕ, then l intersects $\mathfrak{L}$ in a Poisson process of density $\tau\xi(\phi)$, the distribution function of the associated intersection angle being

$$\int_\phi^{\phi+\theta} \sin(\psi - \phi)\, dH(\psi)/\xi(\phi) \qquad (0 \leqslant \theta < \pi),$$

where θ increases in an anticlockwise sense and in the integral $H(\pi + \theta) = 1 + H(\theta)$. Theorems 3 and 5 carry over if S is replaced by $S^* = \pi\int_0^\pi W(\theta) dH(\theta)$, where the "width" $W(\theta)$ is the length of the orthogonal projection of the figure (Theorem 3) or polygon (Theorem 5) onto a line orthogonal to lines with orientation θ. Observe that S^* for a polygon, unlike N, S, and A, depends on more than simply its size and shape. In general, the new probability distributions depend on $H(\theta)$, but *Theorem* 4 *generalizes as it stands.* The (known) important first and second-order moments are:

$$E[N] = 4 \qquad E[S] = 4/\zeta\tau \qquad E[S^*] = 2\pi/\tau \qquad E[A] = 2/\zeta\tau^2 \tag{6}$$

$$\left.\begin{aligned} E[N^2] &= \zeta\eta + 12 \\ E[SN] &= 2(\zeta\eta + 4)/\zeta\tau \\ E[S^*N] &= \pi(\zeta\eta + 4)/\tau \\ E[AN] &= 2\eta/\tau^2, E[AS] = 4\eta/\zeta\tau^3, E[AS^*] = 2\pi\eta/\tau^3, E[A^2] = 4\eta/\zeta\tau^4 \end{aligned}\right\} \tag{7}$$

In the isotropic case $\zeta = 2/\pi$, $\eta = \pi^3/4$, and the values in (1) and (2) may be recovered. The orientation distributions generalizing those of Theorem 7 are simply obtained by "weighting" with respect to $H(\theta)$, e.g., assuming the density $h(\theta) = H'(\theta)$ exists, the density of (*iii*)(*a*) becomes

$$\frac{\cos\frac{\alpha^*}{2}\cos\frac{\beta^*}{2}\cos\frac{\gamma^*}{2}\, h(\alpha)\, h(\beta)\, h(\gamma)}{\int_0^\pi\int_0^\pi\int_0^\pi \cos\frac{\alpha^*}{2}\cos\frac{\beta^*}{2}\cos\frac{\gamma^*}{2}\, h(\alpha)\, h(\beta)\, h(\gamma) d\alpha\, d\beta\, d\gamma}.$$

As regards Theorem 8, the revised form of Theorem 2 essentially contains the generalization of (*i*), but it is difficult to frame a satisfactory generalization of (*ii*). It may be remarked that Goudsmit's simplified problem[2] is a special case of anisotropy, in which the orientation has only two possible values, which are orthogonal and equally likely.

For *anisotropic thick lines*, the over-all mean thickness $\bar{w} = \int_0^\pi \bar{w}(\theta)\, dH(\theta)$. *Theorem* 9 *generalizes as it stands.* Theorem 3 remains true if $\tau S/\pi$ is replaced by $\tau(\bar{w} + S^*/\pi)$, and the "uncovered fraction of the plane" is again $e^{-\tau\bar{w}}$.

* Much of the author's contribution to this topic was made in 1960–1961, while supported by a research studentship of the Department of Scientific and Industrial Research (United Kingdom).

[1] Deltheil, R., *Probabilités Géométriques* (Paris, 1926), especially pp. 68–86.

[2] Goudsmit, S. A., "Random distribution of lines in a plane," *Rev. Mod. Phys.*, **17**, 321–322 (1945); reviewed in Kendall, M. G., and P. A. P. Moran, *Geometrical Probability* (New York: Hafner, 1963), chap. 3.

[3] Miles, R. E., Ph.D. thesis, Cambridge, 1961.

[4] Santaló, L. A., *Introduction to Integral Geometry* (Paris, 1953), especially pp. 10–16.

CONTINUOUS MAPPINGS OF ALMOST AUTOMORPHIC AND ALMOST PERIODIC FUNCTIONS*

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We have previously introduced,[1] and then meaningfully applied,[2] a class of functions which are more general than almost periodic ones. We named them almost automorphic functions because they originally presented themselves, in our work in differential geometry, as scalars and tensors on manifolds with (discrete) groups of automorphisms.

The first examples of functions which are almost automorphic, but, demonstrably, not almost periodic, were then constructed by Veech,[3] and he introduced them not on the continuous additive group $R = \{-\infty < t < \infty\}$, but on its discrete subgroup $Z = \{-\infty < n < \infty\}$ which seems to be a natural habitat for such "counterexamples." Recently, H. Furstenberg communicated to us another such example, again on Z, and it is as follows.

THEOREM 1. *If θ is any nonrational real number, then the double sequence*

$$\varphi(n) = \text{signum } (\cos 2\pi n\theta), \qquad -\infty < n < \infty \tag{1}$$

is almost automorphic, but not almost periodic, on Z.

Now, in the present paper we will give our own proof of this theorem, and we will deduce it from general statements about almost automorphic and almost periodic functions which, although very simple, have some interest as such.

In the general statements it will be appropriate to consider functions from and to general spaces. We start out with a general pointset $X = \{x\}$ and a general group $\Gamma = \{\gamma\}$ acting on it; and we denote by $x' = \gamma x$ the image of the point x under the action of the group element γ, each of which represents a transformation of X into itself.

Definition 1: A function

$$y = f(x) \tag{2}$$

from X to a metric space Y is almost periodic (relative to Γ) if any (infinite) sequence

$$\{\gamma_m'\} \tag{3}$$

in Γ—repetitions allowed—contains an (infinite) subsequence

$$\{\gamma_n\} \tag{4}$$