*COMMUTATORS OF SINGULAR INTEGRAL OPERATORS**

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Let

$$A(f) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x - y)f(y)dy,$$

where x, y are points in n-dimensional Euclidean space R^n and $k(x)$ is a homogeneous function of degree $-n$ with mean value zero on $|x| = 1$, and let $B(f) = b(x)f(x)$. It is well known (see ref. 1) that if k and b are sufficiently smooth and b is bounded, then $(AB - BA)(\partial/\partial x_j)$ and $(\partial/\partial x_j)(AB - BA)$ are bounded operators in L^p, $1 < p < \infty$.

The purpose of the present note is to extend and strengthen the preceding result and establish some related facts of independent interest. These are stated in Theorems 2 and 3 below.

Theorem 1. *Let $k(x)$ have locally integrable first-order derivatives in $|x| > 0$, and suppose that the partials of $k(x) + k(-x)$ belong locally to $L \log^+ L$ in $|x| > 0$. Let $b(x)$ have first-order derivatives in L^r, $1 < r \leq \infty$. Then if $1 < p < \infty$, $1 < q < \infty$, $q^{-1} = p^{-1} + r^{-1}$ and f is continuously differentiable and has compact support, we have*

$$\left\|(AB - BA) \frac{\partial}{\partial x_j} f\right\|_q \leq c\|f\|_p, \qquad (a)$$

where c is independent of f. Furthermore, $(AB - BA)f$ has first-order derivatives in L^q and

$$\left\|\frac{\partial}{\partial x_j}(AB - BA)f\right\|_q \leq c\|f\|_p, \qquad (b)$$

where, again, c is independent of f.

Theorem 2. *Let $h(x)$ be homogeneous of degree $-n - 1$ and locally integrable in $|x| > 0$. Let $b(x)$ have first-order derivatives in L^r, $1 < r \leq \infty$. Then, if $1 < p < \infty$, $1 < q < \infty$, $q^{-1} = p^{-1} + r^{-1}$, $h(x)$ is an even function and*

$$C_\epsilon(f) = \int_{|x-y|>\epsilon} h(x - y)[b(x) - b(y)]f(y)dy.$$

C_ϵ maps L^p continuously into L^q and $\|C(f)\|_q \leq c\|\text{grad } b\|_r\|f\|_p \int |h(x)|d\nu$, where the integral is extended over $|x| = 1$, $d\nu$ denotes the surface area of $|x| = 1$, and c depends on p and r but not on ϵ. Furthermore, as ϵ tends to zero $C_\epsilon(f)$ converges in norm in L^q.

A similar result holds if $h(x)$ is odd provided that it belongs locally to $L \log^+ L$ in $|x| > 0$ and that the functions $x_j h(x)$, $j = 1,2, \ldots, n$, have mean value zero on $|x| = 1$. This, however, will not be proved in the present note.

Theorem 3. *Let $F(t + is)$ be analytic in $s > 0$ and belong to H^p, $0 < p < \infty$. Let $S(F)(t) = [\int \chi(t - u,s)|F'(u + is)|^2 du\, ds]^{1/2}$, where $\chi(t,s)$ is the characteristic* *function of the set $s > 0$, $|t| < s$. Then there exist two positive constants c_1 and c_2 depending on p only, such that $c_1\|F(t)\|_p \leq \|S(F)\|_p \leq c_2\|F(t)\|_p$, where $F(t) = \lim_{s\to 0} F(t + is)$.*

The novelty in the preceding statement is the first inequality for $p \leq 1$. A similar result for the function g of Littlewood and Paley when F has no zeros was proved by T. M. Flett (ref. 3), whose method we borrow partially. Actually, only the case $p \geqslant 1$ will be needed in this note, but its proof is no less laborious than that of the general case.

Proof of Theorem 3: We will assume first that $F(t + is)$ is analytic in $s \geqslant 0$ and that $|F|(t^2 + s^2)^k \to 0$ as $(t^2 + s^2) \to \infty$ for every $k > 0$. Then, of course, F belongs to H^p for every $p > 0$. We introduce now some notation. For a function G defined on the real line we write

$$M_p(G) = \left[\int_{-\infty}^{+\infty} G^p\,dt\right]^{1/p}, \qquad p > 0.$$

If G is also defined in the upper half-plane, we write

$$m(G) = \sup_{u,s} \chi(t - u,s)|G(u,s)|, \qquad S(G) = [\textstyle\int \chi(t - u,s)|\text{grad } G|^2 du\, ds]^{1/2},$$

where $\chi(t,s)$ is the characteristic function of the set $s > 0$, $|t| \leq s$. By integration we obtain $M_2{}^2[S(G)] = 2 \int s|\text{grad } G|^2 dt\, ds$. Now if δ is any positive number, we set $G = |F|^\delta$, then a simple calculation gives

$$\Delta(G^2) = 4|\text{grad } G|^2 \tag{0}$$

and an application of Green's formula yields[4]

$$M_2{}^2(G) = 4 \int s|\text{grad } G|^2 dt\, ds = 2\, M_2{}^2[S(G)]. \tag{1}$$

On account of the definition of G and the analyticity of F, we have the following well-known inequality

$$M_p[m(G)] \leq c\, M_p(G), \qquad 0 < p < \infty. \tag{2}$$

Now let $p \geqslant 1$, then

$$S(G^p)^2 = \int \chi(t - u,s)|pG^{p-1} \text{ grad } G|^2 du\, ds \leq p^2 m(G)^{2p-2} S(G)^2,$$

that is,

$$S(G^p) \leq pm(G)^{p-1}S(G), \qquad 1 < p < \infty. \tag{3}$$

Now let $\alpha,\beta > 0$, $0 < \sigma < 1$, $\alpha\sigma + \beta(1 - \sigma) = 1$. Then

$$S(G)^2 = \int \chi(t - u,s)|\text{grad } G|^2 du\, ds = \alpha^{-2\sigma}\beta^{-2(1-\sigma)} \int (\chi|\text{grad } G^\alpha|^2)^\sigma (\chi|\text{grad } G^\beta|^2)^{1-\sigma} du\, ds,$$

whence from Hölder's inequality we obtain

$$S(G) \leq \left[\frac{1}{\alpha}\, S(G^\alpha)\right]^\sigma \left[\frac{1}{\beta}\, S(G^\beta)\right]^{1-\sigma}. \tag{4}$$

Let us assume now that we have the inequality

$$c\, M_r(G) \geq M_r[S(G)] \tag{5}$$

for some r, $r > 0$. Let $0 < q < r$ and $p = r/q$. Then (3) applied to $G^{1/p}$ gives

$$S(G) \leq pm(G^{1/p})^{p-1}S(G^{1/p}) = pm(G)^{(p-1)/p}S(G^{1/p}),$$

whence, applying Hölder's inequality, we get

$$M_q{}^q[S(G)] \leq p^q\, M_1[m(G)^{q(p-1)/p}S(G^{1/p})^q] \leq p^q\, M_{r/q}[S(G^{1/p})^q]M_{r/(r-q)}[m(G)^{q(p-1)/p}]$$
$$= p^q\, M_r{}^q[S(G^{1/p})]M_q{}^{q(p-1)/p}[m(G)]$$

and from the last expression, (2), and (5) applied to $G^{1/p}$ it follows that

$$M_q{}^q[S(G)] \leq cp^q\, M_r{}^q[G^{1/p}]M_q{}^{q(p-1)/p}(G) = cp^q\, M_q{}^{q/p}(G)M_q{}^{q(p-1)/p}(G)$$

or

$$M_q[S(G)] \leq c_q\, M_q(G). \tag{6}$$

On account of (1), (5) holds with $r = 2$. Hence the preceding inequality holds for $0 < q < 2$.

Now we will show that (6) holds for $0 < q < \infty$. Since (5) implies (6) with $q < r$, it is enough to show that (6) holds for $q \geq 4$. Let $h(t) \geq 0$ be any bounded function with compact support. Then

$$\int_{-\infty}^{+\infty} S(G)^2 h dt = \int_{-\infty}^{+\infty} h(t) \int \chi(t-u,s)|\text{grad } G|^2 du\, ds\, dt$$
$$= \int |\text{grad } G|^2 \int_{-\infty}^{+\infty} h(t)\chi(t-u,s)\, dt\, du\, ds.$$

Now we observe that if $P(t,s)$ denotes the Poisson kernel for the half-plane, then $\chi(t,s) \leq c\, sP(t,s)$ and consequently

$$\int_{-\infty}^{+\infty} h(t)\chi(t-u,s)\, dt \leq c\int_{-\infty}^{+\infty} h(t)sP(t-u,s)dt \leq c\, s\, H(u,s),$$

where $H(t,s)$ is the Poisson integral of $h(t)$. Thus,

$$\int_{-\infty}^{+\infty} S(G)^2 h\, dt \leq c \int |\text{grad } G|^2 s\, H(t,s) dt\, ds.$$

Now, from (0) we have

$$\Delta(G^2H) = H\Delta G^2 + 2(\text{grad } G^2)\cdot(\text{grad } H)$$
$$= 4H|\text{grad } G|^2 + 2G(\text{grad } G)\cdot(\text{grad } H)$$
$$\geqslant 4H|\text{grad } G|^2 - 2G|\text{grad } G|\,|\text{grad } H|$$

and

$$\int_{-\infty}^{+\infty} S(G)^2 h\, dt \leq \frac{c}{4}\int s\Delta(G^2H)dt\, ds + \frac{c}{2}\int sG|\text{grad } G|\,|\text{grad } H|\, dt\, ds$$

and applying Green's formula to the first term on the right[4]

$$\int_{-\infty}^{+\infty} S(G)^2 h\, dt \leq \frac{c}{4}\int_{-\infty}^{+\infty} G^2 h\, dt + \frac{c}{4}\int_{-\infty}^{+\infty} dt \int \chi(t-u,s)\, G|\text{grad } G|\ |\text{grad } H|\, du\, ds$$
$$\leq \frac{c}{4}\int_{-\infty}^{+\infty} G^2 h\, dt + \frac{c}{4}\int_{-\infty}^{+\infty} m(G)\, S(G)\, S(H) dt.$$

Now we set $p = q/(q - 1)$ and apply the three-term Hölder inequality with exponents $2q$, $2q$, p to the preceding integrals and get

$$4\int_{-\infty}^{+\infty} S(G)^2 h\, dt \leq cM_{2q}^2(G)M_p(h) + cM_{2q}[m(G)]M_{2q}[S(G)]M_p[S(H)]. \quad (7)$$

Since H is harmonic and $1 < p < \infty$, we have $M_p[S(H)] \leq c_p M_p(h)$, and since $4 \leq q < \infty$, we also have $M_{2q}[m(G)] \leq c_q M_{2q}(G)$. Substituting in the preceding inequality, setting $M_p(h) = 1$, and taking the supremum of the left-hand side over all such h, we find that $M_q[S(G)^2] = M_{2q}^2[S(G)] \leq c\, M_{2q}(G)[M_{2q}(G) + M_{2q}S(G)]$, and this implies that $M_{2q}[S(G)] \leq c'\, M_{2q}(G)$ provided that $M_{2q}[S(G)] < \infty$. To see that this is the case we observe that since $m(G)$ is bounded, (7) holds with $M_\infty[m(G)]$ replacing $M_{2q}[m(G)]$ and $M_q[S(G)]$ replacing $M_{2q}[S(G)]$ and from this, arguing as above, we obtain

$$M_{2q}^2[S(G)] \leq c\, M_{2q}^2(G) + c\, M_\infty[m(G)]M_q[S(G)].$$

Since the right-hand side is finite for $q = 2$, it follows by induction that the left-hand side is finite for arbitrarily large q and hence for all $q \geq 2$. Thus (6) is established for $0 < q < \infty$.

Now we prove the converse inequality. Let $q > 0$. Then (1) and (4) give

$$M_q^q(G) = M_2^2(G^{q/2}) = 2\, M_2^2 S(G^{q/2}) \leq c\, M_1[S(G^{\alpha q/2})^{2\sigma} S(G^{\beta q/2})^{2(1-\sigma)}],$$

where $\alpha = 2q/(q + 2)$, $\beta = 2/q$, $\sigma = (q + 2)/2(q + 1)$, $1 - \sigma = q/2(q + 1)$. Applying Hölder's inequality to the right-hand side we get

$$M_q^q(G) \leq c\, M_{(q+1)/q}[S(G^{\alpha q/2})^{2\sigma}]M_{q+1}[S(G)^{2(1-\sigma)}].$$

But

$$M_{(q+1)/q}[S(G^{\alpha q/2})^{2\sigma}] = M_{(q+2)/q}^{2\sigma}[S(G^{\alpha q/2})]$$
$$M_{q+1}[S(G)^{2(1-\sigma)}] = M_q^{2(1-\sigma)}[S(G)].$$

Applying (6) to the right-hand side of the first of the preceding identities, and observing that $M_{(q+2)/q}[G^{\alpha q/2}] = M_q^{\alpha q/2}(G)$, substitution in the preceding inequality yields

$$M_q^q(G) \leq c_q\, M_q^{\alpha\sigma q}(G)\, M_q^{2(1-\sigma)}[S(G)].$$

Since $q - \alpha\sigma q = 2(1 - \sigma)$, from this it follows that

$$M_q(G) \leq c_q\, M_q[S(G)]. \quad (8)$$

To obtain (6) and (8) for F we set $G = |F|$ and observe that $|\text{grad } G| = |F'|$. Finally, we must remove the conditions we imposed on F at the beginning of the proof. If $F(z)$, $z = t + is$, is analytic in the upper half-plane and belongs to H^p, then $F(z + i/n) = F_n(z)$ is bounded there. Let now $e_m(z) = \exp(-z^\alpha m)$, where $0 < \alpha < 1/4$ and $\arg(z^\alpha)$ is between 0 and $\pi/4$. Then a simple calculation shows that

$$\int_{s>0} |e_m'(t + is)|^2 dt\, ds \leq c^2\alpha,$$

where c is independent of m. Consequently, $S(e_m)^2 \leq c^2\alpha$. Now, the following inequalities can be readily verified:

$$S(F_n e_m)^2 \leq 2[S(F_n)^2 + m(F_n)^2 S(e_m)^2] \leq 2[S(F_n)^2 + c^2 m(F_n)^2\alpha]$$
$$S(F_n e_m)^p \leq 2^p[S(F_n)^p + c^p m(F_n)^p \alpha^{p/2}].$$

Integrating we get

$$M_p^p[S(F_n e_m)] \leq 2^p[M_p^p[S(F_n)] + c^p\alpha^{p/2}M_p^p[m(F_n)]].$$

Since $M_p^p(F_n) = \lim_m M_p^p(F_n e_m)$ and by (8), $M_p^p(F_n e_m) \leq c_p^p M_p^p[S(F_n e_m)]$ from the inequality above we obtain

$$M_p^p(F_n) \leq c_p^p\, 2^p[M_p^p[S(F_n)] + c^p\alpha^{p/2}M_p^p[m(F_n)]],$$

and letting α tend to zero

$$M_p(F_n) \leq c_p 2 M_p[S(F_n)].$$

Finally, as n tends to infinity, $M_p(F_n)$ converges to $M_p(F)$ and $S(F_n)$ increases and converges to $S(F)$. Thus we can pass to the limit in the preceding inequality and obtain half of the desired result. To obtain the other half we observe that, since $(F_n e_m)'$ converges to F_n', we have $S(F_n) = \lim_m \inf S(F_n e_m)$. Thus from (6) applied to $F_n e_m$ and Fatou's lemma we get

$$M_p[S(F_n)] \leq c_p M(F_n),$$

and a passage to the limit completes the proof of the theorem.

Proof of Theorem 2: We begin with the one-dimensional case. Here $h(x)$ becomes simply x^{-2}, and the proof reduces to estimate

$$\int_{-\infty}^{+\infty} C_\epsilon(f)g\, dx = \int_{|x-y|>\epsilon} (x - y)^{-2}[b(x) - b(y)]g(x)f(y)dx\, dy$$

in terms of the norms of f, g, and b'. For this purpose there is no loss of generality in assuming that these functions are infinitely differentiable and have compact support. Let $e(x)$ be the characteristic function of $x > 0$ and $\chi(x)$ that of $|x| > \epsilon$. Then

$$b(x) = \int_{-\infty}^{+\infty} e(x - t)b'(t)dt,$$

and substituting, the integral above becomes

$$\int_{-\infty}^{+\infty} b'(t) \int (x - y)^{-2}\chi(|x - y|)[e(x - t) - e(y - t)]g(x)f(y)dx\, dy\, dt$$

and the problem reduces to studying the class of the function represented by the inner integral. For this purpose we let z be a complex variable and set

$$f_j(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{1}{x - z} f(x)dx, \qquad j = 1 \text{ if } \mathrm{Im}(z) > 0, \qquad j = 2 \text{ if } \mathrm{Im}(z) < 0,$$

and define similarly $g_j(z)$. Then we have $f(x) = f_1(x) - f_2(x)$ and similarly for g. Furthermore, the f_j belong to H^p, $1 < p < \infty$, in the corresponding half-planes and, with the notation of the preceding proof, we have

$$M_p(f_j) \leq c_p M(f), \qquad 1 < p < \infty. \tag{9}$$

Corresponding relations hold also for g and g_j. We will study the contribution of f_1 to the integral in question, an analogous argument being applicable to f_2. Let us introduce the following kernels

$$K_0(x,y,t) = (x - y)^{-2}\chi(|x - y|)[e(x - t) - e(y - t)]$$
$$K_1(x,y,t) = (x - y - i\epsilon)^{-2}[e(x - t) - e(y - t)]$$
$$K_2(x,y,t) = [(x - t)^2 + (y - t)^2 + \epsilon^2]^{-3/2}\epsilon.$$

An easy calculation shows that $|K_0 - K_1| \leq cK_2$ with c independent of ϵ. Now we set

$$k_j(t) = \int K_j(x,y,t)g(x)f_1(y)dx\,dy \qquad k_2(t) = \int K_2(x,y,t)|g(x)\,f_1(y)|dx\,dy.$$

We are interested in estimating k_0. On account of the inequality between the K_j stated above, we have $|k_0| \leq |k_1| + ck_2$ and thus it will suffice to estimate k_1 and k_2. On account of the analyticity of $f_1(y)$ if $x > t$ we have

$$\int_{-\infty}^{+\infty} K_1(x,y,t)f_1(y)\,dy = \int_{-\infty}^{t} (x - y - i\epsilon)^{-2}f_1(y)\,dy$$
$$= -\int_{s=0}^{+\infty} [(t + is) - (x - i\epsilon)]^{-2}f_1(t + is)\,d(is).$$

As readily seen, for $x < t$ the integral on the left above is also given by this last expression. Thus,

$$k_1(t) = -\int_{-\infty}^{+\infty} g(x)\int_{s=0}^{+\infty} [(t + is) - (x - i\epsilon)]^{-2}f_1(t + is)\,d(is),$$

and interchanging the order of integration we get

$$k_1(t) = -\int_{s=0}^{+\infty} f_1(t + is)\int_{-\infty}^{+\infty} [(t + is) - (x - i\epsilon)]^{-2}\,g(x)\,dx\,d(is).$$

Since $g(x) = g_1(x) - g_2(x)$ and $g_2(z)$ is analytic in $\mathrm{Im}(z) < 0$, its contribution to the inner integral above is zero and the value of this reduces to $2\pi i g_1'(t + is + i\epsilon)$. Thus we have

$$k_1(t) = -2\pi i\int_{s=0}^{+\infty} f_1(t + is)\,g_1'(t + is + i\epsilon)\,d(is).$$

Let us introduce now

$$F(z) = -2\pi i \int_{s=0}^{+\infty} f_1(z + is)g_1'(z + is + i\epsilon)d(is).$$

Then we have $k_1(t) = F(t)$. Furthermore, since f_1 and g_1' are bounded and $O(z^{-1})$ and $O(z^{-2})$, respectively, $F(z)$ belongs to H^p, $p \geq 1$, and with the notation of the preceding proof we have

$$(2\pi)^{-1} S(F) \leq m(f_1)S(g_1(z + i\epsilon)) \leq m(f_1)S(g_1)$$

and if $q^{-1} = p^{-1} + r^{-1}$, $1 < p, q < \infty$, $r \leq \infty$, then by Theorem 3 and (9) we have

$$M_{r/r-1}(k_1) = M_{r/r-1}(F) \leq c\, M_{r/r-1}[S(F)] \leq c\, M_p[m(f_1)]M_{q/q-1}[S(g_1)]$$
$$\leq c\, M_p(f_1)M_{q/q-1}(g_1) \leq c\, M_p(f)M_{q/q-1}(g). \quad (10)$$

Now we estimate k_2. We have

$$\int_{-\infty}^{+\infty} K_2(x,y,t)\,|f(y)|\,dy \leq \epsilon[(x - t)^2 + \epsilon^2]^{-1} \sup_{\delta} \delta^2 \int_{-\infty}^{+\infty} [(y - t)^2 + \delta^2]^{-3/2}|f(y)|\,dy \leq c\,\epsilon[(x - t)^2 + \epsilon^2]^{-1}\bar{f}(t),$$

where $\bar{f}$ is the maximal function of Hardy and Littlewood associated with $|f|$. Consequently,

$$|k_2(t)| \leq c\bar{f}(t) \sup_{\epsilon} \epsilon \int_{-\infty}^{+\infty} [(x - t)^2 + \epsilon^2]^{-1}\,|g(x)|\,dx \leq c\bar{f}(t)\,\bar{g}(t).$$

$$M_{r/r-1}(k_2) \leq c\, M_p(\bar{f})\, M_{q/q-1}(\bar{g}) \leq c\, M_p(f)\, M_{q/q-1}(g).$$

This combined with (10) shows that $M_{r/r-1}(k_0) \leq c\, M_p(f)\, M_{q/q-1}(g)$ where c depends on p and r but not on ϵ. As readily seen, this implies that $M_q[C_\epsilon(f)] \leq c\, M_r(b')\, M_p(f)$.

We now pass to discuss the n-dimensional case. As before, we assume that f and the partial derivatives b_j of b are infinitely differentiable and have compact support. We denote by ν a unit vector in R^n and by E its orthogonal complement and fix ϵ, $\epsilon > 0$. Let s be a real variable and

$$k(x,\nu) = \int_{|s|>\epsilon} h(\nu)\, s^{-2}[b(x) - b(x + \nu s)]f(x + \nu s)\,ds.$$

Then setting $y = x + \nu s$, integration in polar coordinates shows that

$$C_\epsilon(f) = {}^1/_2 \int k(x,\nu)\,d\nu, \quad (11)$$

where $d\nu$ denotes the surface area element of the unit sphere in R^n. We now fix ν and set $x = z + \nu t$, where $z \epsilon E$. Then from the inequality for the one-dimensional case established above we get

$$\int_{-\infty}^{+\infty} k(z + \nu t,\nu)^q dt \leq c\left[\int_{-\infty}^{+\infty} |\text{grad}\, b\,(z + \nu t,\nu)|^r\,dt\right]^{q/r} \times \left[\int_{-\infty}^{+\infty} |f(z + \nu t,\nu)|^p\,dt\right]^{q/p} |h(\nu)|.$$

Integrating with respect to z over E and applying Hölder's inequality to the right-hand side, we obtain

$$[\int |k(x,\nu)|^q\, dx]^{1/q} \leq c\, [\int |\text{grad}\, b|^r\, dx]^{1/r}[\int |f(x)|^p\, dx]^{1/p}\, |h(\nu)|.$$

From this and Minkowski's integral inequality applied to (11) we obtain

$$||C_\epsilon(f)||_q \leq c\, ||\text{grad}\, b||_r\, ||f||_p \int |h(\nu)|\, d\nu,$$

where c depends on p, q, and r but not on ϵ.

Concerning the convergence of $C_\epsilon(f)$ as ϵ tends to zero we merely observe that our assertion obviously holds if f and the b_j are assumed to be infinitely differentiable and have compact support, whence the general case follows from the inequality above by approximation.

Proof of Theorem 1: Since (b) can readily be obtained from (a) by duality, we shall only prove the latter. Let us consider first the case when $k(x)$ is an odd function. There will be no loss in generality in assuming that $k(x)$ is infinitely differentiable in $|x| > 0$ and that f and the b_j are infinitely differentiable and have compact support. Let f_j, b_j, and k_j denote the jth partial derivatives of f, b, and k, respectively. Then integration by parts yields

$$\int_{|x-y|>\epsilon} k(x-y)[b(x) - b(y)]f_j(y)\, dy = \int_{|x-y|>\epsilon} k(x-y)\, b_j(y)\, f(y)\, d-y$$

$$+ \int_{|x-y|>\epsilon} k_j(x-y)[b(x) - b(y)]f(y)\, dy$$

$$- \int k(\nu\epsilon)[b(x) - b(x+\nu\epsilon)]\, f(x+\nu\epsilon)\nu_j\epsilon^{n-1}\, d\nu,$$

where ν_j denotes the jth component of the unit vector ν and $d\nu$ denotes the surface area element of the unit sphere in R^n. Now, the first term on the right represents an ordinary truncated singular integral and its norm in L^q can be estimated in terms of the norms of b_j and f. To estimate the norm of the second term we use Theorem 2, and in the last term we replace $b(x) - b(x+\nu\epsilon)$ by

$$-\int_0^1 \Sigma b_j(x + t\nu\epsilon)\nu_j\epsilon\, dt$$

and apply Minkowski's integral inequality to the resulting integral. Collecting results and letting ϵ tend to zero, (a) follows.

In the case when $k(x)$ is even, the operator A can be represented as a finite sum of operators of the form A_1A_2 where A_1 and A_2 have odd kernels and satisfy the hypothesis of the theorem (see ref. 2). Since $\partial/\partial x_j$ commutes with A_2, we have

$$(A_1A_2B - BA_1A_2)\frac{\partial}{\partial x_j} = A_1(A_2B - BA_2)\frac{\partial}{\partial x_j} + (A_1B - BA_1)\frac{\partial}{\partial x_j}A_2,$$

since A_1 and A_2 are bounded in L^p for every p, $1 < p < \infty$, the desired result follows.

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[1] Calderón, A. P., and A. Zygmund, "Singular integral operators and differential equations," *Am. J. Math.*, **79**, 901–921 (1957).

[2] *Ibid.*, "On singular integrals," **78**, 289–309 (1956).

[3] Flett, T. M., "On some theorems of Littlewood and Paley," *J. London Math. Soc.*, **31**, 336–344 (1956).

[4] To obtain Green's formula for the half-plane under our assumptions we apply it to $n \cos(n^{-1}t) \sin(n^{-1}s)$ and the function G^2 or G^2H over the square $-n\frac{\pi}{2} \leq t \leq n\frac{\pi}{2}$, $0 \leq s \leq n\pi$, and let n tend to infinity.