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## Partial differential equation transform — Variational formulation and Fourier analysis

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### Abstract

Nonlinear partial differential equation (PDE) models are established approaches for image/signal processing, data analysis and surface construction. Most previous geometric PDEs are utilized as low-pass filters which give rise to image trend information. In an earlier work, we introduced mode decomposition evolution equations (MoDEEs), which behave like high-pass filters and are able to systematically provide intrinsic mode functions (IMFs) of signals and images. Due to their tunable time-frequency localization and perfect reconstruction, the operation of MoDEEs is called a PDE transform. By appropriate selection of PDE transform parameters, we can tune IMFs into trends, edges, textures, noise etc., which can be further utilized in the secondary processing for various purposes. This work introduces the variational formulation, performs the Fourier analysis, and conducts biomedical and biological applications of the proposed PDE transform. The variational formulation offers an algorithm to incorporate two image functions and two sets of low-pass PDE operators in the total energy functional. Two low-pass PDE operators have different signs, leading to energy disparity, while a coupling term, acting as a relative fidelity of two image functions, is introduced to reduce the disparity of two energy components. We construct variational PDE transforms by using Euler-Lagrange equation and artificial time propagation. Fourier analysis of a simplified PDE transform is presented to shed light on the filter properties of high order PDE transforms. Such an analysis also offers insight on the parameter selection of the PDE transform. The proposed PDE transform algorithm is validated by numerous benchmark tests. In one selected challenging example, we illustrate the ability of PDE transform to separate two adjacent frequencies of  $\sin(x)$  and  $\sin(1.1x)$ . Such an ability is due to PDE transform's controllable frequency localization obtained by adjusting the order of PDEs. The frequency selection is achieved either by diffusion coefficients or by propagation time. Finally, we explore a large number of practical applications to further demonstrate the utility of proposed PDE transform.

### Keywords

Mode decomposition; Evolution equations; Anisotropic diffusion; Total variation; High-pass filter; Partial differential equation transform

### 1 Introduction

Processing and analysis of signal, sound, data, image, surface or video frame have been an important issue in a general category of subjects, including computer vision, face recognition, feature detection, medical diagnosis, remote sensing, machine vision, and

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artificial intelligence.<sup>13,20,47,51</sup> Signal, image, surface and data processing are realized via a family of operations and algorithms, such as filtering, smoothing, denoising (or deblurring), enhancement, segmentation, reconstruction, pattern recognition, surface analysis and construction. The advance in signal, image, surface and data analysis also benefits many related fields, such as optical sorting, automatic control, augmented reality, robotics, sonar, radar, communication, navigation, biomolecular modeling and a variety of imaging modalities. There are a variety of algorithms for signal, image, surface and data processing. The essential idea is the use of filters, such as linear, nonlinear, active, passive, low-pass, high-pass, Fourier, wavelet, Chebyshev, Gaussian, Kalman, Wiener and conjugate filters.<sup>27,52,53,69</sup> Decomposition of the original signal, image, surface and data into various modes according to their mathematical features and frequency distributions is a key procedure in all the aforementioned applications. Usually, the subsequent analysis, also called secondary processing, on individual mode components enables us to achieve our goal of signal, image, surface and data analysis. Therefore, mode decomposition is a fundamental process in information processing and data analysis.

Fourier or spectral analysis is a classical technique for mode decomposition and remains to be a powerful tool in signal, image and data processing.<sup>13,36,42,51,55</sup> However, Fourier analysis is not suitable for studying data of non-stationary nature, and it may be difficult to choose a suitable window size to satisfy the conflicting requirements of localizing an event in time and resolving its frequency distribution. Moreover, the method is not data adaptive. In many applications in signal processing, one usually desires the information of detailed position and momentum relation, which is also lack in Fourier spectral analysis. Besides, Fourier analysis is less ideal for applications in image processing when localized visual features like edges need to be detected. Most importantly, when the signal or image involves abundantly many modes, the subsequent analysis or secondary processing becomes awkward if it is not fully automatic.

Wavelet transform is another popular technique for mode decomposition and shows a great power in analyzing unsteady and non-stationary data.<sup>13,17,20,28,33,39,40,51,60</sup> The fundamental idea behind wavelet filters is to analyze data according to their scales. In this sense, wavelets realize the locality by resorting to localized multiscale bases, which are usually more appropriate than sines and cosines in the Fourier basis, and decomposing signal into various sub-bands. Since individual wavelet functions can be localized in space while Fourier sine and cosine functions are not, many functions become “sparse” in wavelet bases. By dilation and translation, wavelet transform usually yields much fewer number of non-trivial sub-bands than the number of Fourier modes, and thus results in many useful applications such as data compression, detecting features in images, and removing noise from time series. Wavelet methods thus provide an efficient approach for image mode decomposition and secondary processing. Most modes separated by wavelet transform are usually closely related to the edge information or global feature of the whole image. Therefore, wavelets are efficient for image compression. Overall, the wavelet methods are more adaptive than Fourier analysis. However, wavelet analysis is basically a linear analysis and suffers from many limitations. The down sides include uniformly poor resolution, sometimes counter-intuitive interpretation, and non-data adaptive nature as the same wavelet basis is used to analyze all the data.<sup>30</sup> Moreover, wavelet methods inherit many drawbacks of the Fourier analysis since many commonly used wavelets were originated from Fourier analysis.

Witkin introduced the diffusion equation for image denoising in 1983.<sup>63</sup> The central idea is that the evolution of an image under a diffusion operator is formally equivalent to the standard Gaussian low-pass filter, which is commonly used for image denoising. In 1990, Perona and Malik proposed an anisotropic diffusion equation<sup>44</sup> which is able to remove

noise without smearing the image edges too much. In the anisotropic diffusion equation, the diffusion coefficient is replaced by a function of image gradients such that the diffusion coefficient is small at the image edges.<sup>11,44,50,59,61</sup> Therefore, the Perona-Malik equation is nonlinear. Despite of better image edge protection, the nonlinear anisotropic diffusion operator may break down when the gradient generated by noise is comparable to image edges and features.<sup>8,41</sup> Application of a pre-convolution with a smoothing function to the image can practically alleviate the instability and reduce grey scale oscillation, but the image quality is often degraded. One alternative solution introduced by Wei<sup>59</sup> is to statistically discriminate noise from image edges by a measure based on the local statistical variance of the image. Such a local statistical variance based edge-stopping algorithm works very well for image restoration. An advantage of this anisotropic diffusion approach is that geometric properties, such as curvature, can be easily embedded in partial differential equation (PDE) operators to achieve desirable effects in image and surface. The level set method devised by Osher and Sethian is a typical example of using the mean curvature flow for image<sup>43,46,48</sup> and surface analysis.<sup>3</sup> For this reason, this class of nonlinear evolution equations is also called geometric PDEs or geometric flows. Mathematically, the study of geometric PDEs has enormous impact on geometry analysis, manifold theory, topology, geometric measure theory, and certainly PDEs. In the past two decades, PDE based image processing approaches have raised a strong interest in the mathematical community and have opened new approaches for image de-noising, enhancement, edge detection, restoration, segmentation, etc.<sup>1,3,4,7,25,32,35,45,47,49,64</sup> Recently, geometric PDEs have been utilized as a means for the molecular surface construction of macromolecules<sup>3</sup> and the multiscale modeling of nano-bio systems.<sup>15,16,58</sup>

Variational method is a most powerful mathematical tool. Total variation based edge-preserving image restoration models were pioneered by Rudin, Osher, and Fatemi.<sup>46</sup> The variational PDE technique has been widely used in numerous applications such as image segmentation,<sup>6,12,67</sup> restoration,<sup>46</sup> compression,<sup>19</sup> and image inpainting.<sup>10</sup> The growing impact of variational PDE techniques in image processing is mainly due to their capability of controlling the geometrical features of images and data. The essential idea is that signals and images with extra and possibly spurious components have a relatively large variation or gradient. As such, image processing can be formulated as a problem of minimizing the total energy defined as a functional of the gradient of the image, while preserving important image contents such as edges. The minimization of the total energy by the Euler-Lagrange equation leads to a mean curvature term which helps retain sharp edges in image restoration. Central to most variational PDE methods, the goal of the total energy variation is to reconstruct an image with the best fidelity and the least noise. Regularization procedures and PDE analysis are usually employed in total variation models.<sup>9</sup>

Both the Perona-Malik equation<sup>44</sup> and earlier total variation models employ second order nonlinear PDEs for image or surface analysis. The Willmore flow, proposed in 1920s, is a fourth order geometric PDE and has also been used for surface analysis. In 1999, Wei introduced the first family of arbitrarily high order nonlinear PDEs to more efficiently remove image noise in edge-preserving image restoration.<sup>59</sup> The same research group also proposed an arbitrarily high order geometric PDE for the surface formation and evolution of protein and other molecules.<sup>2</sup> In the past decade, high order nonlinear PDEs, particularly fourth order nonlinear PDEs, have attracted much attention in image analysis.<sup>2,9,11,23,24,38,54,59,68</sup> Compared with second order PDEs, the higher order PDEs are able to suppress high frequency components, including noise, at fast rates. Mathematical analysis of fourth order nonlinear PDEs in Sobolev space was carried out by Bertozzi and Greer,<sup>5,23,24</sup> who proved the existence and uniqueness of the solution to a case with  $H^1$  initial data and a regularized operator. Xu and Zhou<sup>65</sup> performed similar analysis. Recently, Jin and Yang have proved the existence of the strong solution of Wei's fourth order

equation, which is in fact mathematically different from other fourth order PDEs.<sup>34</sup> In general, however, high order PDEs are subject to strict stability constraints in their numerical solutions. Witelski and Bowen designed alternating-direction implicit (ADI) schemes for high order nonlinear PDEs.<sup>62</sup> ADI and other implicit approaches are particularly necessary in signal processing.

Image processing PDEs of both low order type and high order type are mostly designed to function as nonlinear low-pass filters. In 2002, Wei and Jia<sup>61</sup> introduced PDE based band-pass or high-pass filters for image edge detection. Two evolution PDEs are coupled via relative fidelity terms and the difference of the solutions provides image edges. PDE based high-pass filters work because when two PDE low-pass filters evolve at dramatically different speeds, the difference of two low-pass PDE operators gives rise to a band-pass filter. The speeds of evolution in the coupled PDEs are controlled by the appropriate selection of their diffusion coefficients. In the extreme case, one can set one of the PDE operator to an identity operator, i.e., setting the diffusion and fidelity coefficients to zero. Consequently, one has a PDE based all-pass filter. The difference between an all-pass filter and a low-pass filter is a high-pass filter.<sup>61</sup> Nonlinear PDE-based edge detection operators have been shown to work extremely well for images with large amount of textures, compared to classical Sobel, Prewitt, and Canny operators.<sup>52,61</sup>

Although nonlinear PDE based edge detector operators have been introduced for near a decade, it was not clear whether PDEs can be utilized to perform all tasks in the image processing for a long time. More specifically, it was not clear whether one can devise PDEs to perform a full-scale mode decomposition. For all practical purposes, the secondary processing becomes easy when all the mode components are available. Therefore, mode decomposition is the real fundamental issue. This question was not answered until our recent introduction of a family of mode decomposition evolution equations (MoDEE),<sup>56</sup> which are able to generate various mode components for secondary processing. Similar to wavelets, MoDEEs have controllable time-frequency localization and allow perfect reconstruction, and are thus also called a PDE transform.<sup>56</sup> All of the important building blocks for the PDE transform were developed in our earlier work, i.e., arbitrarily high order PDE filters<sup>59</sup> and PDE based band-pass or high-pass filters.<sup>61</sup> The PDE transform requires the re-initialization of the initial value in subsequent applications of the PDE operator during the mode decomposition, a procedure inspired by our mode decomposition via iterative filtering.<sup>37,57</sup> The PDE transform is able to generate physically meaningful modes, also referred to as functional modes. By functional modes, we mean the components which share same band of frequency as well as same category, i.e., trend, edge, texture, noise etc. Obviously, functional modes differ from the pure frequency modes obtained by the Fourier analysis. Based on the functional modes obtained by the PDE transform, secondary processing, or post-processing, can be carried out to achieve desirable processing tasks, such as trend estimation, edge detection, feature extraction, enhancement and denoising, texture analysis, segmentation, pattern recognition, etc. In fact, the primary and secondary processing are unified in the sense that the same set of PDE transform operators are used with different parameter settings.<sup>56</sup>

The objective of the present work is to explore the variational formulation of PDE transforms, to analyze their spectral features and to consider their biomedical applications. There is no doubt that variational approaches are essential to fundamental sciences, such as physics, mechanics, electrodynamics, chemistry, and biology. As a mathematical tool, variational methods play crucial roles in differential geometry, topology, algebra, PDE, etc. Total variation models<sup>46</sup> are the most important ones in image analysis. Therefore, it is useful to investigate the variation formulation of our PDE transform. The present variational PDE transform has to incorporate two unique aspects, namely, high order nonlinear PDE

operators and dynamical coupling of two evolution PDEs. Indeed, the proposed variational formulation of the PDE transform is different from our earlier PDE transforms.<sup>56</sup> Additionally, the Fourier analysis is a valuable approach for the in-depth understanding of the proposed PDE transform. In particular, due to the generality, the proposed PDE transforms appear quite complicated with many parameters. The selection of appropriate PDE transform parameters is nontrivial. Fortunately, the present Fourier analysis sheds light on the behavior of the present PDE transform and thus simplifies the parameter selection. By adjusting the highest order of the PDEs, the proposed PDE transform admits controllable frequency locations, which is desirable in extracting functional modes and closely adjacent frequencies. Finally, the need of technological innovation in biomedical and biological sciences is one of the original motivations for us to design PDE transforms. It is always worthwhile to demonstrate the utility and illustrate the robustness of our new method for practical applications.

The rest of the present paper is organized as follows. Section II is devoted to the theory and formulation of PDE transforms. We first review the essence of PDE based low-pass filters and PDE based band/high-pass filters. We then discuss one form of our earlier PDE transforms. The variational formulation of PDE transforms is proposed in Section II.D. As mentioned, these new variational PDE transforms have a different structure compared to that of our earlier PDE transforms. The Fourier analysis is carried out in Section II.E. To simplify our discussion, we consider a high order linear evolution PDE in one spatial dimension so that the analytical solution can be obtained. Numerical tests and validations are presented in Section III. High order PDE transforms are applied to separate two modes with very close frequencies, which is regarded very difficult using other mode decomposition methods. In Section IV, PDE transforms are applied to enhance images obtained with various medical imaging modalities. Such applications include the detection of early lung cancers, noise removing in optical image obtained by the retinal functioning imager, screening of breast cancer using diffuse optical tomography, and image enhancement on electron microscope photography of neuron cells. This paper ends with some concluding remarks.

## II Theory and formulation

This section discusses the rationale and variational formulation of PDE transforms. To establish notation and illustrate concepts, we start by a brief review of arbitrarily high order PDE based nonlinear low-pass filters introduced by Wei.<sup>59</sup> The central idea is the construction of PDE based nonlinear high-pass filters introduced by Wei and Jia.<sup>61</sup> We construct PDE transforms by the combination of arbitrarily high order PDEs and PDE based high-pass filters. The performance of PDE transforms depends crucially on the use of high-order PDEs in the high-pass filters, which gives rise to the desired frequency localization. In this section, it is also shown that while the exact form of the PDE transform introduced in Ref.<sup>56</sup> can not be derived in a variational formalism, a similar family of PDE transforms can be obtained. The Fourier analysis of a linearized high-order PDE sheds light on the performance of the proposed PDE transform.

### II.A High order PDE based low-pass filters

High order evolution PDEs are widely used in mathematical modeling of material, interface, and curvature.<sup>2</sup> For example, the Willmore flow is often used in the morphological description of membrane. As a high order generalization of the Perona-Malik equation,<sup>44</sup> Wei introduced the first family of arbitrarily high order nonlinear PDEs for image processing in 1999.<sup>59</sup>

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = \sum_q \nabla \cdot [d_q(u(\mathbf{r}), |\nabla u(\mathbf{r}), t) \nabla \nabla^{2q} u(\mathbf{r}, t)] + e(u(\mathbf{r}), |\nabla u(\mathbf{r}), t), \quad q=0, 1, 2, \dots \quad (1)$$

which has since been applied to image denoising and restoration by many researchers.<sup>21,22,38,59</sup> In Eq. (1),  $u(\mathbf{r}, t)$  is the image function,  $d_q(u(\mathbf{r}), |\nabla u(\mathbf{r}), t)$  and  $e(u(\mathbf{r}), |\nabla u(\mathbf{r}), t)$  are edge sensitive diffusion coefficients and enhancement operator respectively. The Perona-Malik equation is recovered at  $q=0$  and  $e(u(\mathbf{r}), |\nabla u(\mathbf{r}), t) = 0$ . As in the original Perona-Malik equation, the hyper-diffusion coefficients  $d_q(u(\mathbf{r}), |\nabla u(\mathbf{r}), t)$  in Eq. (1) can be chosen in many different ways. For instance, one can set

$$d_q(u(\mathbf{r}), |\nabla u(\mathbf{r}), t) = d_{q0} \exp \left[ -\frac{|\nabla u|^2}{2\sigma_q^2} \right], \quad (2)$$

where the values of constants  $d_{q0}$  depend on the noise level, and  $\sigma_0$  and  $\sigma_1$  are chosen as the local statistical variance of  $u$  and  $\nabla u$

$$\sigma_q^2(\mathbf{r}) = \overline{|\nabla^q u - \overline{\nabla^q u}|^2} \quad (q=0, 1). \quad (3)$$

The notation  $\overline{Y(\mathbf{r})}$  above denotes the local average of  $Y(\mathbf{r})$  centered at position  $\mathbf{r}$ . In this algorithm, the statistical measure based on the variance is important for discriminating image edges from noise. As such, one can bypass the image preprocessing, i.e., the convolution of the noise image with a test function or smooth mask.

Recently, we have proposed arbitrarily high order geometric PDEs for surface formation and evolution with application in the surface generation of proteins and other biomolecules,<sup>2</sup>

$$\frac{\partial S}{\partial t} = (-1)^q \sqrt{g(|\nabla \nabla^{2q} S|)} \nabla \cdot \left( \frac{\nabla(\nabla^{2q} S)}{\sqrt{g(|\nabla \nabla^{2q} S|)}} \right) + P(S, |\nabla S|), \quad (4)$$

where  $S(\mathbf{r}, t)$  is a hypersurface function,  $g(|\nabla \nabla^{2q} S|) = 1 + |\nabla \nabla^{2q} S|^2$  is the generalized Gram determinant and  $P$  is a generalized force term, including the potential interactions which are important to the biomolecular surface formation.<sup>2</sup> When  $q=0$  and  $P=0$ , Eq. (4) reduces to the mean curvature flow equation used in our earlier formulation of minimal molecular surfaces,<sup>2</sup> while when  $q=1$  and  $P=0$ , it is a surface diffusion flow.<sup>2</sup> Equation (4) can also be regarded as a variation of Wei's earlier arbitrarily high order PDE (1). The molecular surface generated by Eq. (4) has a distinct morphology.<sup>2</sup>

There has been considerable interest in the high order nonlinear PDE filtering in the past decade.<sup>18,21,22,38</sup> The well-posedness of Eq. (1) was analyzed in terms of the existence and uniqueness of the solution by many researchers.<sup>5,23,24,34,65</sup> In fact, unlike other high-order nonlinear PDEs, Eq. (1) was not derived from a variation formulation. The mathematical properties of the generalized Perona-Malik equation differ from those of other high order PDEs.<sup>34</sup>

## II.B Nonlinear PDE based band-pass and high-pass filters

Image edge detection and texture extraction require the use of high-pass filters. There are a variety of high-pass filters, including the Sobel, Prewitt, Canny operators, and discrete singular convolution edge detector.<sup>29</sup> To construct PDE based edge detectors, Wei and Jia introduced a pair of weakly coupled nonlinear evolution equations<sup>61</sup>

$$u_t(\mathbf{r}, t) = F_1(u, \nabla u, \nabla^2 u, \dots) + \varepsilon_u(v - u) \quad (5)$$

$$v_t(\mathbf{r}, t) = F_2(v, \nabla v, \nabla^2 v, \dots) + \varepsilon_v(u - v) \quad (6)$$

where  $u(\mathbf{r}, t)$  and  $v(\mathbf{r}, t)$  are scalar fields,  $\varepsilon_u$  and  $\varepsilon_v$  are coupling strengths. In the equation  $F_1$  and  $F_2$  are general nonlinear diffusion operators, and can be chosen as the Perona-Malik operator:  $F_1 = \nabla \cdot d_1(|\nabla u|)\nabla$  and  $F_2 = \nabla \cdot d_2(|\nabla v|)\nabla$ .

The initial values for both scalar fields are chosen to be the same image of interest, i.e.,  $u(\mathbf{r}, 0) = v(\mathbf{r}, 0) = X(\mathbf{r})$ . In the theory of nonlinear dynamics, Eqs. (5) and (6) constitute a synchronization system. In terms of image processing, Eqs. (5) and (6) are both nonlinear PDE based low-pass filters. The coupling terms play the role of relative fidelities. To obtain good image edges, two dynamical systems must evolve on the (dramatically) different time scales, i.e.,  $d_{20} \gg d_{10}$ . Therefore, when we choose  $d_{10} \sim 0$ , we have  $u(\mathbf{r}, t) \sim X(\mathbf{r})$ . As such, the coupling terms become the conventional fidelity. In general, low coupling strengths are used so that  $u$  or  $v$  is mostly modified by the diffusion process during the time evolution and image contrast can be maintained. The image edge was defined as the difference of two diffusion systems<sup>61</sup>

$$w(\mathbf{r}, t) = u(\mathbf{r}, t) - v(\mathbf{r}, t). \quad (7)$$

Eq. (7) behaves like a band-pass filter when  $d_{20} \gg d_{10} \sim 0$ . Without loss of generality, one can set  $d_{10} = 0$ <sup>52</sup> and let

$$w(\mathbf{r}, t) = X(\mathbf{r}) - v(\mathbf{r}, t). \quad (8)$$

where  $v(\mathbf{r}, t)$  is the solution of

$$v_t(\mathbf{r}, t) = F_2(v, \nabla v, \nabla^2 v, \dots) + \varepsilon_v(X - v). \quad (9)$$

Equation (8) works as a PDE based high-pass filter. The nonlinear PDE based edge detector described here has been shown to outperform many other classical edge detectors such as Sobel, Prewitt and Canny operators.<sup>61</sup> An additional important feature of the nonlinear PDE based edge detector is the ability to extract image texture with high frequency details.<sup>52,61</sup>

## II.C Partial differential equation transform (PDE transform)

The band-pass and high-pass filters described in the previous section do not automatically serve as mode decomposition filters. It takes two more components to perform the mode decomposition. First, commonly used second order PDE low-pass filters do not have a good frequency localization. Consequently, they cannot be directly used for mode decomposition, although they may be utilized in the IFD algorithm as shown in our earlier work.<sup>57</sup> To achieve frequency localization, we have introduced the use of the high order nonlinear PDEs in our PDE transforms.<sup>56</sup> In general, we can define two nonlinear PDE filters by using the high order nonlinear PDE operator introduced by Wei<sup>59</sup>

$$\frac{\partial}{\partial t} \begin{pmatrix} u_m \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{m-1} \nabla \cdot d_{uj}(|\nabla u_m|) \nabla^2 j - \varepsilon_{u_m}(|\nabla u_m|) & \varepsilon_{v_n}(|\nabla v_n|) \\ \varepsilon_{u_m}(|\nabla u_m|) & \sum_{j=0}^{n-1} \nabla \cdot d_{vj}(|\nabla v_n|) \nabla^2 j - \varepsilon_{v_n}(|\nabla v_n|) \end{pmatrix} \begin{pmatrix} u_m \\ v_n \end{pmatrix},$$

where  $\varepsilon_{u_m}(|\nabla u_m|)$  and  $\varepsilon_{v_n}(|\nabla v_n|)$  are made edge sensitive. As low-pass filters, both  $d_{u_j}(|\nabla u_m|)$  and  $d_{v_j}(|\nabla v_n|)$  are 0 when  $j$  is even. Similarly, both  $d_{u_j}(|\nabla u_m|)$  and  $d_{v_j}(|\nabla v_n|)$  are 0 when  $j$  is odd. We can define a general edge function as

$$w_{m,n}(\mathbf{r}, t) = u_m(\mathbf{r}, t) - v_n(\mathbf{r}, t) = H_{mn}(\mathbf{r}, t)X(\mathbf{r}), \quad (11)$$

where  $H_{mn}(\mathbf{r}, t)$  is a coupled nonlinear PDE operator. For Eq. (11) to work properly, we choose  $|d_{v_j}(|\nabla v_n|)| \gg |d_{u_j}(|\nabla u_m|)|$ . As shown in our earlier work, by increasing the order of the highest derivative, one can increase frequency localization and accuracy of the PDE transform for mode decomposition.<sup>56</sup> The frequency selection of  $w_{m,n}(\mathbf{r}, t)$  also depends on the evolution time. This aspect is investigated in a later section.

In the PDE transform algorithm, intrinsic mode functions are systematically extracted from residues, i.e.,

$$w_{mn}^k = H_{mn}X_{mn}^k, \quad \forall k=1, 2, \dots \quad (12)$$

where  $w_{mn}^k$  is the  $k$ th mode function. Here the residue function is given by

$$X_{mn}^1 = X(\mathbf{r})$$

and

$$X_{mn}^k = X_{mn}^1 - \sum_{j=1}^{k-1} w_{mn}^j, \quad \forall k=2, 3, \dots \quad (13)$$

Therefore,  $X = \sum_{j=1}^{k-1} w_{mn}^j + X_{mn}^k$  is a perfect reconstruction of  $X$  in terms of all the mode functions and the last residue. The mode decomposition algorithm given in Eq. (12) is essentially nonlinear, even if a linear PDE operator might be used.

## II.D Variational formulation of PDE transforms

Variational models have been extremely successful for a wide variety of science and engineering problems<sup>3,14–16,18,46,58,66</sup> and have been an active area of research in applied mathematics. In the field of image analysis, Rudin, Osher, and Fatemi pioneered the total variation based image restoration models for edge preserving image denoising.<sup>46</sup> In the present work, we derive a set of high order nonlinear PDE transforms by variational approach for mode decomposition. Similar variation formulation of high order nonlinear PDE has been discussed by Didas et al.<sup>18</sup> Let us denote  $\mathbf{r} = (x_1, x_2, x_3) \in \mathbb{R}^3$  and define a derivative operator

$$D^j u = \sum_{j_1+j_2+j_3=j} \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \partial_{x_3}^{j_3} u,$$

where the summation is over all possible nontrivial combinations of  $j_1 + j_2 + j_3 = j$ . We define the energy functional as



$$E(u, D^1u, \dots, D^m u, v, D^1v, \dots, D^n v) = \int \left[ \Lambda_u \left( \sum_{j=1}^m |D^j u|^2 \right) - \Lambda_v \left( \sum_{j=1}^n |D^j v|^2 \right) + \varepsilon(v - u)^2 \right] d\Omega \quad (14)$$

where  $\Lambda_u(\cdot)$  and  $\Lambda_v(\cdot)$  are appropriate penalty functions.<sup>18,38</sup> Minimizing the energy functional (14) by the Euler-Lagrange equation, we have

$$\sum_{j=1}^m D^j \Lambda_{uj} \left( \sum_{j=1}^m |D^j u|^2 \right) D^j u + \varepsilon(v - u) = 0 \quad (15)$$

$$\sum_{j=1}^n D^j \Lambda_{vj} \left( \sum_{j=1}^n |D^j v|^2 \right) D^j v + \varepsilon(u - v) = 0 \quad (16)$$

where  $\Lambda_{uj}(\cdot) = (-1)^{j+1} \Lambda_{uj} D^j u$  and  $\Lambda_{vj}(\cdot) = (-1)^{j+1} \Lambda_{vj} D^j v$ .

The energy functional is minimized when Eqs. (15) and (16) are satisfied. To this end, we make use of the steepest descent algorithm with an artificial time and convert Eqs. (15) and (16) into two time dependent PDE transforms

$$\partial_t u = \sum_{j=1}^m D^j \Lambda_{uj} \left( \sum_{j=1}^m |D^j u|^2 \right) D^j u + \varepsilon(v - u) \quad (17)$$

$$\partial_t v = \sum_{j=1}^n D^j \Lambda_{vj} \left( \sum_{j=1}^n |D^j v|^2 \right) D^j v + \varepsilon(u - v). \quad (18)$$

Although the mathematical properties of these high-order PDE transforms may differ from those in Eq. (10), Eqs. (17) and (18) are essentially equivalent to the earlier PDE transforms. Specifically, the magnitudes and signs of  $\Lambda_{uj}$  and  $\Lambda_{vj}$  should be chosen in the same manner as for  $d_{uj}$  and  $d_{vj}$ , respectively. It is convenient, though not necessary, to set  $\varepsilon \sim 0$  for mode decomposition. To construct intrinsic mode functions, Eq. (12) is used with the PDE transform operator  $H_{mn}$  being replaced by Eqs. (17) and (18).

## II.E Fourier analysis of PDE transforms

To analyze the behavior of high order PDE transforms, let us consider a high order linear PDE in one spatial dimension

$$\partial_t v = \sum_{j=1}^n (-1)^{j+1} d_j \partial_x^{2j} v + \varepsilon(X - v), \quad t \geq 0, \quad (19)$$

where  $d_j > 0$ ,  $\varepsilon \sim 0$  and  $X$  is the initial signal. Equation (19) is subject to initial value  $v(x, 0) = X$ . The exact solution of Eq. (19) in the Fourier representation is

$$\widehat{v}(t) = \widehat{L} \widehat{X} \quad (20)$$

where  $\widehat{v}$  and  $\widehat{X}$  are the Fourier transforms of  $v$  and  $X$ , respectively. Here  $\widehat{L}$  is a frequency response function

$$\widehat{L}(\varepsilon, t, d_1, d_2, \dots, d_n) = e^{-\left(\sum_{j=1}^n d_j w^{2j+\varepsilon}\right)t} + \frac{\varepsilon}{\sum_{j=1}^n d_j w^{2j+\varepsilon}} \left(1 - e^{-\left(\sum_{j=1}^n d_j w^{2j+\varepsilon}\right)t}\right). \quad (21)$$

As expected,  $\widehat{L}$  is a low-pass filter. The Fourier representation of the IMF can be given as

$$\widehat{w}(t) = \widehat{H}\widehat{X} \quad (22)$$

where  $\widehat{H}$  is another frequency response function

$$\widehat{H}(\varepsilon, t, d_1, d_2, \dots, d_n) = 1 - e^{-\left(\sum_{j=1}^n d_j w^{2j+\varepsilon}\right)t} - \frac{\varepsilon}{\sum_{j=1}^n d_j w^{2j+\varepsilon}} \left(1 - e^{-\left(\sum_{j=1}^n d_j w^{2j+\varepsilon}\right)t}\right). \quad (23)$$

Obviously,  $\widehat{H} = 1 - \widehat{L}$  is a high-pass filter. The effects of highest order  $2n$ , propagation time  $t$  and coupling strength  $\varepsilon$  on the frequency response behavior of high-pass filter  $\widehat{H}$  is given in Figures 1 and 2. Qualitatively, the behavior of  $\widehat{L}$  and  $\widehat{H}$  can be analyzed as

$\widehat{L}(\varepsilon, t, Z) = e^{-Z^2 t + \frac{\varepsilon}{Z}(e^{-Z^2 t} - 1)}$  and  $\widehat{H}(\varepsilon, t, Z) = 1 - e^{-Z^2 t + \frac{\varepsilon}{Z}(e^{-Z^2 t} - 1)}$  where  $Z^2 = \sum_{j=1}^n d_j w^{2j+\varepsilon}$ . In particular, the impact of higher order, i.e., large  $2m$  and  $2n$  values, should be examined separately.

It is also interesting to investigate a band-pass filter given by the difference of solutions of two high-order PDEs. To this end, one can simply set  $\varepsilon = 0$  and define

$$\widehat{B}(t, d_{11}, d_{12}, \dots, d_{1m}, d_{21}, d_{22}, \dots, d_{2n}) = e^{-\left(\sum_{j=1}^m d_{1j} w^{2j}\right)t} - e^{-\left(\sum_{j=1}^m d_{2j} w^{2j}\right)t}, \quad (24)$$

where we choose  $d_{2j} \gg d_{1j} \sim 0$  and  $t_2 \ll t_1$ . Qualitatively, the behavior of  $\widehat{B}$  can be analyzed as  $\widehat{B}(t, Z_m, Z_n) = e^{-Z_m^2 t} - e^{-Z_n^2 t}$  with  $Z_n^2 = \sum_{j=1}^m d_{2j} w^{2j} \gg Z_m^2 = \sum_{j=1}^m d_{1j} w^{2j}$ .

In Figures 1 and 2, Fourier analysis for the low-order PDE based low-pass filter in Eq. (21) and high-pass filter in Eq. (23) is given. We illustrate the control of the frequency localization by parameter  $2n$ , the highest order of the PDE transform. The effects of coupling strength  $\varepsilon$  (i.e., fidelity term) on the frequency response behavior of the low-pass filter  $\widehat{L}$  corresponding to the diffusion equation (with  $2n = 2$  being the highest order PDE) are given by the plots in Figure 1(a). The black curve shows the frequency response of the low-pass filter corresponding to the diffusion equation without fidelity term, i.e.,  $\varepsilon = 0$ . The blue dashed and red circled curves correspond to the same low-pass filter with different values of  $\varepsilon = 0.4$  and  $1$ , respectively. If the main purpose of the signal processing is to separate modes with close frequencies, a small  $\varepsilon$  value is preferred. On the other hand, as demonstrated in this paper, higher order PDEs provide much higher frequency localization and are more efficient in separating modes with closely adjacent frequencies. In Figure 1(b), the highest order of the PDE used is  $2n = 80$ . The frequency responses approach to the ideal shape of a step function which provides higher precision in separating modes with closely adjacent frequencies.

The effect of  $\varepsilon$ , however, is different from that in the low order PDE case. The three curves of black, blue, and red color are frequency responses of the same low-pass filters with different values of  $\varepsilon = 0, 1$ , and  $10$ . The large coupling strength (i.e., large value of  $\varepsilon$ ) does not sacrifice the accuracy of mode decomposition as it does for the low order PDE; rather,

the larger the value of  $\epsilon$ , the higher shift of the dividing line of the mode frequency separation. In Figures 1(c) and 1(d), the symmetric plots of the high-pass filters are shown corresponding to the low-pass filters in Figures 1(a) and 1(b), respectively.

In Figure 2, the control of the time-frequency localization by  $2n$  and the frequency selection by  $t$  of the highest order of the PDE, on the high-pass filter (23) are explored. The value of  $2n$  in Figure 2(a) is equal to 2, 4 and 80 in the red circled, blue dashed, and black solid curves, respectively. As the order increases, the frequency response function becomes steeper and thus represents a better frequency localization, which is needed in the separation of closely adjacent frequency modes. In Figure 2(b) effect of propagation time is examined on the frequency response of the high order PDE filter in Eq. (23). All the three curves show the frequency response functions corresponding to different values of  $t$  with the same PDE. In the figure, black solid, blue dashed and red circled lines correspond to the numerical results using  $t = 1, 10^{-30}$  and  $10^{-60}$ , respectively. Therefore, one can gradually propagate the coupled high order PDEs in the PDE transform algorithm to systematically extract various modes. The longer propagation time of PDE transforms, the lower frequency of the decomposed mode. The PDE transform algorithm is truly robust and automatic in decomposing signal into series of functional modes or intrinsic modes.

### III Numerical tests and validations

Having analyzed the frequency response of a linear PDE transform, we validate the ability of PDE transform to separate IMFs from signals, and demonstrate such an ability with applications to signals containing closely adjacent frequency modes which have been practically very challenging.

In Figure 3, the PDE transform is applied to the separation of the three modes (Figure 3(a)) from the signal  $\cos(5x) + \cos(1.1x) + \cos(x)$ . The highest order PDE used in the PDE transform is  $2n = 32$ . The three modes are sequentially separated from the original signal using the same PDE transform algorithm with different values of propagation time  $t$ . This is a challenging case due to the extreme closeness of modes  $\cos(1.1x)$  and  $\cos(x)$ . Numerical results for three modes are compared with exact values in Figures 3(b) through 3(d). In Figure 3(b), results for Mode 1,  $\cos(5x)$ , is shown. Black solid line shows the exact value, and red circles show the numerical values obtained by using the PDE transform algorithm with the 32nd order PDE. The  $L_\infty$  norm of the difference between the exact and numerical curves is negligible (smaller than  $10^{-6}$ ). Similarly in Figure 3(c), result of Mode 2,  $\cos(1.1x)$ , is shown. The  $L_\infty$  norm of the difference between the exact and numerical curves is 0.06. In Figure 3(d), results of Mode 3,  $\cos(x)$ , is shown. The  $L_\infty$  norms of the difference between the exact and numerical curves is 0.07 such that two closely adjacent modes  $\cos(1.1x)$  and  $\cos(x)$  are well separated. Additionally, higher order PDE can be easily applied to achieve even better accuracy as illustrated in Figure 4.

In Figure 4, convergence is checked with respect to the value of  $2n$ , which is the highest order PDE used in the PDE transform. A series of signals are designed as composition of the lower frequency mode  $\sin(x)$  with the higher frequency one  $\sin(mx)$ , where  $m$  varies from 1.3 to 1.1 in the figure. In the main window of the figure,  $x$ -axis is the highest order of PDE employed in the PDE transform algorithm, and  $y$ -axis is the  $L_2$  norm of the error of the high frequency mode numerically extracted compared with the exact values of  $\sin(mx)$ . Three different signals of  $\sin(x) + \sin(1.1x)$ ,  $\sin(x) + \sin(1.2x)$  and  $\sin(x) + \sin(1.3x)$  are decomposed by the PDE transform algorithm. The numerical results are shown by black square, blue triangle and red circled curves. The signal  $\sin(x) + \sin(1.1x)$  with two closest adjacent modes requires much higher order of PDEs to be included in the PDE transform algorithm in order to achieve the same level of accuracy (or error) as for the other two

signals. For example, in order to achieve numerical accuracy with  $L_2$  error bounded below 0.1, a very high order PDE ( $2n=64$ ) is required for mode decomposition of the signal with  $m = 1.1$  compared to (roughly) 36th and 24th order PDEs needed for signals with  $m = 1.2$  and  $m = 1.3$ , respectively. For better viewing and clearer quantitative comparison, numerical errors are also plotted in log-scale in the embedded small window in Figure 4. It is clearly observed that the PDE transform with higher order PDE achieves higher accuracy and better resolution in mode decomposition.

More detailed exploration of the convergence of the numerical results shown in Figure 4 is illustrated in Figure 5. The same PDE transform is used to decomposing three signals,  $\sin(x) + \sin(1.5x)$ ,  $\sin(x) + \sin(1.5x)$  and  $\sin(x) + \sin(1.5x)$  are shown in the 1st, 2nd and 3rd row in Figure 5, respectively. In the 1st row, the figure on the left shows the numerical results compared to the exact results for the higher frequency mode  $\sin(1.5x)$  separated out from the signal using high-pass filter designed for the high order PDE transform algorithm. Exact results are shown by the black curve, while green squares, blue triangles and red circled curves show the numerical results using highest order PDE with  $2n=4, 8$  and  $16$ , respectively. Similar results are shown in the 2nd row of Figure 5 where the PDE transform is applied to decompose signal  $\sin(x)+\sin(1.3x)$ . In the 3rd row, due to the closeness of the two modes  $\sin(x)$  and  $\sin(1.1x)$ , higher order PDEs are required to achieve accurate decomposition. In Figures 5(e) and 5(f), green squares, blue triangles and red circled curves show the numerical results using highest order PDE with  $2n=8, 16$  and  $32$ , respectively. For all signals, it is clearly demonstrated that the PDE transform with larger value of  $n$  converges better to the exact results. In addition, the figures on the right of all the three rows in Figure 5 show the shape of the frequency responses for the corresponding low-pass filters (i.e. equivalently for the corresponding high-pass filters) designed by the PDE transform algorithm. The larger value  $n$ , the steeper the rising of frequency response functions. In the limit of  $n \rightarrow \infty$ , the frequency response function would become a step function which ideally divide the frequency domain into high frequency and low frequency regions.

Lastly, in Figure 6, the effect of total propagation time  $t$  as in Eq. (23) on the frequency response functions is explored in details. The same PDE transform algorithm employing the same highest order PDE with  $2n = 48$  is applied to separate out all the modes from the signal  $\sin(x) + \sin(2x) + \sin(3x) + \sin(4x) + \sin(5x) + \sin(6x) + \sin(7x) + \sin(8x) + \sin(9x) + \sin(10x)$ . The 10 modes are separated out, in the order of decreasing frequency peak, by varying propagation time  $t$  only. When the same set of PDEs composing the PDE transform algorithm are propagated by  $t = 4.3 \times 10^{-48}$ , highest frequency mode  $\sin(10x)$  is decomposed; after another propagation time  $t = 9.4 \times 10^{-46}$ , next highest frequency mode  $\sin(9x)$  is decomposed, etc. It is worthwhile to point out that, though the propagation time ranges from  $10^{-50}$  to 1, no specific temporal discretization is needed in the current PDE scheme since the time integral is numerically calculated using one time step. In Figure 6(b), values of propagation time  $t$  are plotted as the function of the frequency peaks of the modes being decomposed. The numerical values of  $t$  (labeled by the black circles) fits well into the analytical estimation  $1/w^{48}$  shown by the red solid curve, where the value of 48 corresponds to the value  $2n = 48$  of the highest order PDE used herein.

The PDE transform algorithm has been designed for the mode decomposition of arbitrary dimensional signals or images. In previous Figures, one-dimensional (1D) signals have been used for illustration and validation purposes. In Figure 7, a two-dimensional (2D) signal  $\sin(4x) \sin(2y) + \sin(4.2x) \sin(2.2y)$  is decomposed by the PDE transform algorithm into the two modes  $\sin(4x) \sin(2y)$  and  $\sin(4.2x) \sin(2.2y)$ . A relatively low value of  $2n = 16$  for the highest order PDE is sufficient to achieve good enough numerical results of decomposed signal.  $L_2$  norm error is 0.1 compared to the amplitude 2.0. By increasing value  $2n$  of the highest order PDEs employed in the PDE transform algorithm, similar convergence can be

achieved as for 1D signals. The PDE transform algorithm has been designed to work naturally and equally efficient for arbitrarily dimensional signals and images.

## IV Applications to biomedical and biological images

In this section, we apply the PDE transform algorithm to several different types of medical images for image denoising and enhancement.

### IV.A Early cancer detection

In the initial period of the cancer development, the patient may not exhibit noticeable symptoms such that it would be too late even if they quickly pick up alerting messages when the cancer has developed for some time already. It is critical to detect cancer in its early state in order to provide a successful cure or surgery. Various medical imaging techniques have been developed to help diagnose and detect cancer. In addition to the hardware-type imaging techniques, image postprocessing can be applied to render a clearer image showing the important details of the cancer images. In this section, imaging techniques and processing are discussed with application to lung cancer and breast cancer.

The general technique for early lung cancer is screening chest with X-ray or spiral computed tomography scan. Such a technique is useful for detecting the lung cancer in its more localized phase which implies it is more likely to be curable. One important goal of imaging and image processing is to decide whether the tumor is resectable and whether it should be a lobectomy or a pneumonectomy. Lobectomy means surgical excision of a lobe of the lung. On the other hand, a pneumonectomy is a surgical procedure to remove the whole lung. Clearly the lobectomy approach is a better and safer choice than pneumonectomy if the former is possible. The restriction for performing a lobectomy of the lung is that the surgery can not be performed on patients with lung cancers spreading to other parts of the tissue. Details from the medical images are therefore helpful in making the decision. In medical imaging of internal organs or brain, pictures are usually taken along one or many of the three planes (see Figure 8): axial plane which refers to the slice parallel to the feet of the patient, coronal plane which is the slice perpendicular to the feet and parallel to the shoulder line of the patient, and sagittal plane which is the vertical plane passing from front to rear and dividing the body into right and left sections. Images taken on the three planes together give three dimensional information of the internal organ of the human body. In Figures 9(e) and 9(f), images of the patient's lung taken along the axial and coronal planes are shown respectively. To test the effect of the PDE transform algorithm, random (Gaussian-type) noise are added in Figures 9(a) and 9(b). The PDE transform algorithm with up to 4th order PDE is used to smooth and enhance the images, results of which are shown in 9(c) and 9(d). In the axial image 9(c), one can see the shape of tumor near the fissure on the left. However, this information alone is not enough for making decision whether the tumor is resectable and whether it should be a lobectomy or a pneumonectomy. The image in 9(d) taken along the coronal plane clearly shows there is no transfissural growth. As such, the tumor is localized and does not cross the fissure which divides the superior lobe from the middle lobe. Therefore lobectomy is a possibility. The lung of another patient is imaged along coronal and sagittal planes as shown in Figure 10. Similarly, white noise is added to the images and the PDE transform algorithm with up to 4th order PDE is performed to smooth the images. Comparing Figure 9(c) with 10(c), transfissural growth is observed in the latter case. The picture taken along the sagittal plane in Figure 10(d) further demonstrates that the tumor crosses the horizontal fissure of the lung. Therefore, lobectomy is not possible according to the images shown in these Figures.

Detection of breast cancer is another important application of medical imaging and image processing. One of the widely used exams for early detection of breast cancer is

mammogram, which take picture of breast with a safe, low-dose X-ray machine. Diffuse optical tomography (DOT) has recently been used to create high-resolution images from a scattering of infrared and visible light for the early detection of breast cancer. DOT is a three-dimensional method where photon density waves are launched from a source and then travel in a banana-shaped path due to multiple scattering, which is different from the straight paths followed by X-rays in mammogram. Compared to the use of X-rays in mammogram imaging, DOT is relatively less expensive, safer and more comfortable. However, the photon density waves used in DOT are more difficult to track because of the scattering and absorption. Image processing is usually performed to remove noise and enhance image quality. In Figure 12(a), a high resolution image is shown created by the DOT technology. The distorted region in the black circle in the image indicates the area of abnormalities associated with the breast cancer. Image shown in Figure 12(b) contains combination of two types of noises: highly oscillatory random noises (with up to 200 oscillations in each direction of the  $256 \times 256$ -pixel image and with random amplitude in each oscillation) and Gaussian-type random noise with standard deviation equal to 30 (see Figure 12(d)). The fine lines of image are therefore distorted and contaminated. In Figure 12(c), an enhanced image is generated from the Figure 12(b) by applying the PDE transform algorithm using up to 4th order PDE. The fine details are well recovered. Such image enhancement can be combined with other mathematical models in development by other groups to make the DOT a better imaging techniques for detecting smaller breast cancer with lower cost and higher safety and reliability.

#### IV.B Optical images from retinal functioning imager

Besides the important biological function of vision, eyes are also helpful for the diagnosis of diseases such as diabetes as well as for the assessment and identification of various neurological effects. A number of technological progresses has advanced the retinal imaging for research and clinical applications, e.g. Many imaging techniques including retinal function imager (RFI), positron emission tomography (PET) and functional magnetic resonance imaging (fMRI) have been developed to provide high resolution images. In particular, RFI is an adaptive noninvasive approach to study the single red blood cells moving through capillaries<sup>26,31</sup> and it enables direct visualization of retinal blood dynamics without the injection of contrast agents. The motion of individual red blood cells and blood cell clusters can be revealed with much clarity which is important towards the quantitative detection of abnormal blood flow velocity in capillaries and arterioles. Like many other medical imaging techniques, poor signal-to-noise ratio in in-vivo applications is always a concern. Various post image processing techniques are generally applied to render enhanced images. In Figure 11, qualitative oximetry with the RFI is applied to detect the perfusion deficits and abnormalities in a patient with sickle cell retinopathy. The area with abnormalities appear as regions of grey scale distinct from their surroundings. Figure 11(a) shows a high resolution image with good resolution. In Figure 11(b), noise is added which is composed of both a highly oscillatory noise (with up to 30 oscillations in each direction of the  $256 \times 256$ -pixel image) with random amplitudes and Gaussian-type random noise (with standard deviation equal to 30). The PDE transform algorithm using up to 4th order PDE is applied for denoising the image. Result is shown in Figure 11(c). Such image enhancement technique would be useful in further improving the image quality of high resolution RFI.

#### IV.C Electron microscope photography of neuron systems

With an intricately elaborate dendritic arbor, Purkinje neurons are characterized by a large number of dendritic spines. Purkinje neurons are some of the largest of the 100 billion neurons in human brain. Neuron cells are the masters of motor coordination in the cerebellar cortex. These large dendritic arbors form nearly two-dimensional layers through which parallel fibers from the deeper-layers pass. Toxic exposure to such as alcohol and lithium

and various immune or neurodegenerative diseases such as autism and Alzheimer's disease can negatively affect human Purkinje cells and their shape and structure. Therefore, a clear visualization of the detailed shape of Purkinje neurons would be useful in the study and diagnosis of various diseases. In Figure 13, electron microscope photography of Purkinje neurons is shown. Electron microscope is a type of microscope that uses a particle beam of electrons to illuminate the sample and to generate a magnified image. Electron microscopes usually have a greater resolving power than a light-powered optical microscope due to the much smaller wavelength of electrons. Using electron microscope imaging technique, the details of vast amount of dendrites on the neuron cells. Noise removal using image processing can further provide a better quality image. The PDE transform using up to 4th order PDE is used to denoise the image 13(b) to generate a better resolved one in 13(c) in which the details of cell structure and geometry of the dendrites are better rendered compared to the original high resolution picture 13(a).

## V Concluding remarks

Geometric partial differential equations (PDEs) are of tremendous importance in signal and image processing, data analysis and surface construction. Mode decomposition is an elementary process in a variety of signal/image processing tasks. However, despite the great progresses made in geometric PDEs accompanying the broad interests in the past two decades, the use of geometric PDEs for mode decomposition was not realized until a recent advance in the field. The resulting models, called PDE transform, has been introduced in our earlier work<sup>56</sup> based on arbitrarily high order nonlinear PDE introduced by Wei<sup>59</sup> and PDE based image edge operator introduced by Wei and Jia.<sup>61</sup> The present work introduces the variational formulation, provides the Fourier analysis and carries out the biomedical and biological applications of the PDE transform.

In the variational formulation of PDE transform, we construct a total energy functional consisting of two image functions and two sets of geometric operators. Due to the dynamical coupling of two operators, the relative signs of two operators are different, which contributes to the energy disparity. The dynamical coupling of two image functions reduces the energy disparity. The energy minimization by using the Euler-Lagrange equation and the introduction of artificial propagation time by the steepest descent algorithm leads to a system of two coupled nonlinear PDE transforms. As discussed in our earlier work,<sup>59</sup> these two equations have to evolve at dramatically different time scales to generate sufficient disparity in the two image functions. Their difference corresponds to the desirable intrinsic mode functions when appropriate model parameters and initial values are provided. To aid the parameter selection, we carried out detailed Fourier analysis on a prototype linear PDE transform, which admits an analytical solution in the Fourier domain. As such, we analyze the filter properties of arbitrarily high order PDE transforms. This analysis provides a guidance for the parameter selection of the proposed PDE transforms. For instance, the order of the highest PDE can be selected to control the frequency localization, while the propagation time or the amplitude of diffusion coefficients of the PDE transform can be adjusted to control the frequency location or precision. Many challenging test examples are explored to validate the proposed PDE transforms for decomposing intrinsic mode functions. We demonstrate usefulness of the proposed PDE transforms by biomedical and biological applications. Image denoising, enhancement and restoration are considered in the present work for early lung cancer detection, optical image, diffuse optical tomography, and electron microscope photography. Numerical experiments indicate that the proposed PDE transform works well.

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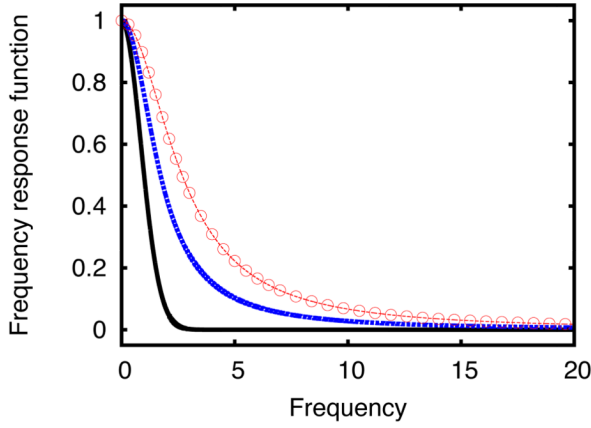
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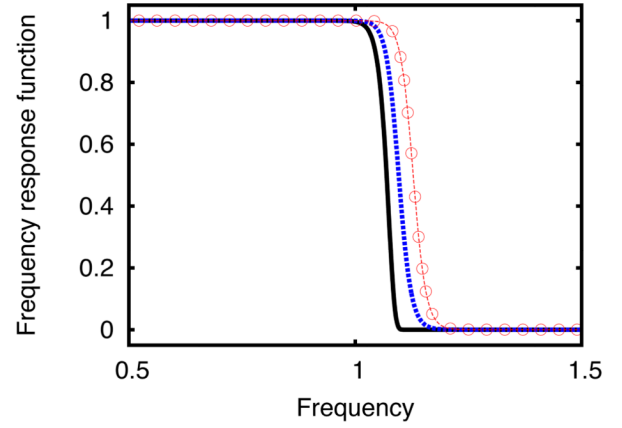


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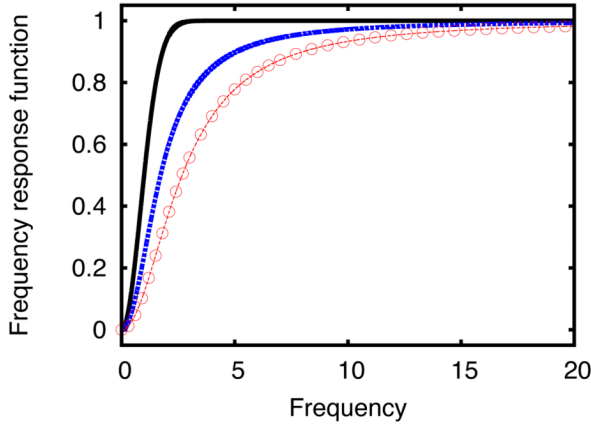
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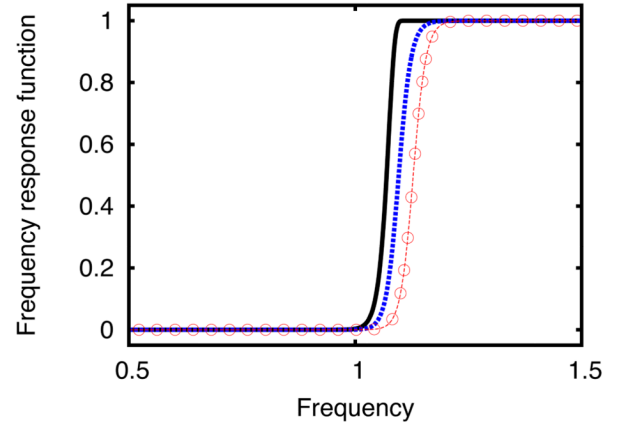
(a) Effect of coupling strength  $\epsilon$  on the frequency response of the PDE transform low-pass filter in Eq. (23) with  $2n = 2$  (and correspondingly  $d_1 = 0.14, t = 5$ ). The black solid, blue dashed, red circled curves correspond to the same low-pass filter with different values of  $\epsilon=0, 0.4$  and  $1$ , respectively.



(b) Effect of coupling strength  $\epsilon$  on the frequency response of the PDE transform high-pass filter in Eq. (23) with  $2n = 80$  (and correspondingly  $d_{40} = 6.5 \times 10^{-4}, t = 5$ ). The black solid, blue dashed and red circled curves correspond to the same low-pass filter with different values of  $\epsilon=0, 1$  and  $10$ , respectively.

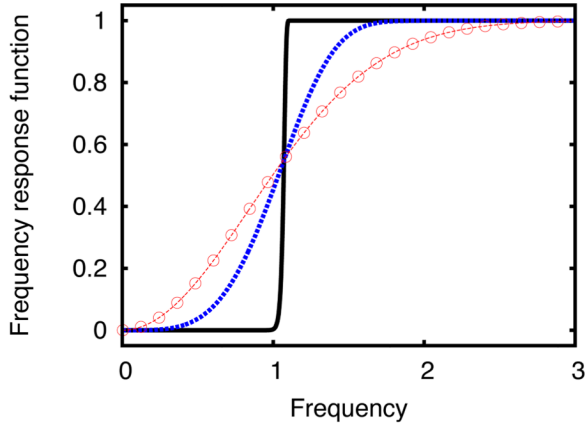


(c) Similar to Figure 1(a) except that the corresponding high-pass filter is shown in this plot.

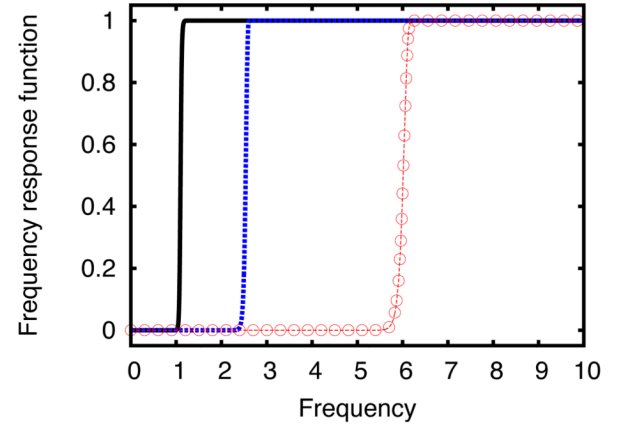


(d) Similar to Figure 1(b) except that the corresponding high-pass filter is shown in this plot.

**Figure 1.** Fourier analysis of the high-order PDE based low- and high-pass filters in Eqs. (21) and (23). Obviously, the time-frequency localization is controlled by parameter  $2n$ , the highest order of the PDE transform. The effects of coupling strength  $\epsilon$  (i.e. fidelity term) on the frequency responses  $\hat{L}$  and  $\hat{H}$  are given. The values of the parameters in Eqs. (21) and (23) are given in the caption below each subfigures.



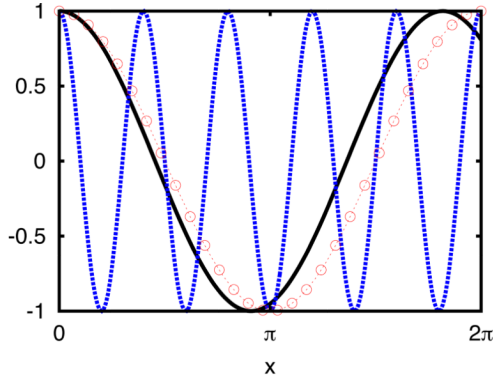
(a) Effect of the highest order of the PDE on the time-frequency localization of the high-pass filter designed in Eq. (23). The value of  $2n$  in Eq. (23), which is the highest order of the PDE used, is 2, 4 and 80 for the red circled, blue dashed, and black solid curves respectively with  $d_1 = 0.14$ ,  $d_2 = 0.121$ ,  $d_{40} = 6.5 \times 10^{-4}$ . In all the three curves,  $\epsilon = 0$  and  $t = 5$ .



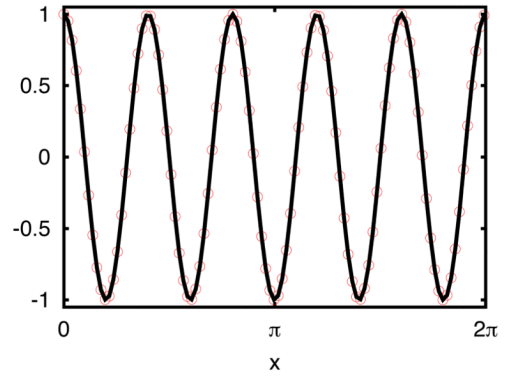
(b) Effect of propagation time on the frequency location (or frequency selection) of the high order PDE filter in Eq. (23). All the three curves show the frequency response functions corresponding to the same value of  $d_j$  in Eq. (23) and PDE with same highest order  $2n = 80$  and  $d_{40} = 6.5 \times 10^{-4}$ . In the figure, black solid, blue dashed and red circled lines correspond to the numerical results using propagation time  $t = 1, 10^{-30}$  and  $10^{-60}$  respectively.

**Figure 2.**

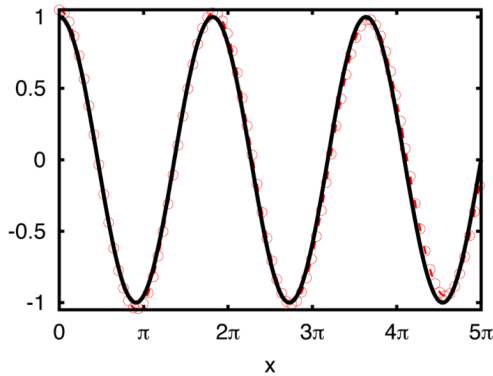
The control of the time-frequency localization by  $2n$  and the frequency location (precision) by  $t$  of the high-order PDE high-pass filter in Eq. (23).



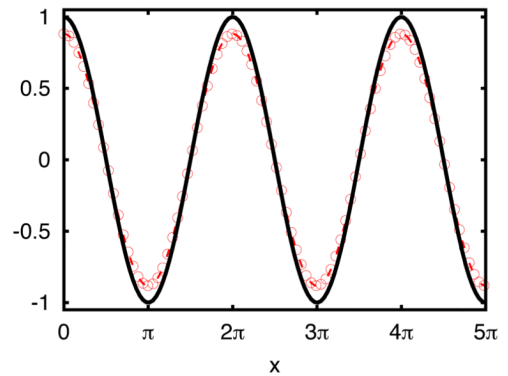
(a) The signal is composed of three frequency modes of  $\cos(5x)$ ,  $\cos(1.1x)$ , and  $\sin(x)$  plotted in blue dashed, black solid and red circled lines, respectively.



(b) Comparison of numerical results with exact values of Mode 1 of  $\cos(5x)$ . Black solid line shows the exact value, and red circled curve shows the numerical values obtained through the PDE transform algorithm using 32rd order PDE with  $d_{16} = -3.5 \times 10^{-22}$  and  $t = 5$ . The  $L_\infty$  norm of the difference between the exact and numerical curves is smaller than  $10^{-7}$ , which is negligible.



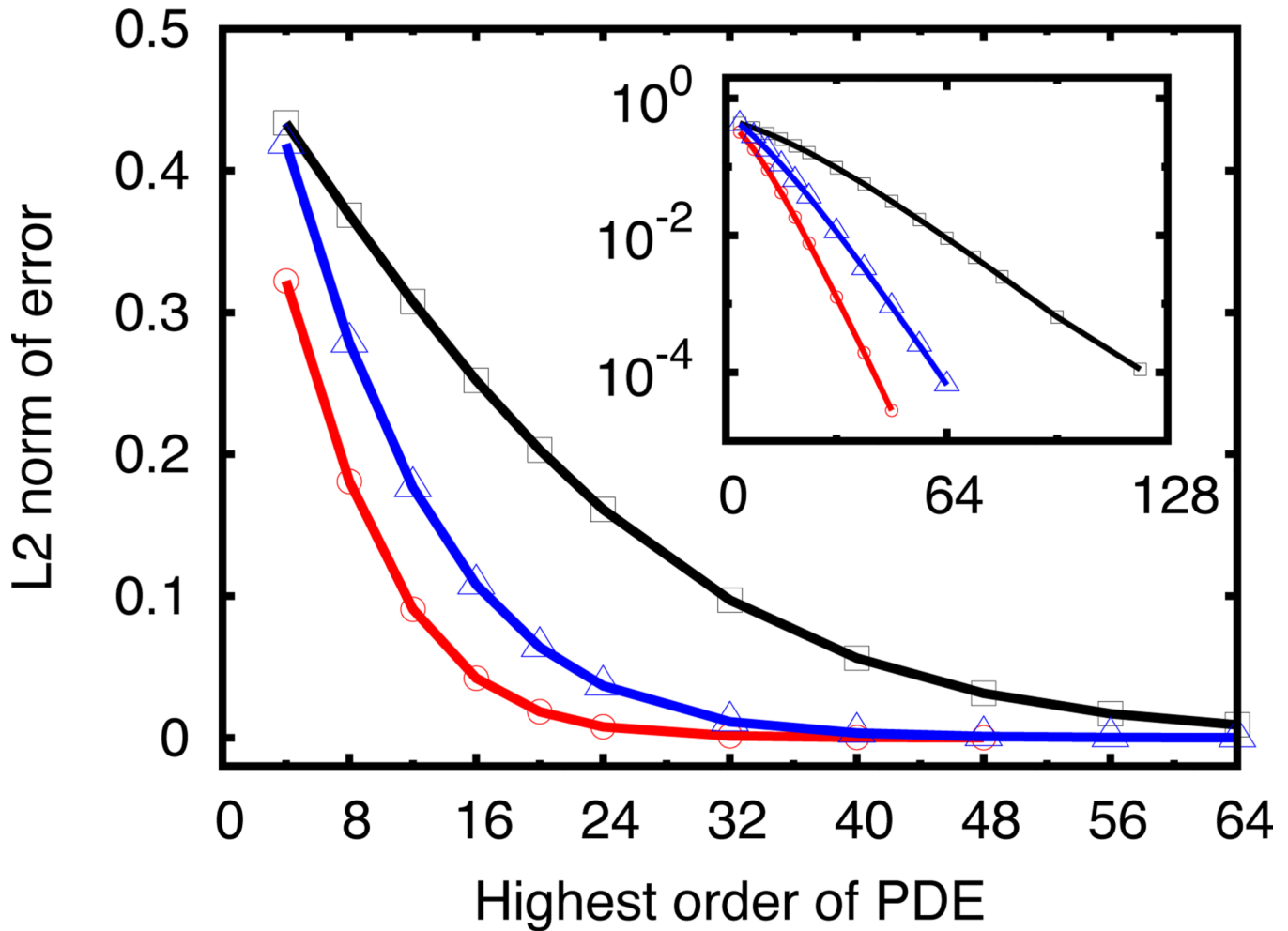
(c) Comparison of numerical results with exact values of Mode 2 of  $\cos(1.1x)$ . The colors and labels of the curves in this subfigure are same as those in 3(b). Parameters in Eq. (23) are chosen to be  $d_{16} = 2.5 \times 10^{-2}$  and  $t = 5$ . The  $L_\infty$  norm of the difference between the exact and numerical curves is 0.06, which is larger than the error for Mode 1 of  $\cos(5x)$  in subfigures 3(b) due to the close overlapping of  $\cos(1.1x)$  and  $\cos(x)$ .



(d) Comparison of numerical results with exact values of Mode 3 of  $\cos(x)$ . Parameters in Eq. (23) are chosen to be  $d_{16} = 10$  and  $t = 5$ . The  $L_\infty$  norm of the difference between the exact and numerical curves is 0.07.

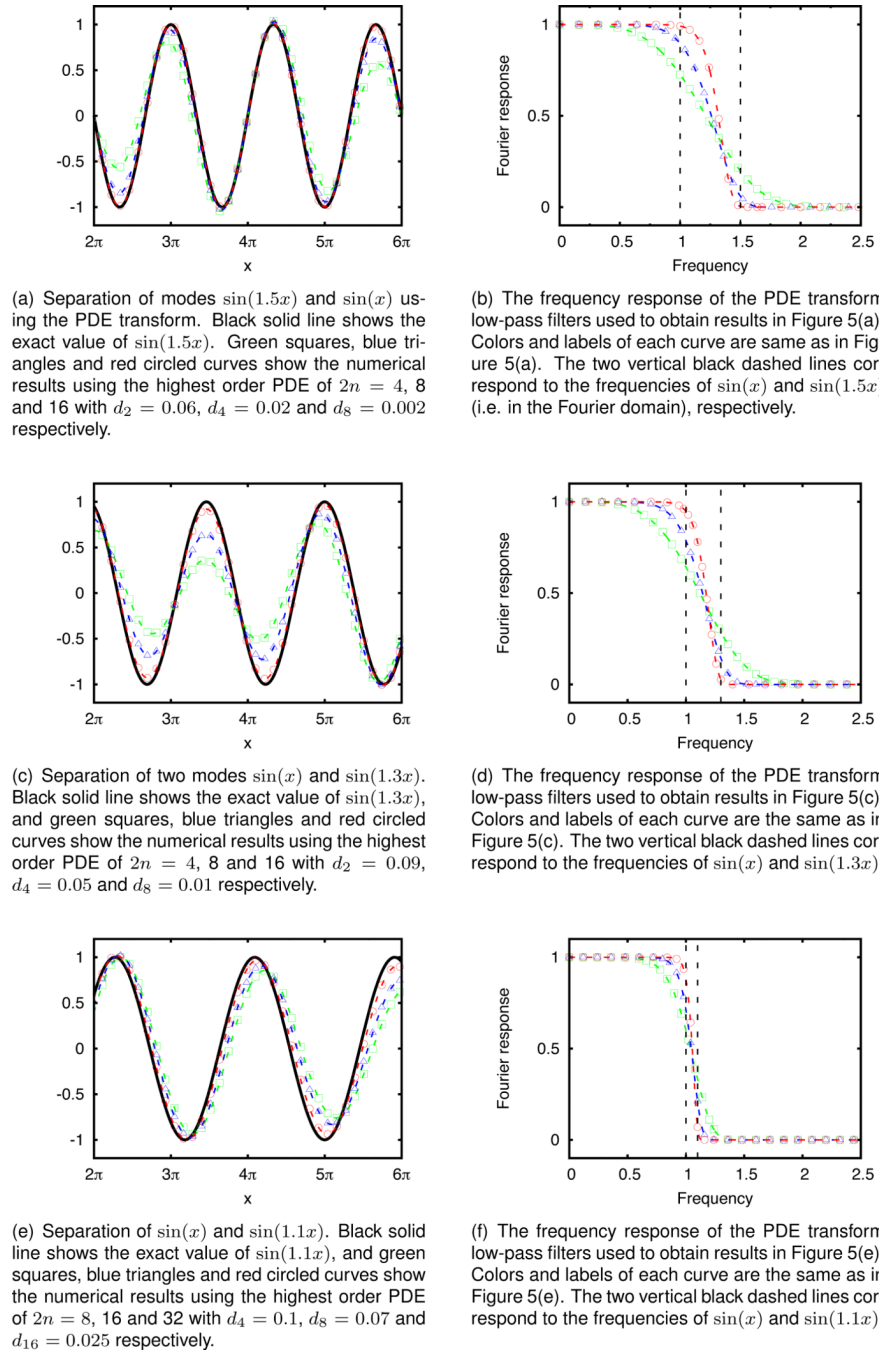
**Figure 3.**

Separation of the three modes from the signal  $\cos(5x) + \cos(1.1x) + \cos(x)$  using the PDE transform algorithm with the PDE of the highest order of  $2n = 32$ . The three modes are sequentially separated from the original signal using the same PDE transform algorithm with different propagation time  $t$ , respectively.

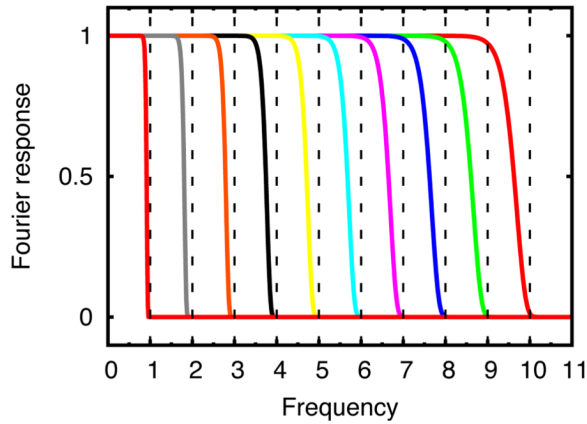


**Figure 4.**

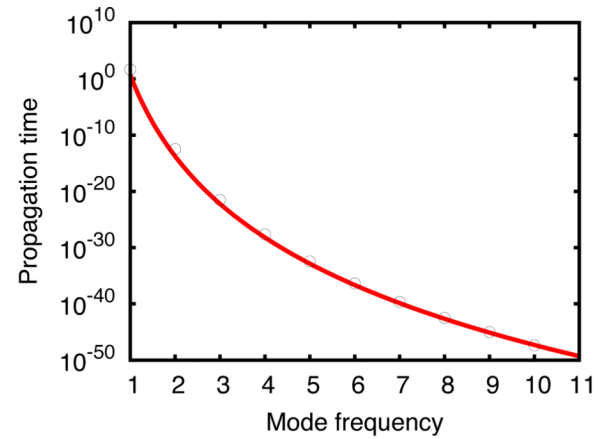
Extraction of higher frequency mode  $\sin(mx)$  from the original signal  $\sin(x) + \sin(mx)$ . In the main window of the figure,  $x$ -axis is the highest order of PDE employed in the PDE transform algorithm, and  $y$ -axis is the  $L_2$  norm error of the high frequency mode numerically extracted compared with the exact values of  $\sin(mx)$ . Three different signals of  $\sin(x) + \sin(1.1x)$ ,  $\sin(x) + \sin(1.2x)$  and  $\sin(x) + \sin(1.3x)$  are decomposed by the PDE transform algorithm, and their numerical results are shown by black square, blue triangle and red circled curves. The first signal of  $\sin(x) + \sin(1.1x)$  with closest overlapping modes requires much higher order of PDEs to be included in the PDE transform algorithm in order to achieve the same level of accuracy (or error) as for the other two signals. For the purpose of alternative viewing and better understanding, numerical errors are also plotted in log-scale as in the embedded smaller subfigure in the main window above. It is clearly observed that the PDE transform algorithm with higher order PDE achieves higher accuracy and better resolution in differentiating two modes with closely adjacent frequencies.



**Figure 5.** Separation of the two modes  $\sin(x)$  and  $\sin(mx)$ , where  $m \approx 1$ , from the same signal. Convergence of the PDE transform algorithm with various highest order PDEs are studied. A higher order PDE transform is needed to differentiate modes with closer frequencies. Propagation time  $t = 5$  in all the plots.



(a) The curves from left to right correspond to the frequency responses of low-pass filters (which are the Fourier transform of PDEs in the PDE transforms) applied to separate 10 modes of  $\sin(x)$ ,  $\sin(2x)$ ,  $\dots$  and  $\sin(10x)$  consecutively. The frequency response curves are generated using different values of propagation time growing from  $t \approx 10^{-50}$  to  $t \approx 10^0$ . The exact corresponding values of propagation time is shown in Figure 6(b). For reference, frequencies of each of the 10 modes in the frequency space are indicated by the vertical black dashed lines.

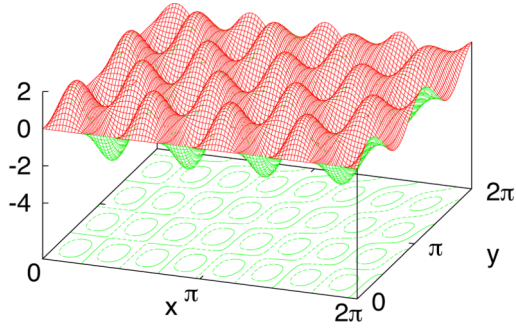


(b) Total effective propagation time  $\tau$  in the PDE transform, where  $\tau$  is the product of super diffusion constant and actual physical propagation time  $t$ . In the simplest case of the PDE transform with the highest PDE of 2nd order only, i.e., the diffusion equation,  $\tau$  is the product of diffusion constant  $D$  and propagation time  $t$ . In the figure, the horizontal axis is the location of mode in frequency space, e.g., mode frequency of  $\sin(3x)$  is 3. The red curve is the analytical plot for  $1/w^{48}$  which approximately fits the numerical values labeled by black circles.

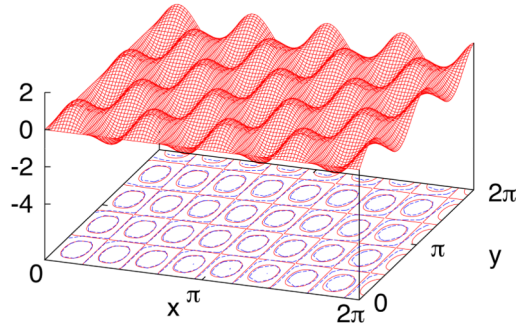
**Figure 6.**

Mode decomposition of the signal  $\sin(x) + \sin(2x) + \sin(3x) + \sin(4x) + \sin(5x) + \sin(6x) + \sin(7x) + \sin(8x) + \sin(9x) + \sin(10x)$  using the PDE transform algorithm including PDE terms up to 48th order with  $d_{24} = 1$ . The figure indicates that separation of lower frequency mode requires larger effective propagation time  $\tau$ , or equivalently speaking larger effective diffusion constant.

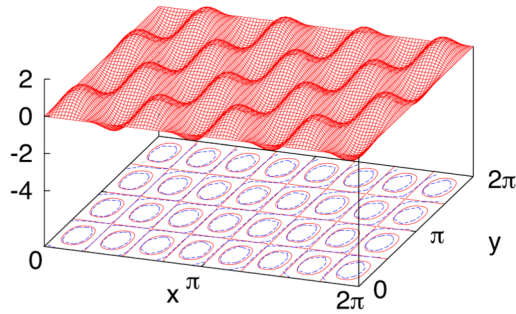




(a) Two-dimensional signal  $\sin(4x) \sin(2y) + \sin(4.2x) \sin(2.2y)$ . The contour plot for the 2D signal is projected on the base of the above subfigure.

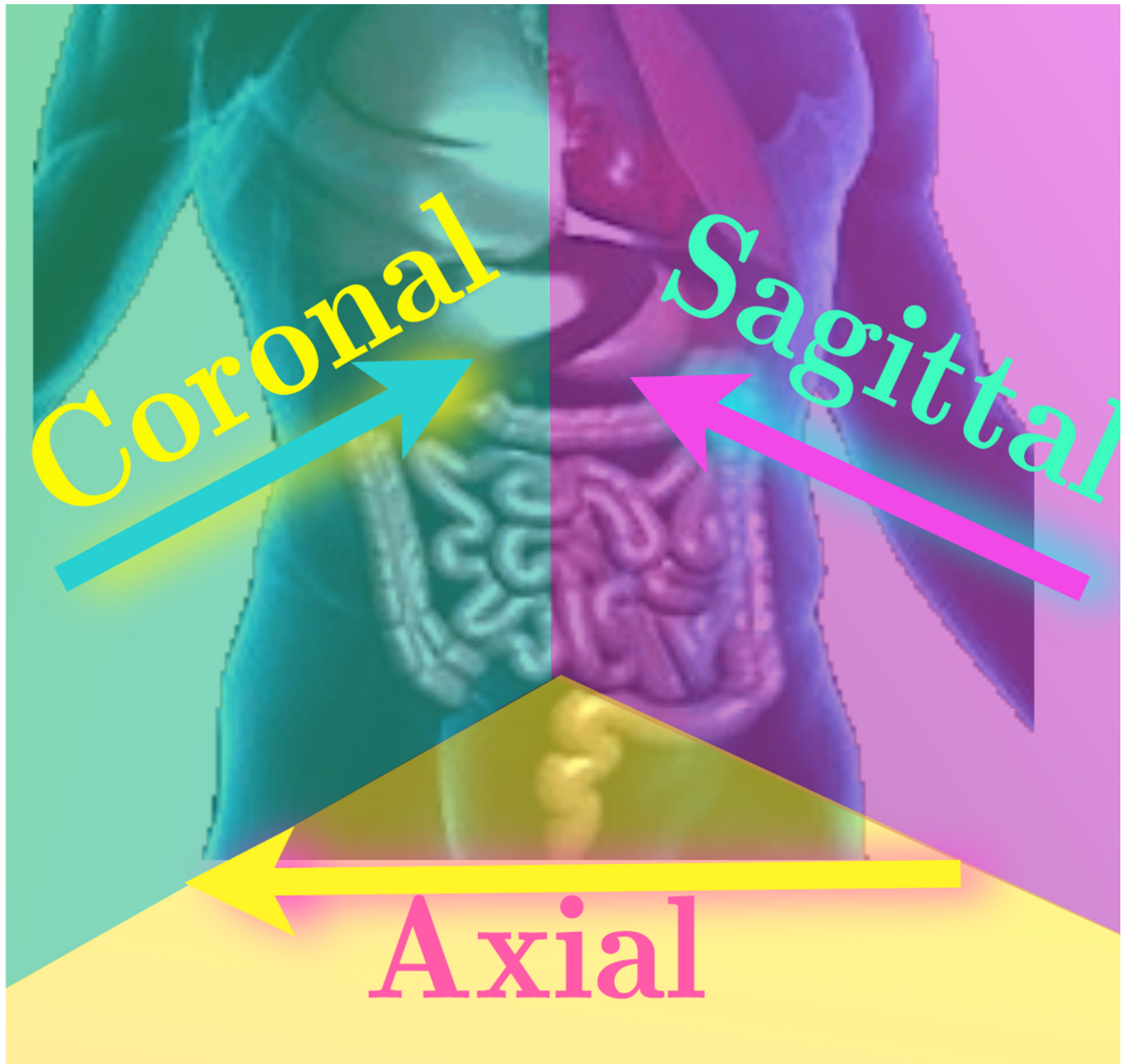


(b) High frequency mode  $\sin(4.2x) \sin(2.2y)$ . The  $L_2$  error of the PDE transform results compared to the exact values is 0.11. To differentiate the small difference between numerical results and exact values, contour plots for both surfaces are projected together on the base. The red and blue colors indicates the contour plots of the PDE transform and exact results, respectively.



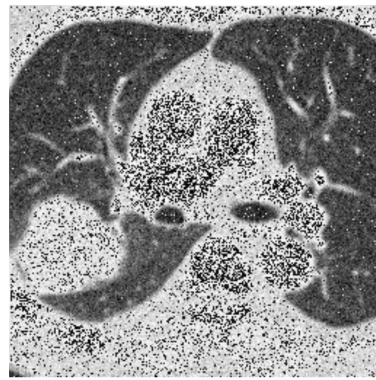
(c) Low frequency mode  $\sin(4x) \sin(2y)$ . Same labels and color sets as in subfigure 7(b) are used for differentiating the contour plot for the PDE transform results from that for the exact mode.

**Figure 7.** Mode decomposition of the signal  $\sin(4x) \sin(2y) + \sin(4.2x) \sin(2.2y)$  using the PDE transform algorithm including PDE terms up to 8th order. Results of two modes  $\sin(4.2x) \sin(2.2y)$  and  $\sin(4x) \sin(2y)$  are shown in subfigures 7(b) and 7(c), respectively.

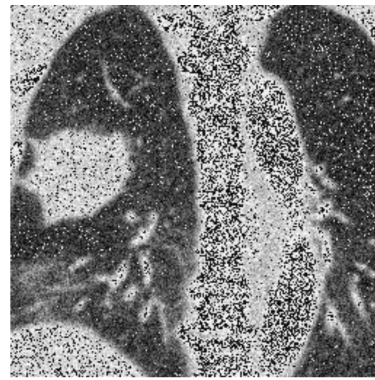


**Figure 8.**

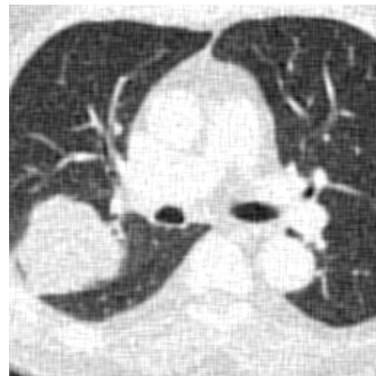
The axial plane is an imaginary plane that divides the body into superior and inferior parts. A coronal plane divides the body into back and front portions. A sagittal plane is a vertical plane which passes from front to rear dividing the body into right and left sections. The three planes are perpendicular to each others.



(a) Axial image of the lung.



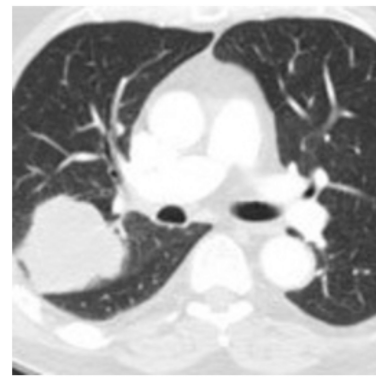
(b) Coronal image of the lung.



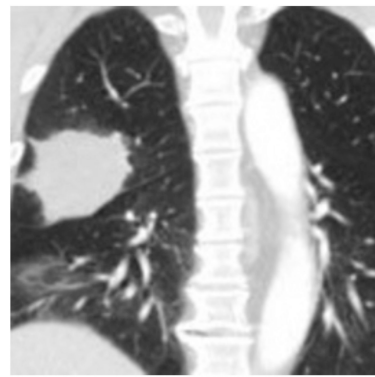
(c) Denoised axial image showing a tumor near the fissure on the left.



(d) Denoised coronal image demonstrating that there was no transfissural growth of the tumor.



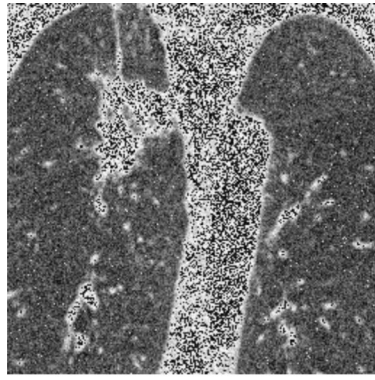
(e) Original noise-free axial image of the lung.



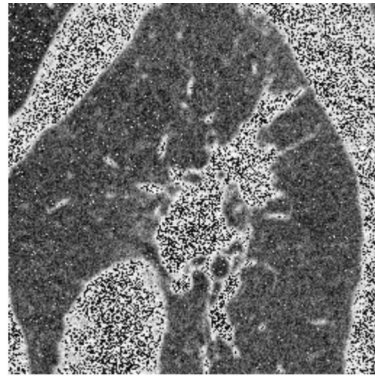
(f) Original noise-free coronal image of the lung.

**Figure 9.**

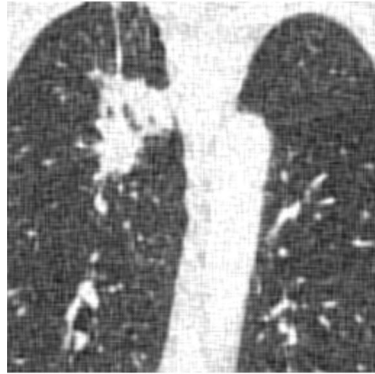
PDE transform algorithm using up to 4th order PDE is used to denoise and enhance the images for lung diagnosis. The tumor is detected in the axial image, and the details shown in the coronal image demonstrates that there was no transfissural growth of the tumor such that lobectomy surgical excision of a lobe only is possible.



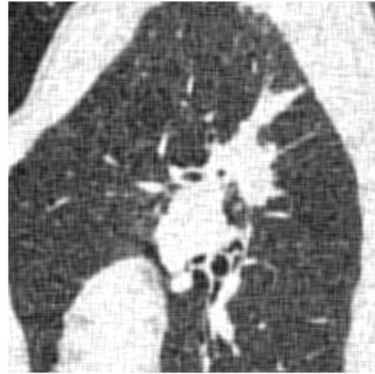
(a) Coronal image of the lung.



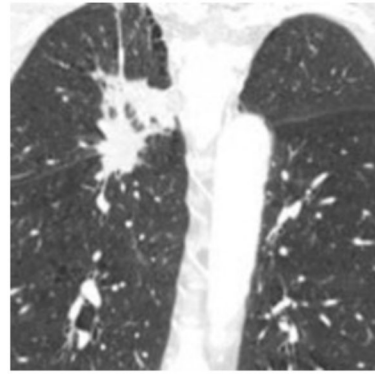
(b) Sagittal image of the lung.



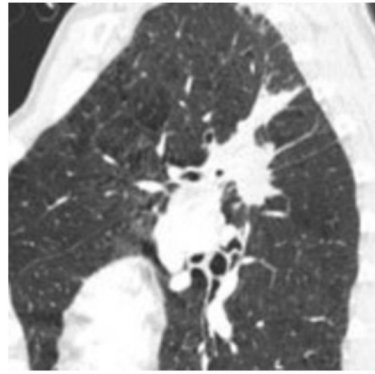
(c) Denoised coronal image showing that tumor is present and is transfissural.



(d) Denoised sagittal image further revealing the details of the transfissural growth of the tumor.



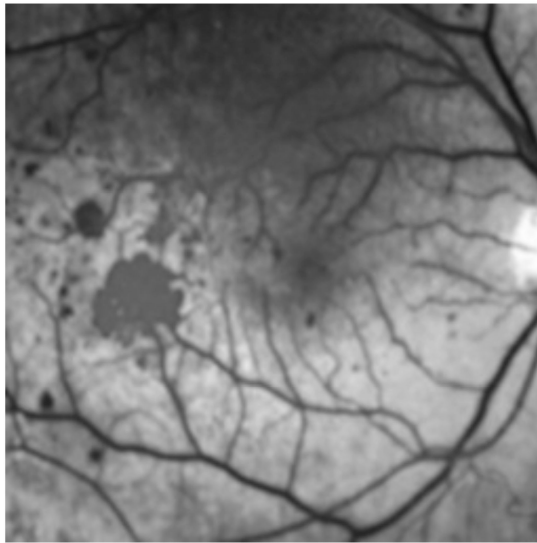
(e) Original noise-free coronal image of the lung.



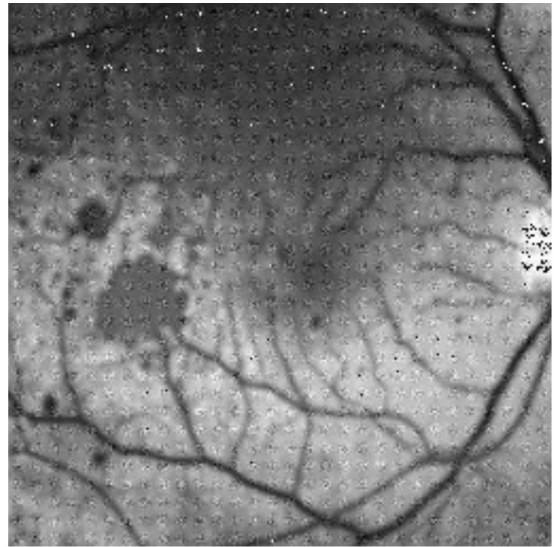
(f) Original noise-free sagittal image of the lung.

**Figure 10.**

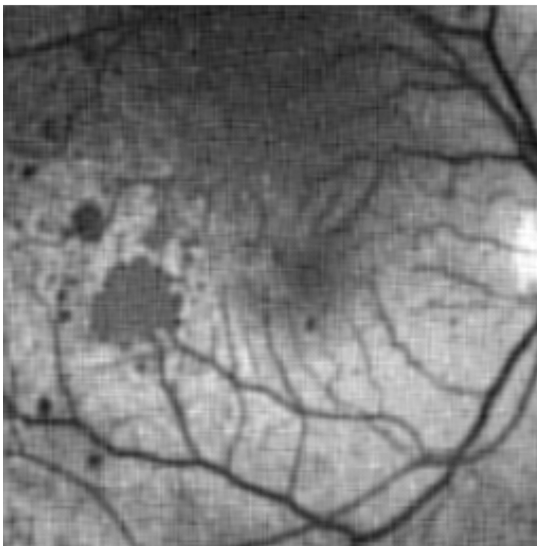
PDE transform algorithm using up to 4th order PDE is used to denoise and enhance the images for lung diagnosis. The coronal image shows that there was transfissural growth of the tumor, and sagittal image reveals more details supporting the diagnosis that lobectomy surgical excision of a lobe only is not possible and pneumonectomy is a surgical procedure might be needed to remove the whole lung.



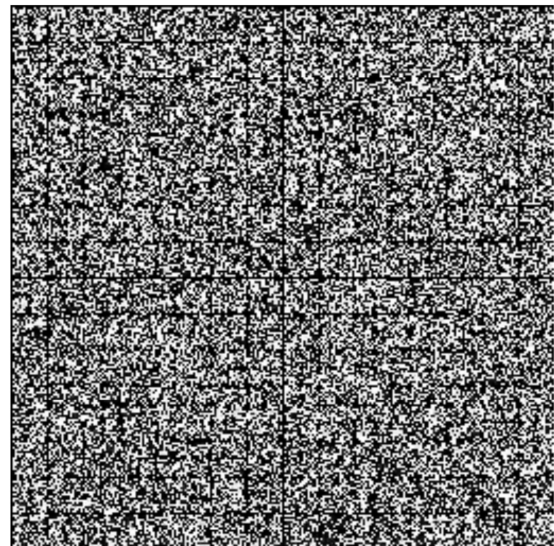
(a) Optical images from retinal functioning imager (RFI) for a patient with sickle cell retinopathy. The area with abnormalities appear as regions of grey scale distinct from their surroundings.



(b) Locally distributed white noise is added to the original image in 11(a) to mimic the RFI noise in the real experiments.



(c) Denoised image using the PDE transform algorithm with the 2nd order PDE.



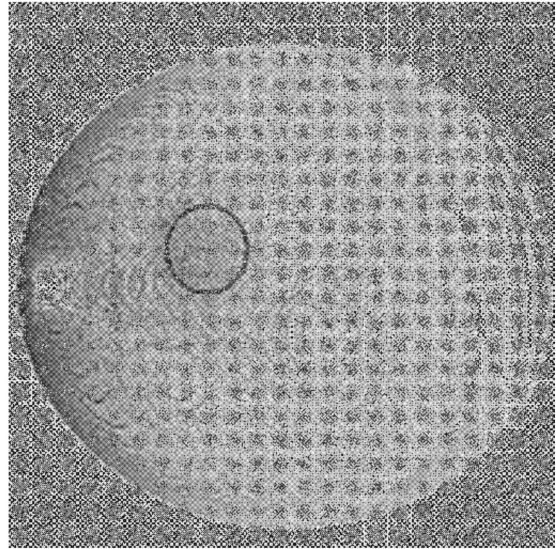
(d) The image showing the noise added to the original image, which is composed of both Gaussian white noise and highly oscillatory random noise.

**Figure 11.**

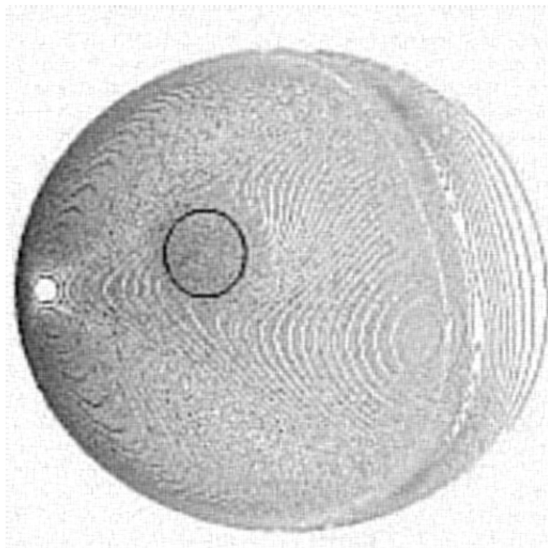
Using qualitative oximetry with optical images from RFI, perfusion deficits and abnormalities in a patient with sickle cell retinopathy is detected. The PDE transform algorithm using up to 4th order PDE is used to denoise and enhance the RFI images.



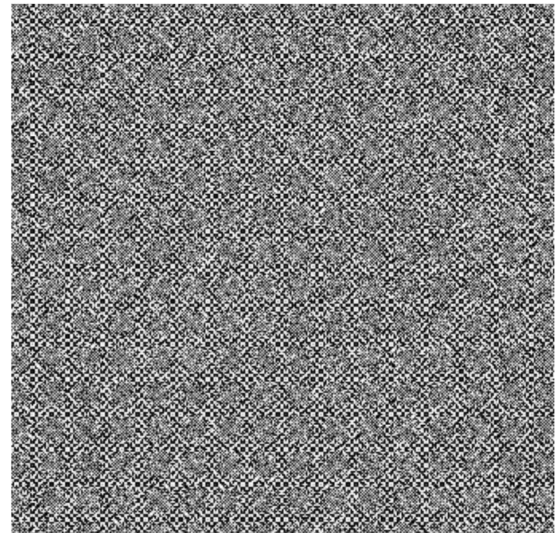
(a) High resolution image created by diffuse optical tomography. The distorted region in the black circle is the area of abnormalities, which is the breast cancer in this case.



(b) The original image is contaminated with highly oscillating and random noise in Figure 12(d).



(c) Denoised image using the PDE transform algorithm. All the fine details and area of abnormalities are recovered from the noise.



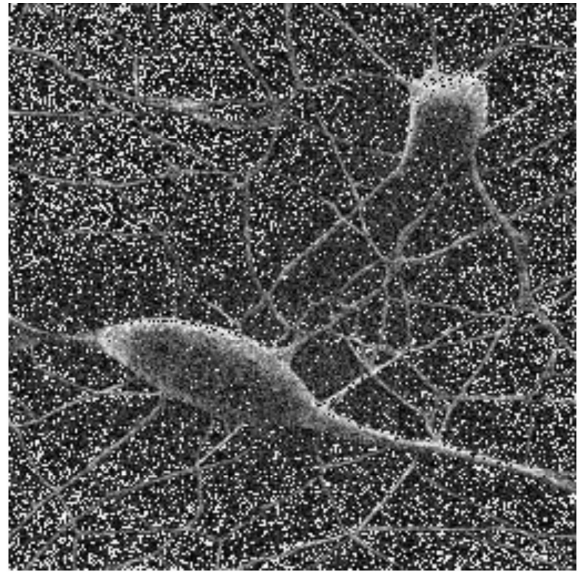
(d) Image showing the noise added to Figure 12(a). The random noise is similar to that in Figure 11(d) except that two different types of highly oscillatory noises are embedded in the Gaussian noise to make the denoising more challenging.

**Figure 12.**

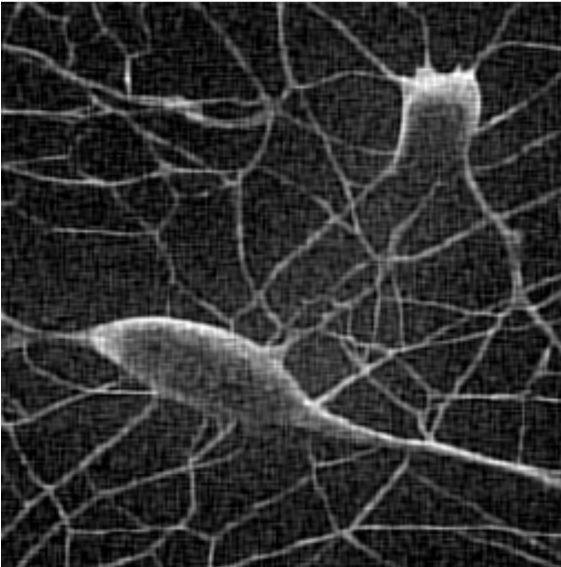
Diffuse optical tomography is used to create extremely high-resolution images from a scattering of infrared and visible light. Such technology can be used for the early detection of breast cancer. As light travels from the source, once it hits the black circle where the tumor is located, the lightwaves become distorted. One practical challenge is that the high resolution imaging is subject to noises, especially highly oscillatory noise which would distort the fine details in the image. The PDE transform can be used to smooth the image and is particularly good for removing high frequency noise and/or oscillatory noises.



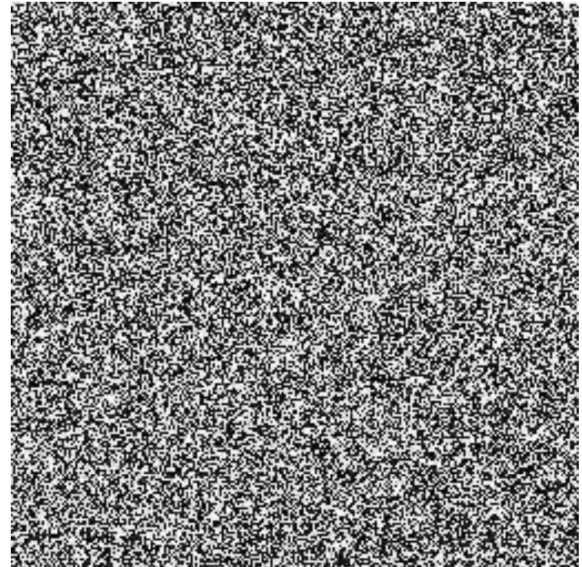
(a) Purkinje neurons are some of the largest of the 100 billion neurons in human brain.



(b) The original image is contaminated with highly oscillatory and random noise in Figure 13(d).



(c) Denoised image using the PDE transform algorithm. Both the shapes and details of dendrites are well recovered from the noise-contaminated image.



(d) Image showing the random noise added to Figure 13(a).

**Figure 13.**

Electron microscope photography of Purkinje neurons showing the details of vast amount of dendrites on the neuron cells. The PDE transform using up to 4th order PDE is used to denoise the image and enhance the details of cell structure and geometry details of the dendrites.