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Statistics

Selecting the highest probability in binomial or multinomial trials

(sequential analysis/k-sample problem)

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ABSTRACT Some sequential procedures are considered for selecting the binomial population with largest success probability or for selecting the multinomial outcome with highest cell probability. Procedures with and without sequential elimination of inferior populations are evaluated with respect to the expected probability of the population selected.

1.

We have $k \geqq 2$ coins. Coin i has probability p_i of heads on each toss. We are allowed to toss the k coins any number of times, the same for each coin. After n tosses of the set of coins we observe that coin i has come up heads $X_i^{(n)}$ times. In order to select a coin with a high p value, we choose in advance some positive integer r and define the stopping times

$$N_i = \text{first } n \geqq r \text{ such that } X_i^{(n)} \geqq X_j^{(n)} + r \text{ for every } j \neq i,$$

and [1]

$$N = \min\{N_1, \ldots, N_k\}.$$

If $N = N_i = n$, we stop with n tosses of the set of coins and select coin i. That is, we terminate sampling as soon as one coin has produced at least r more heads than any of the others.

Let P_i = Prob[select coin i] = $P[N = N_i]$ for this procedure, and let $A = \Sigma_1^k p_i P_i$ = expected p value of the coin selected.

THEOREM 1. *Let the notation be such that* $p_1 \geqq p_2 \geqq \ldots \geqq p_k$. *Then*

$$\frac{P_i}{P_j} \geqq \left(\frac{p_i/q_i}{p_j/q_j}\right)^r \quad \textit{for} \quad i < j, \quad \textit{where} \quad q_i = 1 - p_i, \qquad [2]$$

and equality holds for $k = 2$. *This implies that*

$$P_1 \geqq P_2 \geqq \ldots \geqq P_k$$

and that

$$P_1 \geqq \frac{(p_1/q_1)^r}{\sum_1^k (p_i/q_i)^r} \quad \textit{and} \quad P_k \leqq \frac{(p_k/q_k)^r}{\sum_1^k (p_i/q_i)^r}, \qquad [3]$$

with equality for $k = 2$.

A proof of this theorem is presented in *Section 2*.

By ignoring the "overshoots" $X_i^{(n)} - X_j^{(n)} - r$ on the event $[N = N_i = n]$ we obtain the following approximations:

$$P_i \simeq \frac{(p_i/q_i)^r}{\sum_1^k (p_i/q_i)^r} \equiv \hat{P}_i, \qquad [4]$$

and

$$A \simeq \sum_1^k p_i \hat{P}_i \equiv \hat{A}. \qquad [5]$$

(These approximations are equalities for $k = 2$, by *Theorem 1*.)

The next theorem shows that $\hat{A}$ is always an underestimate of A. We state a somewhat more general result and prove it in *Section 2*.

THEOREM 2. *Let* $\{P_1, \ldots, P_k\}$ *and* $\{\hat{P}_1, \ldots, \hat{P}_k\}$ *be any two sets of positive numbers satisfying* $\Sigma P_i = \Sigma \hat{P}_i = 1$, $\hat{P}_1 \geqq \hat{P}_2 \geqq \ldots \geqq \hat{P}_k$, *and*

$$\frac{P_i}{P_j} \geqq \frac{\hat{P}_i}{\hat{P}_j} \quad \textit{for all} \quad i < j.$$

Let $a_1 \geqq \ldots \geqq a_k$ *be any constants. Then*

$$\sum_1^k a_i P_i \geqq \sum_1^k a_i \hat{P}_i.$$

Taking $\hat{P}_i$ as in [4], *Theorems 1 and 2* imply the following inequalities:

$$A \geqq \sum_1^k p_i \hat{P}_i \qquad [6a]$$

expected selected odds =

$$\sum_1^k (p_i/q_i) P_i \geqq \sum_1^k (p_i/q_i)^{r+1} \Big/ \sum_1^k (p_i/q_i)^r \qquad [6b]$$

Prob[select a coin from among the b best coins]

$$= \sum_1^b P_i \geqq \sum_1^b \hat{P}_i, \quad (b = 1, \ldots, k). \qquad [6c]$$

Monte Carlo simulations lend credence to the conjecture that *Theorem 1 and the inequalities 6 that follow from it remain true even if the selection procedure is modified by dropping any coin j from the contest as soon as*

$$\max_i \{X_i^{(n)}\} \geqq X_j^{(n)} + r.$$

The italicized statement is proved in *Section 2* for the very special case $r = 1$. A rigorous proof in the general case is not known to us at present.

For example, if (a) denotes the original procedure and (b) denotes the modified one, and if $k = r = 3$ with $p_1 = 0.6$, $p_2 = p_3 = 0.5$, then in 10,000 runs we obtained the results in Table 1.

The modified rule substantially reduces the average total number of tosses as well as the average total number of tails ("failures") without much reduction in the average p selected. Further simulations show that a modified procedure can ac-

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Table 1. Selecting the best of three coins with $p_1 = 0.6$, $p_2 = p_3 = 0.5$

Operating characteristic	Monte Carlo estimate for procedure: (a)	(b)	$\hat{P}_i$ in [4] $\hat{A}$ in [5]
P_1	0.683	0.653	0.628
P_2	0.158	0.176	0.186
P_3	0.158	0.170	0.186
A	0.568	0.565	0.563
Average total number of:			
Tosses per run	66.1	50.3	
Tails per run	30.8	23.3	

tually *dominate* an unmodified procedure in the sense that for given $p_1, \ldots, p_k$ there exist integers $r' > r$ such that procedure (*b*) with r' has *smaller* expected total number of tosses and *smaller* expected total number of tails than procedure (*a*) with r, whereas the expected p selected is *larger* with procedure (*b*) than with procedure (*a*).

In the case of a multinomial distribution with cell probabilities $p_1 \geqq \ldots \geqq p_k$ corresponding to k possible outcomes, with $\Sigma_1^k p_i = 1$, let $X_i^{(n)}$ = number of times outcome i occurs during the first n trials. The same procedure [1] can be used to select a cell with high p value. The corresponding *Theorem 1'* states that

$$\frac{P_i}{P_j} \geqq (p_i/p_j)^r \quad \text{for} \quad i<j, \qquad [2']$$

with equality for $k = 2$, and similarly for [3] – [5], where p_i/q_i is replaced by p_i. In particular, the first formula of [3] becomes

$$P_1 \geqq \frac{p_1^r}{\sum_1^k p_i^r}, \qquad [3']$$

which improves for $k > 2$ on an inequality of Alam (1)

$$P_1 \geqq 1 - \sum_2^k \{\theta_i^r/(1+\theta_i^r)\} \quad \text{where} \quad \theta_i = p_i/p_1.$$

Corresponding to [5] we have the approximation

$$A \simeq \left(\sum_1^k p_i^{r+1}\right) \bigg/ \left(\sum_1^k p_i^r\right). \qquad [5']$$

Formula 5' is an exact equality for $r = 1$ and any $k \geqq 2$ and also for $k = 2$ and any $r \geqq 1$. *Theorems 1'* and 2 show that the right-hand side of [5'] is a lower bound for A for any $k > 2$ and $r > 1$.

Approximate formulas for the expected total number of tosses and total number of tails for k coins using procedures (*a*) and (*b*), or for the expected number of trials in the multinomial case, are hard to come by for $k > 2$. For the case of $k = 2$ coins, the expected number of tosses per coin is known (2) to be

$$E[N] = \begin{cases} \dfrac{r}{p_1 - p_2}\left(\dfrac{(p_1/q_1)^r - (p_2/q_2)^r}{(p_1/q_1)^r + (p_2/q_2)^r}\right) & \text{for } p_1 > p_2 \\[2ex] \dfrac{r^2}{2p(1-p)} & \text{for } p_1 = p_2 = p. \end{cases} \qquad [7]$$

For the special cases $k = 3$, $p_1 = \theta p$, $p_2 = p_3 = p$ with $\theta > 1$, the following formula approximates $E[N]$ under procedure (*a*):

$$E[N] \simeq \left[\frac{\hat{P}_1(pq/\pi)^{1/2} + [\hat{P}_1^2(pq/\pi) + 2rp(\theta-1)(3\hat{P}_1 - 1)]^{1/2}}{2p(\theta-1)}\right]^2, \qquad [8]$$

where $\hat{P}_1$ is the lower bound of [3] for P_1,

$$\hat{P}_1 = 1\bigg/\left[1 + 2\left(\frac{p/q}{p_1/q_1}\right)^r\right] = 1\bigg/\left[1 + 2\left(\frac{1-\theta p}{\theta q}\right)^r\right].$$

A heuristic derivation of [8] is given in *Section 2*. In the example cited above, [8] produces $E[N] \simeq 21.46$, leading to an expected total number of tosses = $3E[N] \simeq 64.38$.

In the multinomial case for $k = 2$, [7] becomes

$$E[N] = \begin{cases} \dfrac{r}{p_1 - p_2}\left(\dfrac{p_1^r - p_2^r}{p_1^r + p_2^r}\right) & \text{for } p_1 > p_2 \\[2ex] r^2 & \text{for } p_1 = p_2 = \dfrac{1}{2}. \end{cases} \qquad [7']$$

For $k = 3$ with $p_1 = \theta/(\theta + 2)$, $p_2 = p_3 = 1/(\theta + 2)$, $(\theta > 1)$, [8] becomes

$$E[N] \simeq \left[\frac{\hat{P}_1/[\pi(\theta+2)]^{1/2} + \{\hat{P}_1^2/[\pi(\theta+2)] + [2r(3\hat{P}_1 - 1)(\theta-1)/(\theta+2)]\}^{1/2}}{2(\theta-1)/(\theta+2)}\right]^2, \qquad [8']$$

where $\hat{P}_1 = 1/(1 + 2\theta^{-r})$ is the lower bound of [3'] for P_1.

2.

Proof of *Theorem 1*. Fix $i < j$, and let $\mathbf{p} = (p_1, \ldots, p_k)$ with $p_1 \geqq \ldots \geqq p_k$. Let α denote any particular sequence of outcomes and, to simplify notation, write $X_i = X_i^{(n)}(\alpha)$ for the number of heads with coin i after n tosses. Let

$$f_{\mathbf{p}}^{(n)}(\alpha) = \prod_1^k p_i^{X_i}(1-p_i)^{n-X_i}$$

be the probability function of the sequence α. Let $\mathbf{p}(ij)$ denote the same $\mathbf{p}$ vector but with p_i and p_j interchanged. Now,

$$P_{\mathbf{p}}[N = N_j] = \sum_{n=1}^{\infty} \sum_{[N=N_j=n]} f_{\mathbf{p}}^{(n)}(\alpha),$$

where the second summation is over all sequences α such that $N = N_j = n$. Hence, since

$$\frac{f_{\mathbf{p}}^{(n)}(\alpha)}{f_{\mathbf{p}(ij)}^{(n)}(\alpha)} = \left(\frac{p_i/q_i}{p_j/q_j}\right)^{X_i - X_j},$$

we have

$$\begin{aligned} P_{\mathbf{p}}[N = N_j] &= \sum_{n=1}^{\infty} \sum_{[N=N_j=n]} \left(\frac{p_i/q_i}{p_j/q_j}\right)^{X_i - X_j} f_{\mathbf{p}(ij)}^{(n)}(\alpha) \\ &\leqq \left(\frac{p_i/q_i}{p_j/q_j}\right)^{-r} P_{\mathbf{p}(ij)}[N = N_j] \\ &= \left(\frac{p_i/q_i}{p_j/q_j}\right)^{-r} P_{\mathbf{p}}[N = N_i]. \end{aligned} \qquad [9]$$

The inequality follows because $X_j \geqq X_i + r$ on the event $[N = N_j = n]$, with equality if $k = 2$. The final equality in [9] follows by *symmetry*: the two probabilities $P_{\mathbf{p}}[N = N_i]$ and $P_{\mathbf{p}(ij)}[N = N_j]$ differ only in how we label the coins, whereas the stopping rule is invariant under permutations of the labels. This establishes [2]. The first inequality in [3] follows from

$$P_1 = 1 - \sum_2^k P_j \geqq 1 - P_1 \sum_2^k \left(\frac{p_1/q_1}{p_j/q_j}\right)^{-r},$$

with a similar proof for P_k.

A similar proof holds for *Theorem 1′*, with $f_{\mathbf{p}}^{(n)}(\alpha) = \Pi_1^k p_i^{X_i^{(n)}(\alpha)}$.

Proof of *Theorem 2*. Let $e_i = P_i - \hat{P}_i$ for $i = 1, \ldots, k$. Then $\Sigma_1^k e_i = 0$.

The vector $(e_1, \ldots, e_k)$ possesses the property that if $e_j \geqq 0$ for some j then $e_i \geqq 0$ for all $i < j$. For,

$$\frac{e_i}{e_j} = \frac{P_i - \hat{P}_i}{P_j - \hat{P}_j} \geqq \frac{(P_j\hat{P}_i/\hat{P}_j) - \hat{P}_i}{P_j - \hat{P}_j} = \frac{\hat{P}_i}{\hat{P}_j} \geqq 1.$$

Thus there is an integer m $(1 \leqq m \leqq k-1)$ such that $e_i \geqq 0$ for $i \leqq m$ and $e_i \leqq 0$ for $i > m$. Given $a_1 \geqq \ldots \geqq a_k$ we have

$$\sum_1^k a_i e_i = \sum_1^m a_i e_i - \sum_{m+1}^k a_i|e_i|$$
$$\geqq a_m \sum_1^m e_i - a_{m+1} \sum_{m+1}^k |e_i| \geqq a_m \sum_1^k e_i = 0.$$

This concludes the proof.

Next we prove that *Theorem 1* holds for the modified procedure (*b*) for any $k > 2$ in the special case $r = 1$.

THEOREM 1B. *Let* r = 1 *in procedure* (b) *with* $p_1 \geqq \ldots \geqq p_k$. *Then*

(i) $\dfrac{P_i}{P_j} \geqq \dfrac{p_i/q_i}{p_j/q_j}$ *for* $i < j$, *and*

(ii) P_i *is an increasing function of* p_i.

Proof (by induction on k): For $k = 2$, (*i*) is an equality, while (*ii*) states that $w_i/(w_i + w_j)$ increases in w_i, where $w_i = p_i/q_i$. Assume the theorem is true for $k \geqq m - 1$, and suppose there are $k = m$ coins to start.

Let T denote the toss on which the *first* elimination occurs, and define for $i = 1, \ldots, k$

$$\delta_i = \delta_i^{(T)} = \begin{cases} 1 \text{ if coin } i \text{ is } \textit{not} \text{ eliminated at } n = T \\ 0 \text{ otherwise.} \end{cases}$$

The joint probability function for $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ is

$$P[\boldsymbol{\delta}] = \frac{\prod_i p_i^{\delta_i} q_i^{1-\delta_i}}{1 - \prod_i p_i - \prod_i q_i} \qquad [10]$$

for $\boldsymbol{\delta}$ such that at least one component of $\boldsymbol{\delta}$ is 1 and at least one component is 0. We assert two things about $P[\boldsymbol{\delta}]$. From [**10**],

$$\frac{P[\delta_1, \ldots, \delta_i = 1, \ldots, \delta_j = 0, \ldots, \delta_k]}{P[\delta_1, \ldots, \delta_i = 0, \ldots, \delta_j = 1, \ldots, \delta_k]} = \frac{w_i}{w_j} \quad \text{for any } i \neq j. \qquad [11']$$

Moreover,

$$P[\delta_1, \ldots, \delta_i = 1, \ldots, \delta_k] \quad \text{is increasing in } p_i, \qquad [11'']$$

since

$$P[\delta_1, \ldots, \delta_i = 1, \ldots, \delta_k] = p_i \prod_{j\neq i} p_j^{\delta_j} q_j^{1-\delta_j} \Big/ \left(1 - p_i \prod_{j\neq i} p_j - q_i \prod_{j\neq i} q_j\right)$$

$$= \left(\prod_{j\neq i} p_j^{\delta_j} q_j^{1-\delta_j}\right) p_i \Big/ \left[\left(1 - \prod_{j\neq i} q_j\right) - p_i\left(\prod_{j\neq i} p_j - \prod_{j\neq i} q_j\right)\right]$$

which is increasing in p_i.

Now fix $i < j$ and write

$$P[N = N_i] = \sum_{\substack{\delta_i = 1 \\ \Pi\delta_\ell = 0}} P[N = N_i|\boldsymbol{\delta}]\, P[\boldsymbol{\delta}] \qquad [12]$$

where the summation is over all $\boldsymbol{\delta}$ with $\delta_i = 1$ and $\Pi\delta_\ell = 0$. For $\delta_i = \delta_j = 1$ we have

$$P[N = N_i|\boldsymbol{\delta}] \geqq \frac{w_i}{w_j} P[N = N_j|\boldsymbol{\delta}]$$

by the inductive hypothesis (*i*), since the remainder of the contest $(n > T)$ is equivalent to a new contest with $k < m$ coins. For $\delta_i = 1$, $\delta_j = 0$

$$P[N = N_i|\boldsymbol{\delta}] \geqq P[N = N_j|\boldsymbol{\delta}(ij)]$$

by inductive hypothesis (*ii*), where $\boldsymbol{\delta}(ij)$ denotes $(\delta_1, \ldots, \delta_i = 0, \ldots, \delta_j = 1, \ldots, \delta_k)$. This is because the remainder of the contest given $\boldsymbol{\delta}$ differs from the remainder of the contest given $\boldsymbol{\delta}(ij)$ only by the substitution of coin j for coin i. Hence by [**11′**]

$$P[N = N_i] \geqq \frac{w_i}{w_j} \sum_{\substack{\delta_j = 1 \\ \Pi\delta_\ell = 0}} P[N = N_j|\boldsymbol{\delta}]\, P[\boldsymbol{\delta}] = \frac{w_i}{w_j} P[N = N_j]$$

which establishes (*i*).

For (*ii*) we note that, in [**12**], $P[N = N_i|\delta_1, \ldots, \delta_i = 1, \ldots, \delta_k]$ is increasing in p_i by the inductive hypothesis (*ii*), and this together with [**11″**] completes the proof.

To obtain [8] we proceed as follows. By Wald's lemma,

$$E[X_1^{(N)} - X_2^{(N)}] = (p_1 - p_2)\, E[N].$$

Also, suppressing the superscript of $X_i^{(N)}$,

$$\begin{aligned}
E[X_1 - X_2] &= E[X_1 - X_2 | X_1 > X_2 \geqq X_3]\, P[X_1 > X_2 \geqq X_3] \\
&\quad + E[X_1 - X_2 | X_1 > X_3 > X_2]\, P[X_1 > X_3 > X_2] \\
&\quad + E[X_1 - X_2 | N = N_3]\, P[N = N_3] \\
&\quad + E[X_1 - X_2 | X_2 > X_1 \geqq X_3]\, P[X_2 > X_1 \geqq X_3] \\
&\quad + E[X_1 - X_2 | X_2 > X_3 > X_1]\, P[X_2 > X_3 > X_1] \qquad [13] \\
&= r(P[N = N_1] - P[N = N_2]) \\
&\quad + E[X_3 - X_2 | X_1 > X_3 > X_2]\, P[X_1 > X_3 > X_2] \\
&\quad + E[X_1 - X_2 | N = N_3]\, P[N = N_3] \\
&\quad + E[X_1 - X_3 | X_2 > X_3 > X_1]\, P[X_2 > X_3 > X_1].
\end{aligned}$$

As a first approximation we ignore the final two summands in [**13**]. In the special case $p_2 = p_3 = p$,

$$P[X_1 > X_3 > X_2] \simeq \frac{1}{2} P[N = N_1]$$

and

$$P[N = N_2] = \frac{1}{2}(1 - P[N = N_1]),$$

yielding

$$E[N] \simeq \frac{r(3P[N = N_1] - 1) + E[X_3 - X_2 \mid X_1 > X_3 > X_2]\, P[N = N_1]}{2(p_1 - p_2)}.$$

To evaluate $E[X_3^{(N)} - X_2^{(N)} \mid X_1^{(N)} > X_3^{(N)} > X_2^{(N)}]$, note that for a *fixed* sample size n, $X_3^{(n)} - X_2^{(n)}$ is distributed approximately normally with zero mean and variance $2npq$, so that if $P[X_1^{(n)} > X_3^{(n)} > X_2^{(n)}]$ is near 1,

$$E[X_3^{(n)} - X_2^{(n)} \mid X_1^{(n)} > X_3^{(n)} > X_2^{(n)}] \simeq E[X_3^{(n)} - X_2^{(n)} \mid X_3^{(n)} > X_2^{(n)}] \simeq 2(npq/\pi)^{1/2}.$$

We heuristically replace n by $E[N]$, and solve for $E[N]$ in the resulting quadratic equation

$$E[N] \simeq \frac{r(3P[N = N_1] - 1) + 2(pq/\pi)^{1/2}\, P[N = N_1] \sqrt{E[N]}}{2(p_1 - p_2)},$$

which leads to [8]. We expect the approximations and the final heuristic step to be justified asymptotically as $r \to \infty$. Similar reasoning leads to [8′]. The accuracy of [8] and [8′] remains to be investigated.

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