

Published in final edited form as:

Psychometrika. 2011 October ; 76(4): 537–549. doi:10.1007/s11336-011-9218-4.

Exploratory Bi-factor Analysis

Robert I. Jennrich and **Peter M. Bentler**

University of California at Los Angeles

Abstract

Bi-factor analysis is a form of confirmatory factor analysis originally introduced by Holzinger. The bi-factor model has a general factor and a number of group factors. The purpose of this paper is to introduce an exploratory form of bi-factor analysis. An advantage of using exploratory bi-factor analysis is that one need not provide a specific bi-factor model a priori. The result of an exploratory bi-factor analysis, however, can be used as an aid in defining a specific bi-factor model. Our exploratory bi-factor analysis is simply exploratory factor analysis using a bi-factor rotation criterion. This is a criterion designed to produce perfect cluster structure in all but the first column of a rotated loading matrix. Examples are given to show how exploratory bi-factor analysis can be used with ideal and real data. The relation of exploratory bi-factor analysis to the Schmid-Leiman method is discussed.

Keywords

Bi-factor rotation; general factor; group factor; gradient projection algorithms; Holzinger's bi-factor method; Schmid-Leiman method

1. Introduction

Bi-factor analysis is confirmatory factor analysis using a factor loading matrix of the form

$$\Lambda = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \\ * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{pmatrix}$$

More precisely the loadings in the first column are free parameters and after the first column the loading matrix has at most one free parameter in each row. In bi-factor analysis the first factor is called a general factor and the remaining factors are called group factors.

Bi-factor analysis is a fairly extensively used form of confirmatory factor analysis. Recent references include Chen, West & Sousa (2006), Patrick, Hicks, Nichol & Krueger (2007), Pomplun (2007), and Simms, Grös, Watson & O'Hara (2008). Bi-factor models have also become important in the field of item response theory, (e.g., Reise, Morizot, & Hays, 2007) where the group factors are used to explain departures from unidimensionality.

It is our purpose to introduce an exploratory form of bi-factor analysis. This is done by using exploratory factor analysis with a rotation criterion that loads on the first factor and encourages perfect cluster structure for the loadings on the remaining factors. Exploratory bi-factor analysis is simply exploratory factor analysis using a bi-factor rotation criterion. As such it involves a minor change to an exploratory factor analysis program.

Though it generally isn't, bi-factor analysis might also be called confirmatory bi-factor analysis. This terminology may sometimes be used to emphasize its difference from exploratory bi-factor analysis.

Model building in confirmatory bi-factor analysis consists of separating items into groups. This is usually done using prior knowledge from the field under investigation. But this knowledge is not always available or perhaps is not complete. In these cases exploratory factor analysis methods such as the second order Schmid-Leiman (1957) method have been used to aid in defining the required groups. Exploratory bi-factor analysis provides a more direct and satisfactory approach to bi-factor model building.

2. Holzinger's Bi-factor Method: An Historical Note

Those more interested in the future than the past or more interested in keeping this paper as easy to read as possible should proceed to the next section. Holzinger's methods are no longer used and haven't been for some time.

Bi-factor models and methods were introduced by Holzinger. The most accessible reference is Holzinger and Swineford (1937). Holzinger's method for estimating the parameters in a bi-factor model proceeds as follows.

Assume a covariance matrix has the structure

$$\Sigma = \Lambda \Lambda' + \Psi$$

where Ψ is diagonal and Λ has bi-factor structure. Holzinger shows that the loadings λ on the general factor can be expressed in the form

$$\lambda = Q_1(\Sigma) / Q_2(\Sigma)$$

where $Q_1(\Sigma)$ and $Q_2(\Sigma)$ are quadratic functions of Σ . Replacing Σ by a sample covariance matrix S gives an estimate $\hat{\lambda}$ of λ based on S .

Let λ_r be the unconstrained part of the loadings on the r -th group factor. Holzinger shows that

$$\lambda_r = Q_{1r}(\Omega) / Q_{2r}(\Omega)$$

where $\Omega = \Sigma - \lambda \lambda'$ and $Q_{1r}(\Omega)$ and $Q_{2r}(\Omega)$ are quadratic functions of Ω . An estimate $\hat{\lambda}_r$ of λ_r is obtained by replacing Ω by $S - \hat{\lambda} \hat{\lambda}'$.

Holzinger's estimation algorithm is fairly complex, but does not require iteration or even matrix inversion. Today of course one fits bi-factor models by using a CFA program or a SEM program. These use iterative minimum deviance algorithms for estimation.

3. Bi-factor Rotation

Bi-factor rotation is simply rotation using a bi-factor rotation criterion. We define a bi-factor rotation criterion as an index that measures the departure of a loading matrix from bi-factor structure. Using it for rotation will hopefully produce rotated loading matrices with approximate bi-factor structure.

A $p \times k$ loading matrix Λ has **bi-factor structure** if each row of Λ has at most one nonzero element in its last $k - 1$ columns. Note that bi-factor models generate loading matrices with bi-factor structure.

Definition 1: A rotation criterion $B(\Lambda)$ **measures the departure of Λ from bi-factor structure** if any loading matrix with bi-factor structure minimizes B and all minimizers of B have bi-factor structure.

Definition 2: A rotation criterion $B(\Lambda)$ that measures the departure of Λ from bi-factor structure will be called a **bi-factor rotation criterion**.

Theorem 1: If Λ is an arbitrary $p \times k$ loading matrix and

$$B_q(\Lambda) = \text{qmin}(\Lambda_2)$$

where Λ_2 is the sub-matrix of Λ containing its last $k - 1$ columns and qmin is the quartimin rotation criterion, then B_q is a bi-factor rotation criterion.

Proof: Let Λ be an arbitrary $p \times k$ loading matrix. Then

$$\text{qmin}(\Lambda) = \sum_{i=1}^p \sum_{r=1}^k \sum_{s=r+1}^k \lambda_{ir}^2 \lambda_{is}^2 \quad (1)$$

A loading matrix Λ has perfect cluster structure if and only if $\text{qmin}(\Lambda) = 0$. To see this note that all the terms $\lambda_{ir}^2 \lambda_{is}^2$ on the right in (1) have $r \neq s$. If Λ has perfect cluster structure these terms are all zero and $\text{qmin}(\Lambda) = 0$. Conversely if $\text{qmin}(\Lambda) = 0$ all the terms $\lambda_{ir}^2 \lambda_{is}^2$ must be zero and Λ must have perfect cluster structure. Thus Λ has perfect cluster structure if and only if $\text{qmin}(\Lambda) = 0$.

Turning now to the Λ and Λ_2 in the definition of B , note that $B(\Lambda) = 0$ if and only if $\text{qmin}(\Lambda_2) = 0$ and $\text{qmin}(\Lambda_2) = 0$ if and only if Λ_2 has perfect cluster structure and this happens if and only if Λ has bi-factor structure. Thus $B(\Lambda)$ measures the departure of Λ from bi-factor structure and hence is a bi-factor rotation criterion.

The criterion B_q in Theorem 1 will be called the **bi-quartimin criterion**. Although $B_q(\Lambda)$ does not depend on the first column of Λ , when $B_q(\Lambda)$ is used for rotation, it is all columns of Λ , including its first, that are rotated.

Let A be an initial loading matrix and let $\hat{\Lambda}$ minimize a bi-factor rotation criterion B over all rotations of A . Then $\hat{\Lambda}$ will be called a bi-factor rotation of A corresponding to B . One might use orthogonal or oblique rotation, but unless otherwise stated we will consider only orthogonal rotation in this paper.

An analysis using a deviance function to extract an initial loading matrix A from a sample covariance matrix S followed by a bi-factor rotation of A will be called an **exploratory bi-factor analysis** (EBFA) of S .

Unless otherwise stated in what follows the deviance function will be the normal deviance function, the rotation criterion will be the bi-quartimin criterion, and the rotation method will be orthogonal.

Some care must be used in defining bi-factor rotation criteria to assure they measure departure from bi-factor structure. Bernaards and Jennrich (2003) show that if there is an orthogonal rotation Λ of an initial loading matrix A that has perfect cluster structure, then Λ maximizes the varimax criterion over all orthogonal rotations of A . This might motivate one to use

$$B_v(\Lambda) = -\text{vmax}(\Lambda_2)$$

where vmax is the varimax rotation criterion as a criterion for finding bifactor structure. The problem with using B_v for this purpose is that B_v does not measure departure from bi-factor structure because $-\text{vmax}$ does not measure departure from perfect cluster structure. To see the latter note that

$$\text{vmax}(\Lambda) = \frac{1}{n} \sum_{r=1}^k \left(\sum_{i=1}^p (\lambda_{ir}^2 - \frac{1}{n} \sum_{i=1}^p \lambda_{ir}^2)^2 \right) \quad (2)$$

Let Λ have perfect cluster structure and $\text{vmax}(\Lambda)$ be different from zero. For any scalar α let

$$\Lambda_\alpha = \alpha \Lambda$$

Then Λ_α has perfect cluster structure for all α . Moreover

$$\text{vmax}(\Lambda_\alpha) = \alpha^4 \text{vmax}(\Lambda)$$

Since this is not a constant function of α at least one Λ_α must fail to minimize vmax and hence vmax does not measure departure from perfect cluster structure.

Since $B_v(\Lambda)$ does not measure the departure of Λ from bi-factor structure one might suspect its ability to recover bi-factor structure may be limited. An example in the next section shows it can fail to recover bi-factor structure.

For the computations in this paper the Matlab program `factoran` was used for the extraction of an initial loading matrix A and the gradient projection algorithm of Jennrich (2001) was used for rotation. This algorithm can be found at <http://www.stat.ucla.edu/research/gpa> where it is called `GPForth`.

For those who may be interested in computing formulas for evaluating the bi-quartimin criterion and its derivative, these are given in the Appendix. Please don't view these as part of the definition of bi-factor rotation. The definition is that given above. Also given in the Appendix is Matlab code for implementing the computing formulas and one additional line of Matlab code that can be used to perform bi-quartimin rotation.

The quartimin and varimax formulas in equations (1) and (2) may be found in Harman (1976).

3.1 An Example

Let an initial loading matrix A be the principal components rotation of

$$\Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

That is A is an orthogonal rotation of Λ such that $A'A$ is diagonal. A is called a principal components rotation because the columns of A are principal components of AA' . In this example

$$A = \begin{pmatrix} 1.17 & 0.78 & 0.18 \\ 2.08 & 0.78 & -0.22 \\ 1.17 & 0.78 & 0.18 \\ 2.15 & -0.62 & -0.08 \\ 1.23 & -0.62 & 0.32 \\ 2.15 & -0.62 & -0.08 \end{pmatrix}$$

Using the gradient projection algorithm of Jennrich (2001) to minimize the bi-quartimin criterion over all orthogonal rotations of A gives

$$\hat{\Lambda}_q = \begin{pmatrix} 1.00 & 1.00 & 0 \\ 2.00 & 1.00 & 0 \\ 1.00 & 1.00 & 0 \\ 2.00 & 0 & 1.00 \\ 1.00 & 0 & 1.00 \\ 2.00 & 0 & 1.00 \end{pmatrix}$$

which is a nice result because it is equal to Λ to the precision displayed. Actually $\hat{\Lambda}_q$ and Λ agree to 7.42 decimal places.

Turning to the varimax problem using the criterion B_v to rotate A gave

$$\hat{\Lambda}_v = \begin{pmatrix} 1.28 & 0.53 & 0.29 \\ 1.81 & 1.31 & -0.04 \\ 1.28 & 0.53 & 0.29 \\ 0.67 & 2.13 & 0.08 \\ 0.14 & 1.35 & 0.40 \\ 0.67 & 2.13 & 0.08 \end{pmatrix}$$

which has poor bi-factor structure.

4. Real Data Examples

4.1 The Twenty Four Psychological Tests Data

This is a very well known data set that can be found in Harman (1967, p.125). Harman made a confirmatory bi-factor analysis of it using Holzinger's method. To our knowledge these data have never been analyzed using a bifactor model and modern confirmatory methods.

An exploratory bi-factor analysis of Harman's data gave the loading matrix

$$\hat{\Lambda} = \begin{pmatrix} 0.68 & -0.06 & -0.28 & -0.14 \\ 0.41 & -0.02 & -0.21 & -0.08 \\ 0.46 & -0.00 & -0.37 & -0.10 \\ 0.51 & 0.07 & -0.24 & -0.14 \\ 0.54 & 0.59 & 0.07 & -0.02 \\ 0.51 & 0.64 & -0.05 & 0.08 \\ 0.51 & 0.67 & 0.03 & -0.09 \\ 0.59 & 0.39 & -0.00 & -0.09 \\ 0.51 & 0.69 & -0.07 & 0.08 \\ 0.49 & -0.02 & 0.72 & -0.00 \\ 0.56 & -0.00 & 0.32 & 0.17 \\ 0.57 & -0.20 & 0.43 & -0.15 \\ 0.67 & -0.05 & 0.14 & -0.21 \\ 0.36 & 0.12 & 0.02 & 0.44 \\ 0.37 & 0.03 & -0.03 & 0.41 \\ 0.54 & -0.08 & -0.22 & 0.32 \\ 0.44 & 0.03 & 0.12 & 0.43 \\ 0.57 & -0.15 & 0.06 & 0.24 \\ 0.44 & 0.02 & -0.03 & 0.21 \\ 0.58 & 0.22 & -0.15 & 0.09 \\ 0.63 & -0.04 & 0.10 & -0.03 \\ 0.57 & 0.20 & -0.15 & 0.09 \\ 0.68 & 0.16 & -0.12 & -0.03 \\ 0.61 & 0.18 & 0.29 & 0.09 \end{pmatrix}$$

To simplify viewing this all loadings with absolute value less than 0.3 were set equal to zero giving the rounded loading matrix

$$\widehat{\Lambda}_r = \begin{pmatrix} 0.68 & 0 & 0 & 0 \\ 0.41 & 0 & 0 & 0 \\ 0.46 & 0 & -0.37 & 0 \\ 0.51 & 0 & 0 & 0 \\ 0.54 & 0.59 & 0 & 0 \\ 0.51 & 0.64 & 0 & 0 \\ 0.51 & 0.67 & 0 & 0 \\ 0.59 & 0.39 & 0 & 0 \\ 0.51 & 0.69 & 0 & 0 \\ 0.49 & 0 & 0.72 & 0 \\ 0.56 & 0 & 0.32 & 0 \\ 0.57 & 0 & 0.43 & 0 \\ 0.67 & 0 & 0 & 0 \\ 0.36 & 0 & 0 & 0.44 \\ 0.37 & 0 & 0 & 0.41 \\ 0.54 & 0 & 0 & 0.32 \\ 0.44 & 0 & 0 & 0.43 \\ 0.57 & 0 & 0 & 0 \\ 0.44 & 0 & 0 & 0 \\ 0.58 & 0 & 0 & 0 \\ 0.63 & 0 & 0 & 0 \\ 0.57 & 0 & 0 & 0 \\ 0.68 & 0 & 0 & 0 \\ 0.61 & 0 & 0 & 0 \end{pmatrix}$$

This suggests that the tests are divided into four distinct groups. One loads only on a general factor and the other three load on the first factor and on only one additional factor. We will not attempt to explain in psychological terms what this says about the twenty four psychological tests.

4.1.1 The Corresponding Confirmatory Solution—It is of interest to consider a modern confirmatory solution corresponding to the previous exploratory solution. More precisely consider a confirmatory bi-factor analysis using the rounded exploratory bi-factor loadings above to define the required groups and using minimization of the corresponding maximum likelihood deviance function for estimation. The table below gives the resulting rounded confirmatory loadings $\widehat{\Lambda}_{cr}$ and the rounded exploratory loadings $\widehat{\Lambda}_r$.

Confirmatory $\widehat{\Lambda}_{cr}$				Exploratory $\widehat{\Lambda}_r$			
0.64	0	0	0	0.68	0	0	0
0.41	0	0	0	0.41	0	0	0
0.48	0	0.34	0	0.46	0	0.37	0
0.51	0	0	0	0.51	0	0	0
0.57	0.57	0	0	0.54	0.59	0	0
0.55	0.61	0	0	0.51	0.64	0	0
0.54	0.64	0	0	0.51	0.67	0	0
0.62	0.36	0	0	0.59	0.39	0	0

Confirmatory $\hat{\Lambda}_{cr}$				Exploratory $\hat{\Lambda}_r$			
0.56	0.64	0	0	0.51	0.69	0	0
0.44	0	0.82	0	0.49	0	0.72	0
0.53	0	0.32	0	0.56	0	0.32	0
0.49	0	0.45	0	0.57	0	0.43	0
0.62	0	0	0	0.67	0	0	0
0.37	0	0	0.54	0.36	0	0	0.44
0.35	0	0	0.46	0.37	0	0	0.41
0.52	0	0	0.36	0.54	0	0	0.32
0.43	0	0	0.33	0.44	0	0	0.43
0.54	0	0	0	0.57	0	0	0
0.47	0	0	0	0.44	0	0	0
0.61	0	0	0	0.58	0	0	0
0.63	0	0	0	0.63	0	0	0
0.62	0	0	0	0.57	0	0	0
0.71	0	0	0	0.68	0	0	0
0.63	0	0	0	0.61	0	0	0

The confirmatory and exploratory solutions are quite similar. In this example one might view the confirmatory solution as polish for the exploratory solution.

Although we are not prepared to discuss an oblique form of EBFA at this point, the editor, the associate editor, and a reviewer have asked us to at least show the results for the twenty four psychological tests data when using oblique rotation. This gave the rounded loading matrix

$$\widehat{\Lambda}_{\text{oblq r}} = \begin{pmatrix} 0.65 & 0 & 0.37 & 0 \\ 0.39 & 0 & 0 & 0 \\ 0.42 & 0 & 0.43 & 0 \\ 0.49 & 0 & 0.31 & 0 \\ 0.54 & 0.60 & 0 & 0 \\ 0.51 & 0.64 & 0 & 0 \\ 0.51 & 0.68 & 0 & 0 \\ 0.59 & 0.40 & 0 & 0 \\ 0.50 & 0.69 & 0 & 0 \\ 0.56 & 0 & -0.66 & 0 \\ 0.59 & 0 & 0 & 0 \\ 0.61 & 0 & -0.33 & 0 \\ 0.69 & 0 & 0 & 0 \\ 0.35 & 0 & 0 & 0.45 \\ 0.35 & 0 & 0 & 0.43 \\ 0.50 & 0 & 0 & 0.36 \\ 0.43 & 0 & 0 & 0.44 \\ 0.56 & 0 & 0 & 0 \\ 0.43 & 0 & 0 & 0 \\ 0.55 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0 \\ 0.55 & 0 & 0 & 0 \\ 0.67 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0 \end{pmatrix}$$

Except for the third factor the rounded orthogonal loading matrix $\widehat{\Lambda}_r$ and the rounded oblique loading matrix $\widehat{\Lambda}_{\text{oblq r}}$ are quite similar. We will not attempt to state which matrix suggests a more appropriate bi-factor model for the twenty four psychological tests data.

We are currently working on a paper on oblique EBFA that defines it carefully, deals with a number of technical issues that do not arise in the orthogonal case, and compares its performance with that of the orthogonal case on both constructed and real data examples.

4.2 The Chen, West and Sousa (2006) Data

Chen et al. give a confirmatory bi-factor analysis using a quality of life data set. An exploratory bi-factor analysis of their data gives the loading matrix

$$\hat{\Lambda} = \begin{pmatrix} 0.47 & 0.51 & -0.03 & -0.08 & -0.04 \\ 0.38 & 0.36 & 0.05 & -0.03 & 0.02 \\ 0.48 & 0.57 & 0.02 & 0.05 & 0.03 \\ 0.38 & 0.57 & 0.01 & 0.02 & -0.01 \\ 0.51 & 0.52 & -0.02 & -0.02 & -0.01 \\ 0.52 & 0.01 & 0.36 & -0.00 & -0.07 \\ 0.51 & -0.03 & 0.29 & -0.01 & 0.02 \\ 0.54 & 0.03 & 0.38 & -0.08 & -0.01 \\ 0.55 & -0.03 & 0.31 & 0.17 & -0.02 \\ 0.55 & -0.00 & -0.05 & 0.34 & 0.03 \\ 0.67 & -0.04 & -0.11 & 0.07 & 0.01 \\ 0.54 & -0.05 & 0.04 & 0.30 & -0.00 \\ 0.61 & 0.16 & -0.07 & 0.02 & 0.04 \\ 0.73 & -0.02 & -0.14 & -0.06 & -0.13 \\ 0.65 & -0.00 & -0.07 & -0.07 & 0.55 \\ 0.63 & -0.01 & 0.08 & 0.02 & 0.39 \\ 0.58 & -0.01 & 0.02 & 0.07 & 0.54 \end{pmatrix}$$

To simplify viewing this all numbers with absolute value less than 0.2 were set to zero giving the rounded loading matrix

$$\hat{\Lambda}_r = \begin{pmatrix} 0.47 & 0.51 & 0 & 0 & 0 \\ 0.38 & 0.36 & 0 & 0 & 0 \\ 0.48 & 0.57 & 0 & 0 & 0 \\ 0.38 & 0.57 & 0 & 0 & 0 \\ 0.51 & 0.52 & 0 & 0 & 0 \\ 0.52 & 0 & 0.36 & 0 & 0 \\ 0.51 & 0 & 0.29 & 0 & 0 \\ 0.54 & 0 & 0.38 & 0 & 0 \\ 0.55 & 0 & 0.31 & 0 & 0 \\ 0.55 & 0 & 0 & 0.34 & 0 \\ 0.67 & 0 & 0 & o & 0 \\ 0.54 & 0 & 0 & 0.30 & 0 \\ 0.61 & 0 & 0 & o & 0 \\ 0.73 & 0 & 0 & o & 0 \\ 0.65 & 0 & 0 & 0 & 0.55 \\ 0.63 & 0 & 0 & 0 & 0.39 \\ 0.58 & 0 & 0 & 0 & 0.54 \end{pmatrix}$$

Chen et al. used a standard bi-factor model based on earlier studies of quality of life data. The bi-factor model suggested by $\hat{\Lambda}_r$ agrees exactly with this stand model except for the three loadings on the third factor denoted by small o . These were free loadings in Chen et al. model. In Chen et al.'s confirmatory analysis the loading estimates in these three positions were "insignificant". This motivated Chen et al. to suggest that the third group factor might be absorbed by the general factor and dropped. An alternative suggested by $\hat{\Lambda}_r$ would be to retain the third group factor but add the restriction that the loadings in the positions containing small o be zero.

4.3 Local Minima

All results thus far were those corresponding to the best of 10 random starts of the rotation algorithm. Only the first example produced a non-unique local minimizer. It, however, suggests that a general purpose exploratory bi-factor analysis program should have the option of reporting the best of a number of random starts.

5. The Schmid-Leiman method

At present the primary exploratory method for building bi-factor models is the Schmid-Leiman (SL) method and hence this is the primary competitor to our EBFA method. See Reise, Moore and Haviland (2010) and Chen, West, and Sousa (2006) for a discussion of using the SL method for building bi-factor models.

The exploratory SL method proceeds as follows.

Step 1: Perform an oblique EFA of a sample covariance matrix S to produce factor loadings Λ , factor correlations Φ , and unique variances Ψ .

Step 2: Perform a single factor EFA of Φ to produce a loading vector γ and unique variances Δ .

Step 3: Let

$$\Lambda_{SL} = (\Lambda\gamma, \Lambda\Delta^{1/2}) \quad (3)$$

Then Λ_{SL} and Ψ is called a SL orthogonalization of the oblique analysis in Step 1.

As noted above the SL method has been used to build bi-factor models. This is done by looking at Λ_{SL} and hoping to find approximate bi-factor structure.

There is a serious problem here, however. Note that the first column of Λ_{SL} is a linear combination of its remaining columns. If the data are generated by a bi-factor model whose loading matrix does not satisfy this condition, it is not at all clear that Λ_{SL} will have approximate bi-factor structure. As shown in the next section Λ_{SL} may not even come close to having bi-factor structure.

For a general discussion of SL and other methodologies see Yung et.al (1999). The only SL methodology used in this paper is that outlined above which can be found in the Reise Moore and Haviland, and Chen et.al papers cited there.

To compare the Schmid-Leiman and exploratory bi-factor methods for building bi-factor models we will consider some numerical examples. In these examples quartimin rotation is used in Step 1 of the SL method to produce the factor correlation matrix that is factored in Step 2.

5.1 The Schmid-Leiman method can fail seriously

Consider an orthogonal bi-factor model defined by the loading matrix

$$\Lambda = \begin{pmatrix} 0.70 & 0.50 & 0.00 & 0.00 \\ 0.70 & 0.00 & 0.00 & 0.00 \\ 0.70 & -0.50 & 0.00 & 0.00 \\ 0.70 & 0.00 & 0.50 & 0.00 \\ 0.70 & 0.00 & 0.00 & 0.00 \\ 0.70 & 0.00 & -0.50 & 0.00 \\ 0.70 & 0.00 & 0.00 & 0.50 \\ 0.70 & 0.00 & 0.00 & 0.00 \\ 0.70 & 0.00 & 0.00 & -0.50 \end{pmatrix}$$

and the assumption that its covariance matrix Σ is a correlation matrix. Because the first column of Λ is not a linear combination of its remaining columns a Schmid-Leiman analysis based on data generated by this model cannot consistently estimate Λ and this may compromise its ability to identify a bi-factor model.

A Schmid-Leiman analysis of $S = \Sigma$ gave

$$\Lambda_{SL} = \begin{pmatrix} 0.59 & 0.27 & 0.50 & 0.27 \\ 0.59 & 0.27 & 0.00 & 0.27 \\ 0.59 & 0.27 & -0.50 & 0.27 \\ 0.56 & 0.26 & 0.00 & 0.26 \\ 0.59 & 0.27 & 0.00 & 0.27 \\ 0.56 & 0.26 & 0.00 & 0.26 \\ 0.59 & 0.63 & 0.00 & -0.09 \\ 0.59 & 0.27 & 0.00 & 0.27 \\ 0.59 & -0.08 & 0.00 & 0.63 \end{pmatrix}$$

This has far from bi-factor structure. In this example the Schmid-Leiman method doesn't even come close to suggesting an appropriate bi-factor model.

When EBFA is applied to $S = \Sigma$ it recovers Λ to more than eight decimal places. Thus in this example EBFA performs much better than the Schmid-Leiman method for building an appropriate bi-factor model. This may not of course be true in general. For example if the first column of Λ were approximately a linear combination of its remaining columns, one might expect Λ_{SL} to approximate Λ .

In general, however, it seems reasonable to use EBFA for building bi-factor models and Schmid-Leiman methods for other purposes such as building second order factor analysis models.

5.3 Applying Schmid-Leiman to the Chen et al. data

Applied to the Chen et al. data the Schmid-Leiman method with 4 primary factors gave the rounded Schmid-Leiman loading matrix

$$\Lambda_{SLr} = \begin{pmatrix} 0.45 & 0.53 & 0 & 0 & 0 \\ 0.37 & 0.37 & 0 & 0 & 0 \\ 0.48 & 0.55 & 0 & 0 & 0 \\ 0.37 & 0.56 & 0 & 0 & 0 \\ 0.50 & 0.53 & 0 & 0 & 0 \\ 0.49 & 0 & 0.41 & 0 & 0 \\ 0.48 & 0 & 0.34 & 0 & 0 \\ 0.49 & 0 & 0.42 & 0 & 0 \\ 0.53 & 0 & 0.35 & 0 & 0 \\ 0.55 & 0 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0 & 0 \\ 0.53 & 0 & 0 & 0 & 0 \\ 0.59 & 0 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0 & 0.56 \\ 0.62 & 0 & 0 & 0 & 0.41 \\ 0.59 & 0 & 0 & 0 & 0.53 \end{pmatrix}$$

Here all loadings with absolute values less than 0.3 have been set to zero. This Schmid-Leiman matrix suggests that no items should be assigned to the third group. This is consistent with the comment by Chen et.al that the third group factor might be absorbed by the general factor in future analyses.

We now have three possible models for the Chen et.al data. Chen et.al's original model, the bi-factor corresponding to the EBFA in Section 4.2, and that suggested by the SL analysis above.

If one does not prefer a particular model, one might use confirmatory methods to help decide.

6. Discussion

If nothing else it would seem quite reasonable to add a bi-factor rotation criterion such as the bi-quartimin criterion to libraries of rotation criteria and to add bi-factor rotation as an option in general purpose exploratory factor analysis programs such as those found in SAS, SPSS and STATA. This would make exploratory bi-factor analysis immediately available to data analysts.

Except for one, our EBFA examples have all used orthogonal rotation. One might, however, consider using oblique rotation instead and we are presently looking into the problems that arise when doing this.

We have used only the maximum likelihood deviance function for extracting initial loadings. One might also consider other deviance functions such as least squares or weighted least squares for this purpose.

We have considered only one specific bi-factor rotation criterion namely the bi-quartimin criterion. We have not identified others, but one candidate might be a criterion based on minimum entropy. This issue needs further investigation.

A reviewer has suggested considering statistical properties of bi-factor loadings. For example standard errors for these. This issue also needs further investigation.

It should be mentioned that Carroll (see Harman 1967) has used the name Bi-Quartimin for a rotation criterion that differs from our bi-quartimin criterion. This might motivate us to change the name of our criterion, but bi-quartimin is too appropriate a name for us to do this.

Most authors, including Holzinger, require that in a bi-factor model each row of the loading matrix must have a free loading in its first column and exactly one free loading in the remaining columns. We require at most one free loading in the remaining columns. Chen et al. (2006) call such models incomplete bi-factor models. In this terminology our bi-factor models include both complete and incomplete bi-factor models.

The Schmid-Leiman method has been used to design bi-factor models, but this is problematic. The Schmid-Leiman method is designed to identify two stage models, not bi-factor models. We have shown that its use for the latter purpose can fail. The advantage of EBFA is that it is designed to identify bi-factor models directly and in examples does so quite well. Moreover, it is very simple. EBFA is simply standard exploratory factor analysis using a bi-factor rotation criterion.

Acknowledgments

The research of the second author was supported in part by grants 5P01DA01070-36 and 5K05DA00017-33 from the National Institute on Drug Abuse and by grant 5R44CA137841-3 from the National Cancer Institute. We would like to thank the Editor, Associate Editor, and three anonymous reviewers for helpful suggestions and Steven Reise for suggesting we consider an exploratory approach to bi-factor analysis.

Appendix

Here we will give computing formulas to show how the bi-quartimin criterion might be evaluated and used for rotation. Let

$$B(\Lambda) = \text{qmin}(\Lambda_2)$$

where $\text{qmin}(\Lambda_2)$ is the value of the quartimin criterion at Λ_2 . A simple computing formula for this is given by

$$\text{qmin}(\Lambda_2) = \langle \Lambda_2^2, \Lambda_2^2 N \rangle$$

where Λ_2^2 is the element-wise square of Λ_2 , N is a $(k-1) \times (k-1)$ matrix with zeros on the diagonal and ones elsewhere, and $\langle A, B \rangle$ is the Frobenius product of two equal sized matrices A and B . In formulas

$$\langle A, B \rangle = \sum_i \sum_j A_{ij} B_{ij}$$

In addition to code to evaluate the rotation criterion most general purpose rotation programs also require code to evaluate its derivative. For the bi-quartimin criterion this is

$$\frac{dB}{d\Lambda} = 4(0, \Lambda_2 \cdot (\Lambda_2 N))$$

where 0 is a zero column vector and $A \cdot B$ is the element-wise product of two equal sized matrices A and B .

For rotation one may use any general purpose rotation algorithm. As noted we have used the gradient projection algorithm GPForth which requires code for evaluating $B(\Lambda)$ and $dB/d\Lambda$.

The Matlab the code for computing the value v and gradient G of the bi-quartimin criterion at L is:

```
function [v,G]=vGqmin(L)
[p,k]=size(L);
L2=L.^2;
N=ones(k,k)-eye(k);
v=sum(sum(L2.*(L2*N)));
G=4*L.*(L2*N);
```

To rotate a matrix A download GPForth from <http://www.stat.ucla.edu/research/gpa> and write the code:

```
Lh=GPForth(@vGqmin,A,T);
```

where T is any k by k orthogonal matrix. When this is executed Lh will be the bi-quartimin rotation of A .

References

- Bernaards CA, Jennrich RI. Orthomax rotation and perfect simple structure. *Psychometrika*. 2003; 68:585–588.
- Chen FF, West SG, Sousa KH. A comparison of bifactor and second-order models of the quality of life. *Multivariate Behavioral Research*. 2006; 41:189–225.
- Harman, HH. *Modern Factor Analysis*. Chicago: The University of Chicago Press; 1976.
- Holzinger KJ, Swineford S. The Bi-factor method. *Psychometrika*. 1937; 47:41–54.
- Jennrich RI. A simple general procedure for orthogonal rotation. *Psychometrika*. 2001; 66:289–306.
- Patrick CJ, Hicks BM, Nichol PE, Krueger RF. A bi-factor approach to modeling the structure of the Psychopathy checklist-revisited. *Journal of Personality Disorders*. 2007; 21:118–141. [PubMed: 17492917]
- Pomplun M. A bifactor analysis for a mode-of-administration effect. *Applied Measurement in Education*. 2007; 20:137–152.
- Reise SP, Morizot J, Hays RD. The role of the bifactor model in resolving dimensionality issues in health outcomes measures. *Medical Care*. 2007; 16:19–31.
- Reise SP, Moore TM, Haviland MG. Bi-factor models and rotations: Exploring the extent to which multidimensional data yield univocal scale scores. *Journal of Personality Assessment*. 2010; 92:544–559. [PubMed: 20954056]
- Schmid J, Leiman JM. The development of hierarchical factor solutions. *Psychometrika*. 1957; 22:53–61.
- Simms LJ, Grös DF, Watson D, O'Hara MW. Parsing the general and specific components of depression and anxiety with bifactor modeling. *Depression and Anxiety*. 2008; 25:E34–E46. [PubMed: 18027844]
- Yung Y-F, Thissen D, McLeod LD. On the relation between between the heigher-order factor model and the Hierarchical factor model. *Pshchometrika*. 1999; 64:113–128.