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Oscillatory singular integrals and harmonic analysis on nilpotent groups

(Calderón–Zygmund kernels/nonautomorphic dilations)

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ABSTRACT **Several related classes of operators on nilpotent Lie groups are considered. These operators involve the following features: (*i*) oscillatory factors that are exponentials of imaginary polynomials, (*ii*) convolutions with singular kernels supported on lower-dimensional submanifolds, (*iii*) validity in the general context not requiring the existence of dilations that are automorphisms.**

Section 1. Introduction

Our purpose here is to announce results in several related areas: (*i*) operators defined by singular kernels that are products of an ordinary Calderón–Zygmund kernel, and an oscillatory factor that is the exponential of a purely imaginary polynomial; (*ii*) singular integral transformations on general nilpotent Lie groups (without any assumptions of existence of underlying automorphic dilations on these groups), (*iii*) singular integral transforms on these groups with kernels carried on lower-dimensional varieties.

The oscillatory integrals we consider arose to begin with when one analyzed singular integrals on nilpotent groups, utilizing the abelian Fourier transform on the center or, more generally, when exploiting variants of "twisted convolution." This technique has been used by Ricci (1) and by Geller and Stein (2). In the second of these papers, this idea was developed to study singular integrals carried on some hyperplanes of the Heisenberg group. The oscillatory integrals that occurred were then generalized by Phong and Stein (3, 4) in connection with the study of singular Radon transforms. Certain related results for two-step nilpotent groups had been published by Strichartz (5), and some recent progress has been made by Müller (6), by Christ, (7), and by Greenleaf (8). Of course a model for much of this has been the previously established theory in the abelian case [see, e.g., Stein and Wainger (9)]. Our work here may be viewed as a logical conclusion and unification of several lines of these developments. One of the principal conclusions that arises is that a wide variety of operators that were hitherto studied only on homogeneous nilpotent groups have natural extensions to all (simply-connected) nilpotent Lie groups.

Section 2. Oscillatory Integrals

We begin by formulating a result on additive group on $\mathbf{R}^m$. We let K denote a homogeneous function of degree $-\mu$, with $\mu \leq m$, which is smooth away from the origin, and when $\mu = m$ we also assume that the mean value of K on the unit sphere vanishes. We shall denote by $P(x, y)$ a real polynomial on $\mathbf{R}^m \times \mathbf{R}^m$ of degree $\leq d$, and we consider the operator T given by

$$(Tf)(x) = \text{P.V.} \int_{\mathbf{R}^m} e^{iP(x,y)} K(x - y) f(y) dy \qquad [1]$$

initially defined for $f \in C_0^\infty(\mathbf{R}^m)$.

THEOREM 1. *Suppose* d *is fixed. Then there is a positive* $\varepsilon = \varepsilon(\mathrm{d})$, *so that the following holds: if* $\mathrm{P(x, y)}$ *is a real polynomial of degree* $\leq \mathrm{d}$, *and* P *is not of the form* $\mathrm{P_0(x) + P_1(y)}$, *while* K *is homogeneous of degree* $-\mu$, *with* $\mathrm{m} - \varepsilon < \mu < \mathrm{m}$, *then the operator* [1] *is extendable to a bounded operator on* $\mathrm{L^2(\mathbf{R}^m)}$ *to itself.*

With certain modifications the method of proof of *Theorem 1* also yields the following corollaries:

COROLLARY 1. *Suppose* $1 < \mathrm{p} < \infty$ *and that* K *is homogeneous of degree* $-\mathrm{m}$. *Then the operator* [1] *is extendable to a bounded operator on* $\mathrm{L^p(\mathbf{R}^m)}$ *to itself, whose bounds depends on* p, K, *and the degree of* P *but can otherwise be taken independent of* P.

COROLLARY 2. *The statement of* Corollary 1 *remains valid if we replace* $\mathrm{K(x - y)}$ *in* [1] *by a distribution kernel* $\mathrm{K(x, y)}$ *so that* (a) *the operator* $\mathrm{T_0}$ *given by* $\langle \mathrm{T_0 f, g} \rangle = \iint \mathrm{K(x, y) f(y) g(x) dy dx}$ *is extendable to a bounded operator on* $\mathrm{L^2(\mathbf{R}^m)}$ *and* (b) *away from the diagonal,* K *is a* $\mathrm{C^1}$ *function that satisfies* $|\mathrm{K(x, y)}| \leq \mathrm{A/|x - y|^m}$, $|\nabla_x \mathrm{K(x, y)}| + |\nabla_y \mathrm{K(x, y)}| \leq \mathrm{A/|x - y|^{m+1}}$.

These are analogous results that are valid when the additive structure of $\mathbf{R}^m$ is replaced by that of any nilpotent Lie group N that is homogeneous. We formulate this as follows. Let n be a nilpotent Lie algebra of dimension m and suppose $\{X_i\}$ $1 \leq i \leq m$ is a basis of n. We suppose we are given positive exponents a_i, $1 \leq i \leq m$ and mappings $\delta_t : n \to n$, defined by $\delta_t(X_i) = t^{a_i} X_i$, $t > 0$, which in this section we assume are automorphisms of n. We identify N with n (and hence with $\mathbf{R}^m$) via the exponential map and write δ_t for the corresponding automorphism of the group N. We shall assume that the kernel K as a function on N $(=\mathbf{R}^m)$ is homogeneous of degree $-\mu$ in the sense that $K(\delta_t(x)) = t^{-\mu} K(x)$, with $\mu \leq a$, where $a = a_1 + a_2 \ldots + a_m$. We also suppose that K is smooth away from the origin, and in the critical case (corresponding to $\mu = a$, which arises in the analogue of *Corollary 1*) we assume that the mean value of K on the unit sphere vanishes. The substitute for [1] is then

$$(Tf)(x) = \text{P.V.} \int_{\mathbf{R}^m} e^{iP(x,y)} K(y^{-1} \cdot x) f(y) dy \qquad [1']$$

with $y^{-1} \cdot x$ the corresponding group operations.

COROLLARY 3. *With the modifications described above, the statement of* Theorem 1 *and* Corollary 1 *remain valid when we replace* [1] *by* [1′].

Remarks: (*i*) The proofs of these results involve a combination of four ideas: (*a*) an induction on the degree of P, (*b*)

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certain inequalities for polynomials, (*c*) the method of stationary phase, and (*d*) the reduction of the boundedness of an operator A on L^2 to that of A^*A. The reader may also consult refs. 4 and 10 where the use of such techniques is illustrated in several special cases; in this connection, see also ref. 6. We have been informed by M. Christ (personal communication) that he has also obtained the case $m = 1$, $p = 2$, of *Corollary 1*.

(*ii*) *Corollary 1* (and its analogue for nilpotent Lie groups) can be given a wider scope as we shall see below.

(*iii*) Necessary and sufficient conditions for the boundedness of the operator T_0 arising in *Corollary 2* (i.e., with $P \equiv 0$) have been given by David and Journé (11).

Section 3. Singular Convolution Operators on Nilpotent Lie Groups

As mentioned above oscillatory integrals have been used to prove the boundedness of a variety of convolution operators on homogeneous nilpotent groups. However the theorems above lead directly to certain results where the homogeneity (i.e., the structure of dilations that are automorphisms) is not relevant. A simple example arises for the Heisenberg group $H^1 = \{(x, y, z)\}$, with multiplication law $(x, y, z)\cdot(x', y', z') = (x + x', y + y', z + z' + 2(x'y - xy'))$. Consider the curve $t \to \gamma(t) = (t, t^a, t^b)$, with a and b positive integers, but where we do not assume that $b = a + 1$. Then by the representation theory and Plancherel formula for the group H^1 one can reduce the L^2 boundedness of the "Hilbert transform"

$$T(f)(u) = \text{P.V.} \int_{-\infty}^{\infty} f(u\cdot\gamma(t))\,\frac{dt}{t}, \qquad u \in H^1,$$

to *Corollary 1* above in the case $P(x, y) = \lambda\{(x - y)^b + 4x(x - y)^a - 2(x - y)^{a+1}\}$, $-\infty < \lambda < \infty$.

This observation, together with the general nature of *Theorem 1* and its corollaries, leads one to envisage the possibility that many of the standard results of harmonic analysis on $\mathbf{R}^m$ may in some way be "transferred" to arbitrary nilpotent Lie groups (where no automorphic dilation structure may be present). To this we shall now turn.

As in *Section 2* above, N is a (simply connected) nilpotent Lie group that is identified with $\mathbf{R}^m$ via the exponential map. For a fixed basis of $\{X_i\}$ of its Lie algebra n and fixed positive exponents a_i, we let δ_t denote the linear mappings of n determined by $\delta_t(X_i) = t^{a_i}X_i$ and also denote by δ_t the corresponding mappings of N to itself. *However here we no longer assume that the* δ_t *are automorphisms.* We let $|x|$ denote a norm function with respect to δ_t (i.e., $x \to |x|$ is smooth on $N/\{0\}$, positive, and $|\delta_t x| = t|x|$, all $t > 0$). Note, however, that here the extended triangle inequality $|x\cdot y| \leq c\{|x| + |y|\}$ fails in general. We let K denote a Calderón–Zygmund kernel with respect to the homogeneities δ_t; i.e., $K(\delta_t x) = t^{-a}K(x)a = a_1 + \ldots a_m$, K is smooth away from the origin and the mean value of K vanishes. We define the operator T by

$$T(f) = f*K = \lim_{\varepsilon\to 0} \int_{|y|\geq\varepsilon} f(xy^{-1})K(y)dy,$$

whenever $f \in C_0^\infty(N)$.

THEOREM 2. *The operator* T *defined above is extendable to a bounded operator* $L^p(N)$ *to itself, with* $1 < p < \infty$.

There is also a corresponding result for the maximal function in this setting. We define M by

$$M(f)(x) = \sup_{0<r<\infty} \frac{1}{r^a} \int_{|y|\leq r} |f(xy^{-1})|dy. \qquad [2]$$

THEOREM 3. *The mapping* $f \to M(f)$ *is bounded on* $L^p(N)$ *to itself, when* $1 < p \leq \infty$.

Section 4. Singular Integrals on Submanifolds of Nilpotent Groups

The proofs of *Theorems 2* and *3* require two ideas. First, that we pass from arbitrary nilpotent Lie groups to homogeneous ones, in particular the "free" nilpotent groups (an idea that is implicit in ref. 12), and, second, that we consider operators whose kernels are carried on suitable submanifolds. These considerations lead in fact to an extension of some of the results described in the previous sections and at the same time to their unification in a more general statement. We shall formulate only the generalization of *Theorem 2* to this context, although *Theorem 3* can be similarly extended.

As in the previous section, N denotes an arbitrary simply connected nilpotent Lie group and δ_t denotes a family of dilations that are not necessarily automorphisms.

We begin by fixing an analytic (open) submanifold S of $N/\{0\}$. While we do not assume that S is connected, we do require that it have finitely many components. We also assume that S is homogeneous in the sense that $\delta_t(S) = S$, all $t > 0$. We denote by $d\sigma$ the induced (Lebesgue) measure on S. We fix a function K on S that is smooth and so that K has compact support when restricted to $S \cap \{|x| = 1\}$. We also assume that the measure $K(x)d\sigma(x)$ (which is supported on S) is homogeneous of the critical degree $-a$ in the sense that

$$\int_N \phi(x)K(x)d\sigma(x) = \int_N \phi(\delta_t x)K(x)d\sigma(x), \qquad \text{all} \quad t > 0,$$

for all $\phi \in C_0^\infty(N)$ that vanish near the origin. We assume also that $K(x)d\sigma(x)$ has vanishing mean value in the sense that $\int_{\alpha<|x|<\beta} K(x)d\sigma(x) = 0$, for some fixed $0 < \alpha < \beta$.

Finally we take $P(x, y)$ to be a real polynomial on $N \times N$. The assumptions we have made on K allow us to define the operator T (as a principal-value integral) by

$$(Tf)(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon\leq|y|} f(xy^{-1})e^{iP(x,y)}\,K(y)d\sigma(y) \qquad [3]$$

whenever $f \in C_0^\infty(N)$.

THEOREM 4. *Assume* S *is connected. Then* T *defined by* **[3]** *is extendable to a bounded operator of* $L^p(N)$ *to itself, for* $1 < p < \infty$, *whose norm depends only on* p, K, *and the degree of* P.

The theorem as stated above applies only vacuously to the case when S is one-dimensional because of the assumption that S is connected. When S is not connected, a necessary condition is that broadly speaking each component of S generates the same subgroup of N. Precise sufficient conditions when S is not connected may be formulated as follows:

ADDENDUM. (i) *Assume first that the dilations described above are automorphisms of* N, *that the polynomial* P *in* **[3]** *vanishes identically, and that each component of* S *generates the same subgroup of* N. *Then the operator* **[3]** *is extendable to a bounded operator on* L^p. (ii) *Alternatively, assume that* S *is one-dimensional but* $S \cup \{0\}$ *is real analytic near the origin* (*i.e.,* $S \cup \{0\}$ *is a polynomial curve*). *Here we need not assume that the underlying dilations are automorphisms. Then the conclusions of* Theorem 4 *still hold with general polynomials* P.

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