A positivity result in the theory of Macdonald polynomials

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We outline here a proof that a certain rational function $C_n(q, t)$, which has come to be known as the "q, t-Catalan," is in fact a polynomial with positive integer coefficients. This has been an open problem since 1994. Because $C_n(q, t)$ evaluates to the Catalan number at $t = q = 1$, it has also been an open problem to find a pair of statistics a, b on the collection $\mathcal{D}_n$ of Dyck paths Π of length $2n$ yielding $C_n(q, t) = \sum_\pi t^{a(\Pi)}q^{b(\Pi)}$. Our proof is based on a recursion for $C_n(q, t)$ suggested by a pair of statistics recently proposed by J. Haglund. One of the byproducts of our results is a proof of the validity of Haglund's conjecture.

1. Preliminaries

At the 1988 Alghero meeting of the Lotharingian Seminar, Macdonald introduced a two-parameter symmetric function basis $\{J_\mu[X; q, t]\}_\mu$ that has since proved to be fundamental in the Theory of Symmetric Functions. In recent years the Theory of Symmetric Functions has acquired particular importance because of its relation to the Representation Theory of Hecke algebras and the Symmetric Groups, and has been shown to have applicability in a wide range of scientific and mathematical disciplines. In many of these developments the Macdonald polynomials and some of their specializations have played a central role. In the original paper (1) and in subsequent work (2–7) a number of conjectures have been formulated that assert that certain rational functions in q, t are in fact polynomials with positive integer coefficients. For a decade these conjectures have resisted several various attempts of proof by a wide range of approaches. Although these conjectures lie squarely within the Theory of Symmetric Functions, the approaches range from diagonal actions of the symmetric group on polynomial rings in two sets of variables (2, 3, 5) to the Algebraic Geometry of Hilbert schemes (8). Efforts to resolve these conjectures within the Theory of Symmetric Functions have led to the discovery of a variety of new methods to deal with symmetric function identities (3, 4, 6, 8). In this paper we outline an argument that yields a purely symmetric function proof of one of these conjectures. To state the result we need some definitions and notational conventions.

A partition μ will always be identified with its Ferrers diagram. The partition conjugate to μ will be denoted μ'. By the French convention, if the parts of μ are $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0$, the Ferrers diagram has μ_i lattice cells in the ith row (from the bottom up). Here $|\mu|$ and $l(\mu)$ denote, respectively, the sum of the parts and the number of nonzero parts of μ. The symbol $\mu \vdash n$ will also be used to indicate that $|\mu| = n$. Following Macdonald, the arm, leg, coarm, and coleg of a lattice square s are the parameters $a_\mu(s)$, $l_\mu(s)$, $a'_\mu(s)$, and $l'_\mu(s)$, giving the number of cells of μ that are, respectively, *strictly* east, north, west, and south of s in μ.

Here and after, for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ we set

$$T_\mu = \prod_{s \in \mu} t^{l'_\mu(s)}q^{a'_\mu(s)}, \qquad B_\mu(q, t) = \sum_{s \in \mu} t^{l'_\mu(s)}q^{a'_\mu(s)},$$

$$\Pi_\mu(q, t) = \prod_{s \in \mu} (1 - t^{l'_\mu(s)}q^{a'_\mu(s)}) \qquad \textbf{[1.1]}$$

and

$$\tilde{h}_\mu(q, t) = \prod_{s \in \mu} (q^{a_\mu(s)} - t^{l_\mu(s)+1}),$$

$$\tilde{h}'_\mu(q, t) = \prod_{s \in \mu} (t^{l_\mu(s)} - q^{a_\mu(s)+1}). \qquad \textbf{[1.2]}$$

This given we can show

THEOREM 1.1. *For every $n \geq 1$ the rational function*

$$C_n(q, t) = \sum_{\mu \vdash n} \frac{T_\mu^2(1 - t)(1 - q)B_\mu(q, t)\Pi_\mu(q, t)}{\tilde{h}_\mu(q, t)\tilde{h}'_\mu(q, t)} \qquad \textbf{[1.3]}$$

evaluates to a polynomial with positive integer coefficients.

To show how this relates to Macdonald polynomials and to outline our proof, we need to introduce plethystic notation. This is a very powerful notational device that considerably facilitates the manipulation of symmetric function identities. This device can also be easily implemented in software such as MAPLE or MATHEMATICA when we express symmetric functions in terms of the power sum sequence $\{p_k\}_{k\geq 1}$. To begin with, if $E = E[t_1, t_2, t_3, \ldots]$ is a formal Laurent series in the variables $t_1, t_2, t_3, \ldots$ (which may include the parameters q, t) we set

$$p_k[E] = E[t_1^k, t_2^k, t_3^k, \ldots].$$

More generally, if a certain symmetric function F is expressed as the formal power series

$$F = Q[p_1, p_2, p_3, \ldots]$$

then we simply let

$$F[E] = Q[p_1, p_2, p_3, \ldots]|_{p_k \to E[t_1^k, t_2^k, t_3^k, \ldots]}, \qquad \textbf{[1.4]}$$

and refer to it as "plethystic substitution" of E into the symmetric function F. We also adopt the convention that inside the plethystic bracket X and X_n stand for $X = x_1 + x_2 + \ldots$ and $X_n = x_1 + x_2 + \ldots + x_n$. In particular, a symmetric polynomial $P = P(x_1, x_2, \ldots, x_n)$ may be simply written in the form $P = P[X_n]$. We should mention that the present breakthrough would not have been possible without the insight provided by this notational device.

This given, we will work here with the modified Macdonald polynomial $\tilde{H}_\mu[X; q, t]$ obtained by setting

$$\tilde{H}_\mu[X; q, t] = t^{n(\mu)}J_\mu\left[\frac{X}{(1 - 1/t)}; q, 1/t\right] \qquad \left[\text{with } n(\mu) = \textstyle\sum_{s \in \mu} l'_\mu(s)\right]. \qquad \textbf{[1.5]}$$

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Another important ingredient here is the linear operator ∇ defined by setting for the basis $\{\tilde{H}_\mu[X; q, t]\}_\mu$:

$$\nabla \tilde{H}_\mu[X; q, t] = T_\mu \tilde{H}_\mu[X; q, t]. \quad [1.6]$$

Now it was shown in ref. 6 that the elementary symmetric function e_n has the expansion

$$e_n[X] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t](1-t)(1-q)B_\mu(q, t)\Pi_\mu(q, t)}{\tilde{h}_\mu(q, t)\tilde{h}'_\mu(q, t)} \quad [1.7]$$

so that Eq. **1.6** gives

$$\nabla e_n[X] = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu[X; q, t](1-t)(1-q)B_\mu(q, t)\Pi_\mu(q, t)}{\tilde{h}_\mu(q, t)\tilde{h}'_\mu(q, t)}. \quad [1.8]$$

Equating coefficients of the Schur function S_λ gives

$$\nabla e_n[X]|_{S_\lambda} = \sum_{\mu \vdash n} \frac{T_\mu \tilde{K}_{\lambda\mu}(q, t)(1-t)(1-q)B_\mu(q, t)\Pi_\mu(q, t)}{\tilde{h}_\mu(q, t)\tilde{h}'_\mu(q, t)}, \quad [1.9]$$

where from Eq. **1.5** we derive that $\tilde{K}_{\lambda\mu}(q, t)$ is related to the Macdonald q, t-Kostka coefficient $K_{\lambda\mu}(q, t)$ by the simple reversion

$$\tilde{K}_{\lambda\mu}(q, t) = t^{n(\mu)} K_{\lambda\mu}(q, 1/t).$$

In particular, it follows from Macdonald's work (9) that (for $\mu \vdash n$)

$$\tilde{K}_{1^n, \mu}(q, t) = T_\mu. \quad [1.10]$$

By using this in Eq. **1.9** for $\lambda = 1^n$, Eq. **1.3** becomes

$$C_n(q, t) = \nabla e_n[X]|_{S_{1^n}}. \quad [1.11]$$

Our proof of *Theorem 1.1* is based on this identity. The reader is referred to refs. 6 and 7 for several conjectures concerning the expressions in Eq. **1.9**.

Here it suffices to know that it was shown in ref. 4 that ∇ acts integrally on Schur functions. This implies that all the expressions in Eq. **1.9**, and in particular $C_n(q, t)$, evaluate to polynomials in q, t with integer coefficients. Our proof of *Theorem 1.1* gives the positivity of the latter coefficients as well as a combinatorial interpretation of their values. This is obtained by means of a recursion satisfied by the two-parameter family of polynomials

$$Q_{n,s}(q, t) = t^{n-s} q^{\binom{s}{2}} \nabla e_{n-s}\left[X \frac{1-q^s}{1-q}\right]\Big|_{S_{1^{n-s}}}. \quad [1.12]$$

More precisely we show that

THEOREM 1.2. *For any pair of integers $n \geq s \geq 1$ we have*

$$Q_{n,s}(q, t) = t^{n-s} q^{\binom{s}{2}} \sum_{r=0}^{n-s} \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q Q_{n-s,r}(q, t), \quad [1.13]$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = [(q;q)_n]/[(q;q)_k(q;q)_{n-k}]$ *denotes the q-binomial coefficient.*

Note that since $\nabla 1 = 1$, Eq. **1.12** gives the initial conditions

$$Q_{n,n}(q, t) = q^{\binom{n}{2}}. \quad [1.14]$$

It is then easily seen that Eq. **1.13** yields

$$Q_{n,s}(q, t) \in \mathbf{N}[q, t] \quad \forall\, n \geq s \geq 1.$$

Moreover, Eqs. **1.12** and **1.13** with $n \rightarrow n + 1$ and $s \rightarrow 1$ give

$$C_n(q, t) = \nabla e_n|_{S_{1^n}} = \sum_{r=1}^{n} Q_{n,1}(q, t) \in \mathbf{N}[q, t]. \quad [1.15]$$

A remarkable corollary of Eqs. **1.13** and **1.15** is the combinatorial formula

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} t^{area(\Pi)} q^{maj(\beta(\Pi))}, \quad [1.16]$$

where $\mathcal{D}_n$ is the collection of all Dyck paths of length $2n$, $area(\Pi)$ denotes the area under the path, and $maj(\beta(\Pi))$ denotes the "major index" of a certain path $\beta(\Pi)$ associated to Π. The reader is referred to ref. 6 for a more detailed description of these combinatorial structures.

We must mention that Eq. **1.16** had been previously conjectured by J.H. in an article to appear in the journal Advances in Mathematics and was in fact the starting point of the investigation that led to the present results.

2. Outline of the Argument

Because it can be shown that

$$\begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q = h_r\left[\frac{1-q^s}{1-q}\right] \quad [2.1]$$

we see that Eq. **1.13** simply states that the equality

$$\nabla e_m\left[X \frac{1-z}{1-q}\right]\Big|_{S_{1^m}} = \sum_{r=1}^{m} h_r\left[\frac{1-z}{1-q}\right] t^{m-r} q^{\binom{r}{2}} \nabla e_{m-r}\left[X \frac{1-q^r}{1-q}\right]\Big|_{S_{1^{m-r}}} \quad [2.2]$$

must hold true for $z = q^s$ and all pairs $m, s \geq 1$. This of course implies (and is, in fact, equivalent to) the equality of the two polynomials on both sides of Eq. **1.2**.

Now the polynomials

$$h_r\left[\frac{1-z}{1-q}\right] = \frac{(z;q)_n}{(q;q)_n} \quad [2.3]$$

have the "Taylor" expansion formula:

$$P(z) = \sum_{r \geq 0} \frac{(z;q)_r}{(q;q)_r} q^r (\delta_q^r P(z)|_{z=1}), \quad [2.4]$$

with δ_q the q-difference operator

$$\delta_q P(z) = \frac{P(z) - P(z/q)}{z}. \quad [2.5]$$

By using Eq. **2.4** we immediately derive that Eq. **1.13** holds true if and only if we have

$$\delta_q^k \nabla e_m\left[X\frac{1-z}{1-q}\right]\Big|_{S_{1^m}}\Big|_{z=1}$$

$$= \frac{t^{m-k}}{q^k} q^{\binom{k}{2}} \nabla e_{m-k}\left[X\frac{1-q^k}{1-q}\right]\Big|_{S_{1^{m-k}}} \quad \forall\, k = 1, 2, \ldots, m. \qquad \textbf{[2.6]}$$

This identity is made more amenable to symmetric function manipulations by means of the expansions

$$e_m\left[X\frac{1-q^k}{1-q}\right] = (1-q^k)\sum_{\mu\vdash m}\frac{\tilde{H}_\mu[X;q,t]\Pi_\mu(q,t)h_k[(1-t)B_\mu(q,t)]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)} \qquad \textbf{[2.7]}$$

and

$$h_m\left[X\frac{1-q^k}{1-q}\right] = (-t)^{m-k}q^{k(m-1)}(1-q^k) \cdot \sum_{\mu\vdash m}\frac{\tilde{H}_\mu[X;q,t]\Pi_\mu(q,t)e_k\left[(1-t)B_\mu\left(\frac{1}{q},\frac{1}{t}\right)\right]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}. \qquad \textbf{[2.8]}$$

By using these relations we were able to derive from Eq. **2.6** that *Theorem 1.1* is equivalent to the following:

THEOREM 2.1. *For all* $1 \le k \le m$ *we have*

$$\sum_{i=1}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q q^{\binom{i}{2}-i}(1-q^i)t^{m-i}\sum_{\mu\vdash m}\frac{T_\mu^2\Pi_\mu}{\tilde{h}_\mu\tilde{h}'_\mu}e_i\left[(1-t)B_\mu\left(\frac{1}{q},\frac{1}{t}\right)\right]$$

$$= \frac{t^{m-k}}{q^k}q^{\binom{k}{2}}(1-q^k)\sum_{\nu\vdash m-k}\frac{T_\nu^2\Pi_\nu h_k[(1-t)B_\nu(q,t)]}{\tilde{h}_\nu(q,t)\tilde{h}'_\nu(q,t)}. \qquad \textbf{[2.9]}$$

Note that Eq. **2.9** for $k = 1$ reduces to

$$\sum_{\mu\vdash m}\frac{T_\mu^2\Pi_\mu}{\tilde{h}_\mu\tilde{h}'_\mu}B_\mu\left(\frac{1}{q},\frac{1}{t}\right) = \sum_{\nu\vdash m-1}\frac{T_\nu^2\Pi_\nu}{\tilde{h}_\nu\tilde{h}'_\nu}B_\nu(q,t), \qquad \textbf{[2.10]}$$

and for $k = 2$

$$t(1-t)\sum_{\mu\vdash m}\frac{T_\mu^2\Pi_\mu}{\tilde{h}_\mu\tilde{h}'_\mu}B_\mu\left(\frac{1}{q},\frac{1}{t}\right) + \sum_{\mu\vdash m}\frac{T_\mu^2\Pi_\mu}{\tilde{h}_\mu\tilde{h}'_\mu} \cdot e_2\left[(1-t)B_\mu\left(\frac{1}{q},\frac{1}{t}\right)\right]$$

$$= \sum_{\nu\vdash m-2}\frac{T_\nu^2\Pi_\nu}{\tilde{h}_\nu\tilde{h}'_\nu}h_2[(1-t)B_\nu(q,t)]. \qquad \textbf{[2.11]}$$

To establish these identities we need a basic mechanism for converting sums over partitions of size m to sums over partitions of smaller size. Now, it develops that this can be achieved by summation formulas involving "Pieri" coefficients. The latter are the rational functions $d^f_{\mu\nu}(q,t)$ occurring in multiplication rules of the form.

$$f[X]\tilde{H}_\nu[X;q,t] = \sum_{\substack{\mu\supseteq\nu\\ k\le|\mu|\le k+d}} d^f_{\mu\nu}(q,t)\tilde{H}_\mu[X;q,t], \qquad \textbf{[2.12]}$$

when $\nu \vdash k$ and $f[X]$ is a symmetric function of degree d. Stanley for the Jack symmetric functions case (10) and Macdonald in ref. 1 give explicit formulas for $d^f_{\mu\nu}(q,t)$ when $f = h_d$ or $f = e_d$ for some $d \ge 1$. These formulas may be used to settle Eqs. **2.10** and **2.11**. They should also yield what is needed for Eq. **2.9** as well, because in principle the multiplication rules for any f may be obtained by combining successive multiplications by the elementary (or homogeneous) symmetric functions. However, to carry this out in full generality we run into a task of forbidding complexity.

The breakthrough was the discovery that the necessary summation formulas may be directly obtained through the operator ∇. This shows once more that this remarkable operator somehow encodes within its action a great deal of the combinatorial complexity of Macdonald polynomials [see refs. 2–11] and (ref. 11 available at http://www.emis.de/journals/SLC/). To state our summation formulas we need further notation.

Let us recall that the so called "Hall" scalar product $\langle,\rangle$ is defined by setting for the power basis

$$\langle p_\mu, p_\nu\rangle = \chi(\mu=\nu)z_\mu, \qquad \textbf{[2.13]}$$

where for a partition $\mu = 1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\ldots$ we set $z_\mu = 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^{\alpha_3}\alpha_3!\ldots$ Our versions of the Macdonald polynomials $\tilde{H}_\mu$ are orthogonal with respect to the scalar product $\langle,\rangle_*$ defined by setting

$$\langle p_\mu, p_\nu\rangle_* = (-1)^{|\mu|-l(\mu)}\chi(\mu=\nu)z_\mu p_\mu[(1-t)(1-q)]. \qquad \textbf{[2.14]}$$

To be precise we have

$$\langle\tilde{H}_\mu, \tilde{H}_\nu\rangle_* = \tilde{h}_\mu\tilde{h}'_\mu\chi(\mu=\nu). \qquad \textbf{[2.15]}$$

Now, companions to the Pieri rules in Eq. **2.12** are their dual forms

$$f^\perp\tilde{H}_\mu[X;q,t] = \sum_{\substack{\nu\subseteq\mu\\ m-d\le|\nu|\le m}} c^{f^\perp}_{\mu\nu}(q,t)\tilde{H}_\nu[X;q,t], \qquad \textbf{[2.16]}$$

where $\mu \vdash m$, f is any symmetric function of degree d, and $f^\perp$ denotes the operator that is the Hall-adjoint of multiplication by f. We should note that the Pieri coefficients and their dual counterparts are related by the identity

$$c^{f^\perp}_{\mu\nu}(q,t)\tilde{h}_\nu\tilde{h}'_\nu = d^{\omega f^*}_{\mu\nu}(q,t)\tilde{h}_\mu\tilde{h}'_\mu, \qquad \textbf{[2.17]}$$

where ω as customary denotes the fundamental involution of symmetric functions and for any symmetric polynomial f we set

$$f^*[X] = f\left[\frac{X}{(1-t)(1-q)}\right]. \qquad \textbf{[2.18]}$$

This is an easy consequence of Eq. **2.15** and the definitions in Eqs. **2.12** and **2.16**.

This given, our proof of the recursion in Eq. **1.13** is based on the following two remarkable summation formulas.

THEOREM 2.2. *For g a symmetric polynomial of degree d and* $\mu \vdash m$ *we have*

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$$\sum_{\substack{\nu \subseteq \mu \\ m-d \le |\nu| \le m}} c_{\mu\nu}^{(\omega g)^\perp}(q,t) T_\nu$$

$$= T_\mu G\left[\left(1-\frac{1}{t}\right)\left(1-\frac{1}{q}\right)B_\mu\left(\frac{1}{q},\frac{1}{t}\right)-1\right] \quad \mathbf{[2.19]}$$

with

$$G[X] = \omega\nabla\left(g\left[\frac{X+1}{\tilde{M}}\right]\right). \quad \mathbf{[2.20]}$$

THEOREM 2.3. *For f a symmetric polynomial of degree d and* $\nu \vdash k$ *we have*

$$\sum_{\substack{\mu \supseteq \nu \\ k \le |\mu| \le k+d}} d^f_{\mu,\nu} T_\mu \Pi_\mu = T_\nu \Pi_\nu (\nabla f)[MB_\nu]. \quad \mathbf{[2.21]}$$

We should mention that both Eq. **2.19** and Eq. **2.21** are ultimate consequences of the following result (proved in ref. 11):

THEOREM 2.4. *For a given symmetric function P set*

$$\Pi_P[X; q, t] = \nabla^{-1} P[X - \varepsilon]\big|_{\varepsilon=-1}.$$

Then for all partitions μ *we get*

$$\langle P, \tilde{H}_\mu[X+1; q, t]\rangle_* = \Pi_P[(1-t)(1-q)B_\mu(q,t) - 1; q, t].$$

To give an idea of the manner in which Eqs. **2.19** and **2.21** are used to obtain Eq. **2.9**, we shall use them to prove Eqs. **2.10** and **2.11**.

To begin, the case $\omega g = h_1$ of Eq. **2.19** gives

$$T_\mu B_\mu\left(\frac{1}{q},\frac{1}{t}\right) = \sum_{\nu \to \mu} c_{\mu\nu}^{h_1^\perp}(q,t) T_\nu,$$

where the symbol $\nu \to \mu$ is to indicate that ν is obtained by removing one of the corners of μ. Substituting this in the left hand side of Eq. **2.10** gives

$$\begin{aligned}
\sum_{\mu \vdash m} \frac{T_\mu^2 \Pi_\mu}{\tilde{h}_\mu \tilde{h}'_\mu} B_\mu\left(\frac{1}{q},\frac{1}{t}\right) &= \sum_{\mu \vdash m} \frac{T_\mu \Pi_\mu}{\tilde{h}_\mu \tilde{h}'_\mu} \sum_{\nu \to \mu} c_{\mu\nu}^{h_1^\perp}(q,t) T_\nu \\
&= \sum_{\nu \vdash m-1} \frac{T_\nu}{\tilde{h}_\nu \tilde{h}'_\nu} \sum_{\mu \leftarrow \nu} c_{\mu\nu}^{h_1^\perp} \frac{\tilde{h}_\mu \tilde{h}'_\mu}{\tilde{h}_\mu \tilde{h}'_\mu} T_\mu \Pi_\mu \\
(\text{by Eq. } \mathbf{2.17}) \quad &= \sum_{\nu \vdash m-1} \frac{T_\nu}{\tilde{h}_\nu \tilde{h}'_\nu} \sum_{\mu \leftarrow \nu} d_{\mu\nu}^{e_1^*}(q,t) T_\mu \Pi_\mu.
\end{aligned} \quad \mathbf{[2.22]}$$

Now Eq. **2.21** gives

$$\sum_{\mu \leftarrow \nu} d_{\mu\nu}^{e_1^*}(q,t) T_\mu \Pi_\mu = T_\nu \Pi_\nu B_\nu(q,t),$$

which when substituted in Eq. **2.22** immediately yields Eq. **2.10**.

For Eq. **2.11** we use Eq. **2.19** with

$$\omega g = e_2[(1-t)X]$$

and obtain

$$\frac{1}{q}\sum_{\nu \subseteq_2 \mu} c_{\mu\nu}^{\omega g^\perp} T_\nu = t(1-t) T_\mu B_\mu\left(\frac{1}{t},\frac{1}{q}\right) + T_\mu e_2\left[(1-t)B_\mu\left(\frac{1}{t},\frac{1}{q}\right)\right],$$

where the symbol $\nu \subseteq_2 \mu$ means that ν is obtained by removing two corners from μ. Substituting this in the left hand side of Eq. **2.11** gives

$$\begin{aligned}
\textit{left hand side of Eq. } \mathbf{2.11} &= \sum_{\mu \vdash m} \frac{T_\mu \Pi_\mu}{\tilde{h}_\mu \tilde{h}'_\mu} \frac{1}{q} \sum_{\nu \subseteq_2 \mu} c_{\mu\nu}^{\omega g^\perp} T_\nu \\
&= \frac{1}{q} \sum_{\nu \vdash m-2} \frac{T_\nu}{\tilde{h}_\nu \tilde{h}'_\nu} \sum_{\mu \supseteq_2 \nu} d^f_{\mu\nu} T_\mu \Pi_\mu
\end{aligned}$$

with

$$f = h_2\left[\frac{X}{1-q}\right]$$

and Eq. **2.21** gives

$$\frac{1}{q}\sum_{\mu \supseteq_2 \nu} d^f_{\mu\nu} T_\mu \Pi_\mu = T_\nu \Pi_\nu h_2[(1-t)B_\nu(q,t)]$$

and proves Eq. **2.11**.

The details of all these calculations and the complete proof of *Theorem 1.2* will appear in the proceedings of the September 2000 Montreal Colloquium in Algebraic Combinatorics, which is to be published by Discrete Mathematics.

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