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Mathematics

A natural representation of the Fischer–Griess Monster with the modular function J as character

(vertex operators/finite simple group F_1/Monstrous Moonshine/affine Lie algebras/basic modules)

I. B. FRENKEL*†, J. LEPOWSKY*†, AND A. MEURMAN†

*Department of Mathematics, Rutgers University, New Brunswick, NJ 08903; and †Mathematical Sciences Research Institute, Berkeley, CA 94720

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ABSTRACT **We announce the construction of an irreducible graded module V for an "affine" commutative nonassociative algebra $\hat{\mathfrak{B}}$. This algebra is an "affinization" of a slight variant $\mathfrak{B}$ of the commutative nonassociative algebra B defined by Griess in his construction of the Monster sporadic group F_1. The character of V is given by the modular function $J(q) = q^{-1} + 0 + 196884q + \dots$. We obtain a natural action of the Monster on V compatible with the action of $\hat{\mathfrak{B}}$, thus conceptually explaining a major part of the numerical observations known as Monstrous Moonshine. Our construction starts from ideas in the theory of the basic representations of affine Lie algebras and develops further the calculus of vertex operators. In particular, the homogeneous and principal representations of the simplest affine Lie algebra $A_1^{(1)}$ and the relation between them play an important role in our construction. As a corollary we deduce Griess's results, obtained previously by direct calculation, about the algebra structure of B and the action of F_1 on it. In this work, the Monster, a finite group, is defined and studied by means of a canonical infinite-dimensional representation.**

1. Introduction

Amazing discoveries about the simple group F_1, the Fischer–Griess Monster, appeared before the group was actually born. Most of these results were based on the postulated existence of a 196883-dimensional irreducible representation of F_1. Starting from McKay's observation that 196883 + 1 is a coefficient of the modular function $J(q) = q^{-1} + 0 + 196884q + \dots$, Thompson (1) found further numerology suggesting that there should be a natural graded F_1-module $V = \coprod_{n\geq -1} V_n$ with character $J(q)$. In particular, V_1 would be the direct sum of the trivial and 196883-dimensional irreducible modules.

Collecting and enriching these discoveries, Conway and Norton (2) associated "Thompson series"

$$T_g(q) = \sum_{n\geq -1} H_n(g)q^n,\ q = e^{2\pi i\tau},\ \mathrm{Im}\tau > 0$$

to each conjugacy class g of F_1, with the property that each $T_g(q)$ is the normalized generator of a genus 0 function field corresponding to a discrete subgroup of SL(2,**R**). In particular, $T_1(q) = J(q)$, the field generator corresponding to the modular group SL(2,**Z**). Conway and Norton conjectured that each H_n is a character of F_1, and Atkin, Fong, and Smith proved that the H_n are generalized characters and gave overwhelming evidence that they are characters (3).

In a tremendous piece of work, Griess (4) constructed the desired 196883-dimensional representation of F_1, giving birth to the largest sporadic group, thereby showing that the strange discoveries could make sense. However, instead of illuminating the mysteries, he added a new one, by constructing a peculiar commutative nonassociative algebra B and proving directly that its automorphism group contains F_1.

The present work started as an attempt to explain the appearance of $J(q)$ as well as the structure of B. Our understanding of these phenomena has now led us to conceptual constructions of several objects: V; a variant $\mathfrak{B}$ of B; an "affinization" $\hat{\mathfrak{B}}$ of $\mathfrak{B}$; and F_1, which appears as a group of operators on V and at the same time as a group of algebra automorphisms of $\mathfrak{B}$ and of $\hat{\mathfrak{B}}$. While we were heavily influenced by Griess's constructions, the present infinite-dimensional theory, an extension of the theory of vertex operator constructions of the basic modules of affine Lie algebras, is completely self-contained. Moreover, it predicts from the start the six arbitrary parameters in Griess's approach, and it produces an algebra with an identity element.

A significant portion of sporadic group theory is now brought into the realm of Lie algebras and their representations. A uniform theory of all the finite simple groups may now be on the horizon.

These results were presented at the November 1983 workshop on Vertex Operators in Mathematics and Physics at the Mathematical Sciences Research Institute. The details will appear elsewhere.

2. Vertex Operators and the Space V

By a lattice $(L, \langle\cdot,\cdot\rangle)$ we shall understand a free abelian group L together with a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on L such that $\langle\alpha,\beta\rangle \in (1/r)\,\mathbf{Z}$ and $\langle\alpha,\alpha\rangle \in (2/r)\,\mathbf{Z}$ for all $\alpha, \beta \in L$. Here r is a positive integer. Note that rL is an even lattice.

Let L be a lattice and consider a 2-cocycle ε_0 on L with values in $\mathbf{Z}/s\mathbf{Z}$ for a positive integer s. (In this paper we shall need the cases $r = 1$, 2 and $s = 2$, 4.) To ε_0 is associated a central extension $\hat{L} = \{e_\alpha\kappa^m | \alpha \in L,\ m \in \mathbf{Z}/s\mathbf{Z}\}$ of the group L, the product in $\hat{L}$ being determined by the conditions $\kappa^s = 1$, κ central, and $e_\alpha e_\beta = e_{\alpha+\beta}\,\kappa^{\varepsilon_0(\alpha,\beta)}$. We have an exact sequence $1 \to \langle\kappa\rangle \to \hat{L} \xrightarrow{-} L \to 1$, where $\overline{e_\alpha} = \alpha$.

Let $\mathbf{F}$ be a field of characteristic 0 containing a primitive sth root of unity ω. Denote by $\{e^\alpha | \alpha \in L\}$ the basis of the group algebra $\mathbf{F}[L]$ consisting of group elements, and let $\hat{L}$ act on $\mathbf{F}[L]$ via $e_\alpha\cdot e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta}$, $\kappa\cdot e^\beta = \omega e^\beta$, where $\varepsilon(\alpha,\beta) = \omega^{\varepsilon_0(\alpha,\beta)}$. Also, let T be any $\hat{L}$-module such that κ acts as multiplication by ω. (We shall usually take T finite-dimensional.)

We next associate to L two infinite-dimensional Lie algebras. Set $\mathfrak{h} = L \otimes_{\mathbf{Z}} \mathbf{F}$ and extend $\langle\cdot,\cdot\rangle$ to a symmetric bilinear form on $\mathfrak{h}$. For $Z = \mathbf{Z}$ or $\mathbf{Z} + \frac{1}{2}$ consider

$$\hat{\mathfrak{h}}_Z = \mathfrak{h} \otimes_{\mathbf{F}} \coprod_{n\in Z} \mathbf{F}t^n \oplus \mathbf{F}c,\ c \neq 0,$$

a $\frac{1}{2}\mathbf{Z}$-graded vector space via $\deg \mathfrak{h} \otimes t^n = n$, $\deg \mathbf{F}c = 0$. We shall view $\coprod_{n\in Z} \mathbf{F}t^n$ as a subspace of the algebra $\mathbf{F}[t^{1/2},$

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$t^{-1/2}]$ of Laurent polynomials in an indeterminate $t^{1/2}$. The formulas $[c, \hat{\mathfrak{h}}_Z] = 0$, $[x \otimes t^m, y \otimes t^n] = \langle x, y\rangle\, m\, \delta_{m+n,0}\, c$ for $x, y \in \mathfrak{h}$, $m, n \in Z$ make $\hat{\mathfrak{h}}_Z$ a $\frac{1}{2}\mathbf{Z}$-graded Lie algebra. Consider the abelian subalgebras

$$\mathfrak{b} = \coprod_{\substack{n\in Z\\ n\geq 0}} \mathfrak{h} \otimes t^n \oplus \mathbf{F}\, c, \qquad \hat{\mathfrak{h}}_Z^- = \coprod_{\substack{n\in Z\\ n<0}} \mathfrak{h} \otimes t^n.$$

Make $\mathbf{F}$ a $\mathfrak{b}$-module by taking $(\mathfrak{h} \otimes t^n)\cdot \mathbf{F} = 0$ and letting c act by 1. We identify the induced $\hat{\mathfrak{h}}_Z$-module $S(\hat{\mathfrak{h}}_Z^-) = U(\hat{\mathfrak{h}}_Z) \otimes_{U(\mathfrak{b})} \mathbf{F}$ with the indicated symmetric algebra. This space has a natural $\frac{1}{2}\mathbf{Z}$-grading, and is in fact a graded $\hat{\mathfrak{h}}_Z$-module. For $n \in Z$, let $\alpha(n)$ denote the operator $\alpha \otimes t^n$ on $S(\hat{\mathfrak{h}}_Z^-)$.

We can now introduce the vertex operators $X(\alpha, \zeta)$ in the two cases. First (for $Z = \mathbf{Z}$) set

$$V_L = S(\hat{\mathfrak{h}}_Z^-) \otimes_{\mathbf{F}} \mathbf{F}[L], \qquad [1]$$

$$X(\alpha, \zeta) = \exp\Big(\sum_{\substack{n\in\mathbf{Z}\\ n>0}} \alpha(-n)\zeta^{-n}/n\Big)\exp\Big(-\sum_{\substack{n\in\mathbf{Z}\\ n>0}} \alpha(n)\zeta^{n}/n\Big) \otimes e_\alpha \zeta^{-\alpha-\langle\alpha,\alpha\rangle/2} \qquad [2]$$

for $\alpha \in L$, so that $X(\alpha, \zeta)$ is an operator from V_L to $V_L\{\zeta\} = \{\Sigma v_n\zeta^n | v_n \in V_L\}$ where by definition $\zeta^{-\alpha}\cdot e^\beta = \zeta^{\langle-\alpha,\beta\rangle}\, e^\beta$ for $\beta \in L$. Here $\zeta^{1/2r}$ is viewed as a formal variable; all powers of ζ occurring will be integral powers of $\zeta^{1/2r}$. View V_L as a graded vector space by setting $\deg(S(\hat{\mathfrak{h}}_Z^-)_n \otimes e^\alpha) = n - \langle\alpha, \alpha\rangle/2$, the subscript denoting homogeneous degree. Then the homogeneous components $x_\alpha(n)$ defined by $X(\alpha, \zeta) = \Sigma x_\alpha(n)\, \zeta^n$ are well-defined operators on V_L, homogeneous of degree n. (Note that for $r = 1$, $n \in \mathbf{Z}$ and for $r = 2$, $n \in \frac{1}{2}\mathbf{Z}$.) The Lie algebra $\hat{\mathfrak{h}}_\mathbf{Z}$ also acts on V_L by $\alpha \otimes t^n \mapsto \alpha(n) \otimes 1$ [which we shall also call $\alpha(n)$] for $\alpha \in \mathfrak{h}$, $n \in \mathbf{Z}\backslash\{0\}$; $c \mapsto 1$; and $\alpha = \alpha \otimes t^0 \mapsto 1 \otimes \alpha$, where $\alpha\cdot e^\beta = \langle\alpha, \beta\rangle\, e^\beta$ for $\beta \in L$. Then V_L is a graded $\hat{\mathfrak{h}}_\mathbf{Z}$-module. The space V_L together with the operators $x_\alpha(n)$ and the action of $\hat{\mathfrak{h}}_\mathbf{Z}$ is called the *homogeneous representation* or *picture* (see refs. 5 and 6).

For a sublattice M of L and a coset M_1 of M in L, we set

$$\mathbf{F}[M_1] = \coprod_{\beta\in M_1} \mathbf{F}e^\beta \subset \mathbf{F}[L],$$

$$V_{M_1} = S(\hat{\mathfrak{h}}_Z^-) \otimes_{\mathbf{F}} \mathbf{F}[M_1],$$

by abuse of notation. We also view $X(\alpha, \zeta)$ and $x_\alpha(n)$ for all $\alpha \in M$ and all n as operators on V_{M_1}.

We also set (taking $Z = \mathbf{Z} + \frac{1}{2}$)

$$V_L' = S(\hat{\mathfrak{h}}_{\mathbf{Z}+1/2}^-) \otimes T, \qquad [3]$$

$$X(\alpha, \zeta) = \exp\Big(\sum_{\substack{n\in\mathbf{Z}+1/2\\ n>0}} \alpha(-n)\zeta^{-n}/n\Big)\exp\Big(-\sum_{\substack{n\in\mathbf{Z}+1/2\\ n>0}} \alpha(n)\zeta^{n}/n\Big) \otimes e_\alpha\, 2^{-\langle\alpha,\alpha\rangle} \qquad [4]$$

for $\alpha \in L$, an operator from V_L' to $V_L'\{\zeta\}$. (Here it is assumed that $\mathbf{F}$ contains $2^{-\langle\alpha,\alpha\rangle}$.) We view V_L' as a graded vector space by $(V_L')_n = S(\hat{\mathfrak{h}}_{\mathbf{Z}+1/2}^-)_n \otimes T$. The components $x_\alpha(n)$ given by $X(\alpha, \zeta) = \Sigma x_\alpha(n)\zeta^n$ (where $n \in \frac{1}{2}\mathbf{Z}$) are again well-defined operators on V_L', homogeneous of degree n. The Lie algebra $\hat{\mathfrak{h}}_{\mathbf{Z}+1/2}$ acts on V_L' by $\alpha \otimes t^n \mapsto \alpha(n) \otimes 1 = \alpha(n)$ for $\alpha \in \mathfrak{h}$, $n \in \mathbf{Z} + \frac{1}{2}$; and $c \mapsto 1$. This makes V_L' a graded $\hat{\mathfrak{h}}_{\mathbf{Z}+1/2}$-module. We call V_L' together with the operators $x_\alpha(n)$ and the action of $\hat{\mathfrak{h}}_{\mathbf{Z}+1/2}$ the *twisted representation* or *picture*.

The rank $L = 1$ case of the twisted picture arose in the first differential operator construction of an affine Lie algebra (7). The general case was studied in ref. 8.

We shall later consider the direct sum $W_L = V_L \oplus V_L'$ (see Eqs. **1** and **3**). Let $X(\alpha, \zeta)$ also denote the operator $X(\alpha, \zeta) \oplus X(\alpha, \zeta)$ from W_L to $W_L\{\zeta\}$ (see Eqs. **2** and **4**). The same convention shall also apply to other operators introduced later.

It will be convenient to take $\alpha(n) = 0$ when $n \in \mathbf{Z} + \frac{1}{2}$ in the homogeneous picture and to take $\alpha(n) = 0$ when $n \in \mathbf{Z}$ in the twisted picture. For $h \in \mathfrak{h}$ set $h(\zeta) = \Sigma_{n\in\mathbf{Z}/2} h(n)\zeta^n$ in either picture. For $\alpha \in L$ and $n \in (1/2r)\mathbf{Z}$ define $X^\pm(\alpha, \zeta)$ and $x_\alpha^\pm(n)$ by $X^\pm(\alpha, \zeta) = X(\alpha, \zeta) \pm X(-\alpha, \zeta)$ and $X^\pm(\alpha, \zeta) = \Sigma_n x_\alpha^\pm(n)\zeta^n$.

Physicists have introduced a procedure : : called normal ordering. We shall need the generating series of operators $:\alpha(\zeta)X^-(\beta, \zeta):$ and $:h_1(\zeta)h_2(\zeta):$ for $\alpha, h_1, h_2 \in \mathfrak{h}$ and $\beta \in L$ defined by

$$:\alpha(\zeta)X^-(\beta, \zeta): = \Big(\sum_{\substack{n\in\mathbf{Z}/2\\ n<0}} \alpha(n)\zeta^n + \tfrac{1}{2}\alpha(0)\Big) X^-(\beta, \zeta) + X^-(\beta, \zeta) \cdot \Big(\sum_{\substack{n\in\mathbf{Z}/2\\ n>0}} \alpha(n)\zeta^n + \tfrac{1}{2}\alpha(0)\Big)$$

$$:h_1(\zeta)h_2(\zeta): = \sum_{n\in\mathbf{Z}} \Big(\sum_{k\in\mathbf{Z}/2} :h_1(k)h_2(n-k):\Big)\zeta^n$$

where $:h_1(m)h_2(n): = h_1(m)h_2(n)$ if $m \leq n$ and $:h_1(m)h_2(n): = h_2(n)h_1(m)$ if $n \leq m$. Set $h_1h_2(\zeta) = :h_1(\zeta)h_2(\zeta):$ in the homogeneous picture and $h_1h_2(\zeta) = :h_1(\zeta)h_2(\zeta): + \frac{1}{8}\langle h_1, h_2\rangle$ in the twisted picture. Also let $:\alpha(\zeta)X^-(\beta, \zeta):_n$ and $h_1h_2(n)$ be the coefficients of ζ^n in the generating series $:\alpha(\zeta)X^-(\beta, \zeta):$ and $h_1h_2(\zeta)$, respectively.

THEOREM 1. (See ref. 8.) *Assume that the lattice* L *is even; that* s = 2; *that* $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$, $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$, $\varepsilon(\alpha, \alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$ *for all* $\alpha, \beta, \gamma \in$ L; *and that the module* (T, π) *satisfies the condition* $\pi(e_{2\alpha}) = 1$ *for all* $\alpha \in$ L. *Then in either the homogeneous or the twisted picture we have*

$$[x_\alpha^+(m), x_\beta^+(n)] = \begin{cases} 0 & \text{if } \langle\alpha, \beta\rangle = 0 \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta}^+(m + n) & \text{if } \langle\alpha, \beta\rangle = -1 \\ \varepsilon(\alpha, \beta)\,(m x_{\alpha+\beta}^+(m + n) + :\alpha(\zeta)X^-(\alpha + \beta, \zeta):_{m+n}) & \text{if } \langle\alpha, \beta\rangle = -2 \\ \frac{1}{2}(m - n)\,\alpha^2(m + n) + \frac{1}{3}(m^3 - m)\delta_{m+n,0} & \text{if } \beta = -\alpha \text{ and } \langle\alpha, \alpha\rangle = 4 \end{cases}$$

for $\alpha, \beta \in$ L, m, n $\in \mathbf{Z}$.

We now consider the special case $L = \Lambda$, the Leech lattice (9), which by ref. 10 is the unique positive definite even unimodular lattice of rank 24 that has no vectors with norm square 2. In this case, if $\alpha, \beta \in \Lambda$ and $\langle\alpha, \alpha\rangle = \langle\beta, \beta\rangle = 4$, then $\langle\alpha, \beta\rangle \in \{0, \pm1, \pm2, \pm4\}$. We may and do assume the conditions imposed in *Theorem 1* and also that T is irreducible (and therefore uniquely determined up to isomorphism and of dimension 2^{12}). In this case we shall write T_Λ for T.

Let θ be the involution of $V_\Lambda \oplus V_\Lambda'$ preserving V_Λ and V_Λ' and uniquely determined by the conditions $\theta(1 \otimes e^\alpha) = 1 \otimes e^{-\alpha}$, $\theta(1 \otimes v) = 1 \otimes v$ and $\theta\alpha(n)\theta^{-1} = -\alpha(n)$ for $\alpha \in \Lambda$, $v \in$

T_Λ, and $n \in \frac{1}{2}\mathbf{Z}$. Then $\theta e_\alpha \theta^{-1} = e_{-\alpha}$ and $\theta X(\alpha, \zeta)\theta^{-1} = X(-\alpha, \zeta)$, so that each $x_\alpha^+(n)$ ($n \in \mathbf{Z}$) preserves the ± 1-eigenspaces of θ on V_Λ and on V'_Λ. We denote these eigenspaces by $V_\Lambda^\pm$ and $V'^{\pm}_\Lambda$, respectively. We set

$$V = V_\Lambda^+ \oplus V'^{-}_\Lambda. \qquad [5]$$

This will be our F_1-module (see *Section 4*). Motivated by consideration of modular functions, we translate the gradations of V_Λ and V'_Λ so that $\deg(1 \otimes 1) = 1$ and $\deg(1 \otimes T_\Lambda) = -\frac{1}{2}$. Then $V = \coprod_{n\leq 1} V_n$. (The indices n are the negatives of those used in *Section 1*.)

Write $\mathrm{ch}_q V = \Sigma_n \dim V_{-n}q^n$, q an indeterminate.

THEOREM 2. *We have* $\mathrm{ch}_q V = J(q) = q^{-1} + 0 + 196884q^1 + \dots$, *the modular function* J.

Following ref. 4 (cf. ref. 2), we define a group $\hat{C}$ as follows: For $a \in \mathrm{Aut}(\hat{\Lambda})$, let $\bar{a}$ be the corresponding automorphism of Λ. Recall that π is the representation of $\hat{\Lambda}$ on T_Λ. Set $C_0 = \{a \in \mathrm{Aut}(\hat{\Lambda}) | \bar{a} \in \mathrm{Aut}(\Lambda, \langle\cdot,\cdot\rangle) = \cdot 0\}$. Then C_0 induces a group C_1 of automorphisms of the extraspecial group $\pi(\hat{\Lambda})$ since C_0 preserves Ker π. Let $\phi: C_0 \to C_1$ be the corresponding homomorphism. For $b \in GL(T_\Lambda)$ write $\mathrm{int}(b)$ for conjugation by b. Let $C_* = \{a_T \in GL(T_\Lambda) | a_T(\pi(\hat{\Lambda}))a_T^{-1} = \pi(\hat{\Lambda})$ and $\mathrm{int}(a_T)|_{\pi(\hat{\Lambda})} \in C_1\}$. Denote by C_T the commutator subgroup of C_*, and define the following subgroup of $C_0 \times C_T$: $\hat{C} = \{(a, a_T) \in C_0 \times C_T | \phi(a) = \mathrm{int}(a_T)$ on $\pi(\hat{\Lambda})\}$.

We now determine an action of $\hat{C}$ on $W_\Lambda = V_\Lambda \oplus V'_\Lambda$ (cf. ref. 11) by requiring that (*i*) $g\alpha(n)g^{-1} = (\bar{a}\alpha)(n)$, (*ii*) $ge_\alpha g^{-1} = a(e_\alpha)$, (*iii*) g: $1 \otimes 1 \mapsto 1 \otimes 1$, and (*iv*) $g|_{1\otimes T} = 1 \otimes a_T$ for $g = (a, a_T) \in \hat{C}$, $\alpha \in \Lambda$, $n \in \frac{1}{2}\mathbf{Z}$. Let C be the quotient of $\hat{C}$ acting faithfully on V. Then C has structure $2_+^{1+24}(\cdot 1)$ and C is isomorphic to the group denoted C in ref. 4.

For $g \in C$, define the series

$$T_g(q) = \sum_{n\geq -1} (\mathrm{tr}\, g|_{V_{-n}})q^n.$$

THEOREM 3. *Let* $g = (a, a_T) \in \hat{C}$. *Then*

$$T_g(q) =$$

$$\frac{1}{2}\left(\frac{\theta_{\bar{a}}^+(q)}{\eta_{\bar{a}}(q)} + \frac{\theta_{\bar{a}}^-(q)}{\eta_{-\bar{a}}(q)} + \frac{\mathrm{tr}(a_T)\eta_{\bar{a}}(q)}{\eta_{\bar{a}}(q^{1/2})} + \frac{\mathrm{tr}(-a_T)\eta_{-\bar{a}}(q)}{\eta_{-\bar{a}}(q^{1/2})}\right),$$

where $\eta_{\bar{a}}(q) = \Pi_k\, \eta(q^k)^{p_k}$ *if* $\bar{a}$ *has characteristic polynomial* $\Pi(1 - x^k)^{p_k}$ ($k > 0$, $p_k \in \mathbf{Z}$), $\eta(q) = q^{1/24}\Pi_{n>0}(1 - q^n)$,

$$\theta_{\bar{a}}^\pm(q) = \sum_{\substack{\alpha\in\Lambda \\ \bar{a}\alpha = \pm\alpha}} s(\alpha)q^{\langle\alpha,\alpha\rangle/2}$$

and $s(\alpha) = (-1)^{s_0(\alpha)}$ *where* $a(e_\alpha) = \kappa^{s_0(\alpha)}e_{\bar{a}\alpha}$.

These formulas are different from those discovered by Conway and Norton (2) for the action of C on the conjectured infinite-dimensional F_1-module. The Conway–Norton formulas were used by Kac (11) to construct an infinite-dimensional C-module V whose subspace V_0 was a $24\cdot 2^{12}$-dimensional nontrivial C-module that had to be divided out. Attempts to extend this C-module V/V_0 to F_1 have not been successful.

3. Triality

Let $\{\alpha, x_\alpha, x_{-\alpha}\}$ be the standard basis of $\mathfrak{sl}(2,\mathbf{F})$. Set $x_\alpha^\pm = x_\alpha \pm x_{-\alpha}$, $y_1 = i\alpha$, $y_2 = x_\alpha^-$, and $y_3 = ix_\alpha^+$, where $i = \sqrt{-1}$. In the basis $\{y_k\}$ the commutation relations are $[y_1, y_2] = 2y_3$ and its cyclic permutations, so that there are "manifest" automorphisms permuting the three pairs $\{\pm y_k\}$. We denote by σ_0 the involutive automorphism of $\mathfrak{sl}(2, \mathbf{F})$ such that σ_0: $y_1 \leftrightarrow y_2$, $y_3 \leftrightarrow -y_3$, i.e., σ_0: $i\alpha \leftrightarrow x_\alpha^-$, $x_\alpha^+ \leftrightarrow -x_\alpha^+$.

We shall now consider certain homogeneous and twisted representations based on the rank 1 lattice L generated by α, where we suppose that $\langle\alpha, \alpha\rangle = 2$, so that $L \subset \mathfrak{h} = \mathbf{F}\alpha$. Assume that the cocycle ε satisfies $\varepsilon(\alpha, -\alpha) = 1$. We take the homogeneous module to be $S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}}) \otimes \mathbf{F}[\mathbf{Z}\alpha \pm \alpha/4]$ and the twisted module to be $S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}+\frac{1}{2}}) \otimes T^\pm$, where $T^\pm = \mathbf{F}e^\pm$ is 1-dimensional and $e_\alpha \cdot e^\pm = (\pm i)e^\pm$. The components of the vertex operators $X(\pm\alpha, \zeta)$ generate the Lie algebra

$$\mathfrak{sl}(2, \mathbf{F})^\wedge = \coprod_m \mathbf{F}\alpha(m) \oplus \coprod_n \mathbf{F}x_\alpha^+(n) \oplus \coprod_p \mathbf{F}x_\alpha^-(p) \oplus \mathbf{F}1,$$

where $m \in \mathbf{Z}$, $n, p \in \mathbf{Z} + \frac{1}{2}$ in the homogeneous case and $p \in \mathbf{Z}$, $m, n \in \mathbf{Z} + \frac{1}{2}$ in the twisted case.

THEOREM 4. *There is a unique linear isomorphism* σ_1: $S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}}) \otimes \mathbf{F}[\mathbf{Z}\alpha \pm \alpha/4] \to S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}+\frac{1}{2}}) \otimes T^\pm$ *such that* (*i*) $\sigma_1 \circ x(k) \circ \sigma_1^{-1} = (\sigma_0 x)(k)$ *for* $x \in \{\alpha, x_\alpha^\pm\}$, $k \in \frac{1}{2}\mathbf{Z}$ *and* (*ii*) σ_1: $1 \otimes e^{\pm\alpha/4} \mapsto 1 \otimes e^\pm$.

This theorem underlies our general "triality" results and our extra automorphism σ in F_1.

From now on we shall consider only a certain class of lattices L associated to "codes." Let Ω be a finite set with n elements, $n \in 4\mathbf{Z}$. Let $\mathscr{C}$ be a subspace of $\mathscr{P}(\Omega)$, the power set, which is an $\mathbf{F}_2$-vector space under symmetric difference. Assume that (*i*) $|C| \in 4\mathbf{Z}$ for all $C \in \mathscr{C}$ (so that $|C_1 \cap C_2| \in 2\mathbf{Z}$ for all $C_1, C_2 \in \mathscr{C}$, (*ii*) $\Omega \in \mathscr{C}$. Let $\mathfrak{h}$ be an $\mathbf{F}$-vector space (soon to be viewed as the $\mathbf{F}$-span of a certain lattice L) with a basis $\{\alpha_k | k \in \Omega\}$, and consider the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ such that $\langle\alpha_k, \alpha_l\rangle = 2\delta_{k,l}$. For $C \subset \Omega$ set $\alpha_C = \Sigma_{k\in C}\, \alpha_k$, and let ε_C denote the automorphism of $\mathfrak{h}$ given by ε_C: $\alpha_k \mapsto -\alpha_k$ for $k \in C$, ε_C: $\alpha_k \mapsto \alpha_k$ for $k \in \Omega\backslash C$.

Define the following lattices and cosets:

$$Q = \coprod_{k\in\Omega} \mathbf{Z}\alpha_k,\ L_0 = \left(\sum_{C\in\mathscr{C}} \mathbf{Z}\tfrac{1}{2}\alpha_C\right) + Q,$$

$$L_1 = L_0 + \tfrac{1}{4}\alpha_\Omega,\ L = L_0 \cup L_1. \qquad [6]$$

For the lattice L, we take $r = 2$, $s = 4$ and $\omega = i$ (see *Section 2*).

There exist lattices Ψ such that (*i*) $L \supset \Psi \supset 2L^\perp$, (*ii*) $L/2L^\perp = Q/2L^\perp \oplus \Psi/2L^\perp$, and (*iii*) $\langle\alpha, \beta\rangle \in 2\mathbf{Z}$ for all $\alpha, \beta \in \Psi \cap L_0$, where $L^\perp = \{\alpha \in \mathfrak{h} | \langle\alpha, L\rangle \subset \mathbf{Z}\}$; fix such a lattice Ψ. Let π_Q and π_Ψ denote the projections with respect to the decomposition *ii* and define the 2-cocycle ε_0: $L \times L \to \mathbf{Z}/4\mathbf{Z}$, where $\varepsilon_0(\alpha, \beta) = 2\langle\pi_Q(\alpha + 2L^\perp), \pi_\Psi(\beta + 2L^\perp)\rangle + 4\mathbf{Z}$. Then $\varepsilon(\alpha, \beta) = i^{\varepsilon_0(\alpha,\beta)}$. With this choice of cocycle we define an $\hat{L}$-module $T_L = \mathbf{F}[L/Q] = \coprod_{\beta+Q\in L/Q} \mathbf{F}e^{\beta+Q}$ by $e_\alpha \cdot e^{\beta+Q} = \varepsilon(\alpha, \beta)e^{\alpha+\beta+Q}$, $\kappa \cdot e^{\beta+Q} = ie^{\beta+Q}$, for $\alpha, \beta \in L$.

Consider the space $W_L = V_L \oplus V'_L = \coprod_{0\leq j\leq 3} W_j$ where $W_0 = S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}}) \otimes \mathbf{F}[L_0]$, $W_1 = S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}}) \otimes \mathbf{F}[L_1]$, $W_2 = S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}+\frac{1}{2}}) \otimes \mathbf{F}[L_0/Q]$, $W_3 = S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}+\frac{1}{2}}) \otimes \mathbf{F}[L_1/Q]$. The components of the operators $X(\pm\alpha_k, \zeta)$, for $k \in \Omega$, generate the affine Lie algebra $(\mathfrak{sl}(2, \mathbf{F})^n)^\wedge$, acting on each W_j. Using *Theorem 4*, we obtain the following result, which serves to define a map σ_2 on W_L:

THEOREM 5. *There is a unique involutive linear automorphism* σ_2: $W_L \to W_L$ *such that*

(i) $\sigma_2 W_0 = W_0$, $\sigma_2 W_1 = W_3$, $\sigma_2 W_2 = W_2$

(ii) $\sigma_2 i\alpha_k(\zeta)\sigma_2^{-1} = X^-(\alpha_k, \zeta)$

and $\sigma_2 X^+(\alpha_k, \zeta)\sigma_2^{-1} = -X^+(\alpha_k, \zeta)$ *for* $\kappa \in \Omega$

(iii) σ_2: $1 \otimes e^{\alpha_C/2} \leftrightarrow 2^{-|C|/2} \sum_{T\subset C} (-i)^{|T|}\, 1 \otimes e^{\varepsilon_T\alpha_C/2}$

$$\sigma_2{:}1 \otimes e^{\alpha_C/2+Q} \leftrightarrow (-1)^{|C|/4}\, 1 \otimes e^{\alpha_{C+\Omega}/2+Q}$$
$$\sigma_2{:}1 \otimes e^{\varepsilon_C\alpha_\Omega/4} \leftrightarrow 1 \otimes e^{\varepsilon_C\alpha_\Omega/4+Q} \textit{ for } C \in \mathscr{C}.$$

Equating the characters of W_1 and W_3 gives:

COROLLARY. *We have* $\theta_{L_1}(q)/\eta(q)^n = |\mathscr{C}|\eta(q)^n/\eta(q^{1/2})^n$, *where* $\theta_{L_1}(q) = \Sigma_{\alpha\in L_1} q^{\langle\alpha,\alpha\rangle/2}$ *and* $\eta(q) = q^{1/24}\,\Pi_{m>0}(1 - q^m)$.

Remark. Condition *ii* in *Theorem 5* simply states that σ_2 carries the natural action of $(\mathfrak{sl}(2, \mathbf{F})^n)^\wedge$ on W_L to its "σ_0-transform."

Remark. Over $\mathbf{C}$, $\sigma_2|_{W_0}$ is the action of diag $(2^{-1/2}[\begin{smallmatrix} i & 1\\ -1 & -i\end{smallmatrix}]) \in SL(2, \mathbf{C})^n$.

Remark. The involution σ_2 is part of the action of a symmetric group Σ_3 on W_L that preserves W_0 and permutes the elements of $\{W_1, W_2, W_3\}$ and of $\{y_1^{(k)}(n), y_2^{(k)}(n), y_3^{(k)}(n)\}$ for $k \in \Omega$, $n \in \tfrac{1}{2}\mathbf{Z}$, where $y_1^{(k)}(n) = i\alpha_k(n)$, $y_2^{(k)}(n) = x_{\alpha_k}^-(n)$ and $y_3^{(k)}(n) = ix_{\alpha_k}^+(n)$.

THEOREM 6. *On* W_L, *conjugation by* σ_2 *transforms* $X(\varepsilon_S\tfrac{1}{2}\alpha_C, \zeta)$ *into a linear combination of the operators* $X(\varepsilon_T\tfrac{1}{2}\alpha_C, \zeta)$ *for* $T \subset C$; *more precisely,*

$$\sigma_2(-i)^{|S|}X(\varepsilon_S\tfrac{1}{2}\alpha_C, \zeta)\sigma_2^{-1}$$
$$= 2^{-|C|/2}\sum_{T\subset C}(-1)^{|S\cap T|}(-i)^{|T|}X(\varepsilon_T\tfrac{1}{2}\alpha_C, \zeta)$$

for all $S \subset C$, $C \in \mathscr{C}$.

Now take $n = 24$ and let $\mathscr{C}$ be the binary Golay code. This is characterized among the subspaces of $P(\Omega)$ by the properties: (*i*) $\mathscr{C}$ is self-dual—i.e., $\mathscr{C} = \{C \subset \Omega|\, |C \cap C_1| \in 2\mathbf{Z}$ for all $C_1 \in \mathscr{C}\}$, (*ii*) $|C| \in 4\mathbf{Z}$ for all $C \in \mathscr{C}$, and (*iii*) $|C| \neq 4$ for all $C \in \mathscr{C}$. In this case the lattice L contains the following standard realization of the Leech lattice Λ as a sublattice of index 2: $\Lambda = \mathbf{Z}\text{-span}\{\alpha_k + \alpha_l, \tfrac{1}{2}\alpha_C, \tfrac{1}{4}\alpha_\Omega - \alpha_k\, |k, l \in \Omega, C \in \mathscr{C}\}$.

We shall assume that the lattice Ψ is chosen to satisfy the conditions (*i*) $\Lambda \supset \Psi \supset 2\Lambda$, (*ii*) $\tfrac{1}{4}\alpha_\Omega - \alpha_\infty \in \Psi$, (*iii*) $\langle\alpha, \alpha\rangle \in 4\mathbf{Z}$ for all $\alpha \in \Psi$ in addition to the earlier requirements; this is in fact possible. Here ∞ denotes any fixed element of Ω. The 2^{13}-dimensional module T_L contains the 2^{12}-dimensional module $T_\Lambda = \{v \in T_L | e_{\alpha_\Omega/2-2\alpha_\infty}\cdot v = v\}$, which is (isomorphic to) the module called T_Λ in *Section 2*. Hence the space W_L contains our proposed F_1-module V (see *Section 2*), but σ_2 does not leave V invariant. To remedy this we define $\sigma{:}W_L \to W_L$ by

$$\sigma|_{W_0} = \sigma_2 \circ (-1)^{\alpha_\infty},\ \sigma|_{W_1} = -ie_{\alpha_\infty} \circ \sigma_2,$$
$$\sigma|_{W_2} = -i\sigma_2 \circ e_{\alpha_\infty},\ \sigma|_{W_3} = -i\sigma_2 \circ e_{\alpha_\infty}. \quad [7]$$

Here $(-1)^{\alpha_\infty}$ is the operator on W_0 which takes $p \otimes e^\beta$ to $(-1)^{\langle\alpha_\infty,\beta\rangle} p \otimes e^\beta$ for $p \in S(\hat{\mathfrak{h}}_{\bar{\mathbf{Z}}})$, $\beta \in L_0$. *Theorem 6* and the definition of σ immediately give:

THEOREM 7. (*1*) *The linear automorphism* σ *leaves* V *invariant.*

(*2*) *We have*

$$\sigma(-1)^{|S|/2}X^+(\varepsilon_S\tfrac{1}{2}\alpha_C, \zeta)\sigma^{-1}$$
$$= (-1)^{|C\cap\{\infty\}|}2^{-|C|/2}\sum_{\substack{T\subset C\\ |T|\in 2\mathbf{Z}}}(-1)^{|S\cap T|}(-1)^{|T|/2}X^+(\varepsilon_T\tfrac{1}{2}\alpha_C, \zeta)$$

for all $S \subset C$, $|S| \in 2\mathbf{Z}$, $C \in \mathscr{C}$.

Remark. Theorem 2 follows from the Golay code case of the corollary to *Theorem 5* together with the fact that $\sigma(V_{L_0\cap\Lambda}) = V_{L_0}^+$.

It is convenient to define

$$\tau = ((-1)^{\alpha_\Omega/4-\alpha_\infty}, e_{\alpha_\Omega/4-\alpha_\infty}) \in \hat{C} \quad [8]$$

(see *Section 2*). Then the automorphisms σ and τ of V generate a copy of the symmetric group Σ_3.

We now give our definition of the group F_1:

$$F_1 = \langle C, \sigma\rangle = \langle C, \Sigma_3\rangle \subset GL(V). \quad [9]$$

4. The $(F_1\hat{\mathfrak{B}})$-Module V.

Let $\mathfrak{a}$ be a commutative nonassociative algebra, with multiplication denoted $\times$, over the field $\mathbf{F}$. Suppose that $\mathfrak{a}$ is equipped with a symmetric bilinear form $\langle\cdot,\cdot\rangle$, associative in the sense that $\langle x \times y, z\rangle = \langle x, y, \times z\rangle$ for all $x, y, z \in \mathfrak{a}$. Set

$$\bar{\mathfrak{a}} = \mathfrak{a} \otimes {}_{\mathbf{F}}\mathbf{F}[t, t^{-1}] \text{ and } \hat{\mathfrak{a}} = \bar{\mathfrak{a}} \oplus \mathbf{F}e, \quad [10]$$

where t is an indeterminate and $e \neq 0$. Provide $\hat{\mathfrak{a}}$ with the commutative nonassociative product $\times$ determined by:

$$x \otimes t^m \times y \otimes t^n = (x \times y) \otimes t^{m+n} + \langle x, y\rangle m^2\delta_{m+n,0}e$$
$$e \times e = e \times (x \otimes t^m) = 0 \quad [11]$$

for all $x, y \in \mathfrak{a}$ and $m, n \in \mathbf{Z}$. Also define a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\hat{\mathfrak{a}}$ by

$$\langle x \otimes t^m, y \otimes t^n\rangle = \langle x, y\rangle\delta_{m+n,0},$$
$$\langle e, e\rangle = \langle e, x \otimes t^m\rangle = 0 \quad [12]$$

for all $x, y \in \mathfrak{a}$ and $m, n \in \mathbf{Z}$. Then $\langle\cdot,\cdot\rangle$ is an associative form with respect to $\times$ on $\hat{\mathfrak{a}}$.

Let A be an associative or Lie algebra. Given two sequences $x = (x(m))_{m\in\mathbf{Z}}$, $y = (y(n))_{n\in\mathbf{Z}}$ of elements of A, define the function

$$[x \times y]{:}\ \mathbf{Z} \times \mathbf{Z} \to A$$
$$[x \times y](m, n) = \tfrac{1}{2}([x(m + 1), y(n - 1)]$$
$$+ [y(n + 1), x(m - 1)]). \quad [13]$$

By abuse of notation, we shall sometimes write this expression $[x(m) \times y(n)]$.

The symbol $[\cdot \times \cdot]$ is read "cross bracket" or "cross" for short, because it is made up of two brackets that "cross." The significance of this operation is that, in interesting cases, $[x(m) \times y(n)]$ can be expressed in terms of a singly indexed sequence $(z(p))_{p\in\mathbf{Z}}$ of elements of A. As explained in ref. 8, the cross bracket is an example of a product defined by "correction factors" (cf. *Section 1* of ref. 12).

By a *graded* $\hat{\mathfrak{a}}$*-module* we mean a $\mathbf{Z}$-graded vector space $V = \amalg_{n\in\mathbf{Z}}V_n$ together with a linear map $\pi{:}\ \hat{\mathfrak{a}} \to \text{End } V$ such that $\pi(x \otimes t^m)$ is homogeneous of degree m for all $x \in \mathfrak{a}$, $m \in \mathbf{Z}$ and such that

$$\pi(x \otimes t^m \times y \otimes t^n) = [\pi(x \otimes t^m) \times \pi(y \otimes t^n)] \quad [14]$$

for all $x, y \in \mathfrak{a}$ and $m, n \in \mathbf{Z}$, where on the right-hand side we are considering the cross bracket of the two sequences $(\pi(x \otimes t^m))_{m\in\mathbf{Z}}$ and $(\pi(y \otimes t^n))_{n\in\mathbf{Z}}$ (see formula **13**).

Let (V, π) be a graded $\hat{\mathfrak{a}}$-module, and suppose that G is a group that acts as linear automorphisms of $\mathfrak{a}$ and V, preserving V_n for all $n \in \mathbf{Z}$. Then we call (V, π) a *graded* $(G, \hat{\mathfrak{a}})$*-module* if

$$g\pi(x \otimes t^n)g^{-1} = \pi(g\cdot x \otimes t^n) \quad [15]$$

for all $g \in G$, $x \in \mathfrak{a}$, and $n \in \mathbf{Z}$.

Our first example of a cross-bracket algebra comes from *Section 2* (*Theorem 1*).

THEOREM 8. (See ref. 8.) *The sequences* $(h^2(n))_{n\in\mathbf{Z}}$ *for* $h \in \mathfrak{h}$ *and* $(x_\alpha^+(n))_{n\in\mathbf{Z}}$ *for* $\alpha \in \Lambda$ *with* $\langle\alpha, \alpha\rangle = 4$ *"span an algebra which is closed under cross-bracket." More precisely, set*

$$\mathfrak{k} = S^2(\mathfrak{h}) \oplus \sum_{\substack{\alpha\in\Lambda \\ \langle\alpha,\alpha\rangle=4}} \mathbf{F}x_\alpha^+, \qquad [16]$$

where each $x_\alpha^+ \neq 0$, *and we impose the relations* $x_\alpha^+ = x_{-\alpha}^+$ *and no others. Then there is a commutative nonassociative algebra structure* $\times$ *and a nonsingular symmetric bilinear form* $\langle\cdot, \cdot\rangle$ *on* $\mathfrak{k}$ *such that* V (*see formula* **5**) *is a graded* $\hat{\mathfrak{k}}$*-module* (*see Eqs.* **11**, **12**, *and* **14**) *under the action* $\pi:\hat{\mathfrak{k}} \to$ *End* V *defined by*

$$\pi\colon h^2 \otimes t^n \mapsto h^2(n), \qquad x_\alpha^+ \otimes t^n \mapsto x_\alpha^+(n), \qquad e \mapsto 1 \qquad [17]$$

for all $h \in \mathfrak{h}$, $n \in \mathbf{Z}$, *and* $\alpha \in \Lambda$ *with* $\langle\alpha, \alpha\rangle = 4$. [*Recall that* $S^2(\mathfrak{h})$ *is spanned by the squares of the elements of* $\mathfrak{h}$.] *The bilinear maps* $\times$ *and* $\langle\cdot, \cdot\rangle$ *are uniquely determined by the requirement that* (V, π) *be a graded* $\hat{\mathfrak{k}}$*-module.*

Remark. The actual computation of $\times$ and $\langle\cdot, \cdot\rangle$ is immediate from *Theorem 1*. The result is written in detail in ref. 8, except that our present form $\langle\cdot, \cdot\rangle$ is ½ the form used in ref. 8.

Formula **5** gives the decomposition of V into irreducible components for $\hat{\mathfrak{k}}$. We now apply the results of *Section 3* in order to extend $\hat{\mathfrak{k}}$ to a larger algebra $\hat{\mathfrak{B}}$, which will act irreducibly on V. The extra operators will be "the Σ_3-conjugates of the sequences $(\pi(x \otimes t^n))$ for $x \in \mathfrak{k}$." (Recall that $\Sigma_3 = \langle\sigma, \tau\rangle$, where σ, τ are defined in Eqs. **7** and **8**.) One convenient way of making this precise is the following:

The degree -1 subspace V_{-1} of V equals $S^2(\mathfrak{h}(-1)) \oplus \Sigma\mathbf{F}(1 \otimes e^\alpha + 1 \otimes e^{-\alpha}) \oplus \mathfrak{h}(-½) \otimes T)$ where the sum ranges over all $\alpha \in \Lambda$ with $\langle\alpha, \alpha\rangle = 4$. Motivated by this decomposition, we define the **F**-vector space

$$\mathfrak{B} = \mathfrak{k} \oplus (\mathfrak{h} \otimes T), \qquad [18]$$

with $\mathfrak{k}$ as in formula **16**, and we identify $\mathfrak{B}$ with V_{-1} via the linear map

$$\iota:\mathfrak{B} \to V_{-1} \qquad [19]$$

such that $x \mapsto \pi(x \otimes t^{-2})\cdot 1 \otimes 1$ for $x \in \mathfrak{k}$ and $h \otimes v \mapsto h(-½) \otimes v$ for $h \in \mathfrak{h}$, $v \in T$. Give $\mathfrak{B}$ the F_1-module structure which makes ι an F_1-isomorphism (see formula **9**). Define the vector space $\hat{\mathfrak{B}}$ from $\mathfrak{B}$ as in formula **10**.

To extend π (see formula **17**) from $\hat{\mathfrak{k}}$ to $\hat{\mathfrak{B}}$, write

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}_1, \qquad [20]$$

$$\mathfrak{l} = S^2(\mathfrak{h}) \oplus \sum_{\alpha\in L_0\cap\Lambda} \mathbf{F}x_\alpha^+, \qquad [21]$$

$$\mathfrak{p}_1 = \sum_{\alpha\in L_1\cap\Lambda} \mathbf{F}x_\alpha^+, \qquad [22]$$

$$\mathfrak{p}_2 = \sigma\mathfrak{p}_1,\ \mathfrak{p}_3 = \tau\sigma\mathfrak{p}_1. \qquad [23]$$

(See Eqs. **6**.) Then

$$\mathfrak{h} \otimes T = \mathfrak{p}_2 \oplus \mathfrak{p}_3. \qquad [24]$$

Define the operators $(\sigma p_1)(n) = \sigma \circ \pi(p_1 \otimes t^n) \circ \sigma^{-1}$, $(\tau\sigma p_1)(n) = \tau\sigma \circ \pi(p_1 \otimes t^n) \circ (\tau\sigma)^{-1}$ for $p_1 \in \mathfrak{p}_1$, $n \in \mathbf{Z}$. Extend π from $\hat{\mathfrak{k}}$ to a linear map $\pi:\hat{\mathfrak{B}} \to$ End V by setting $\pi(p_j \otimes t^n) = p_j(n)$ for $p_j \in \mathfrak{p}_j$ (j = 2, 3) and $n \in \mathbf{Z}$. Then

$$\iota(x) = \pi(x \otimes t^{-2})\cdot 1 \otimes 1$$

for all $x \in \mathfrak{B}$ (see formulas **18** and **19**). We are now ready to state:

THEOREM 9. *The sequences* $(h^2(n))_{n\in\mathbf{Z}}$ *for* $h \in \mathfrak{h}$ *and* $(x_\alpha^+(n))_{n\in\mathbf{Z}}$ *for* $\alpha \in \Lambda$ *with* $\langle\alpha, \alpha\rangle = 4$, *together with their transforms by the symmetric group* $\Sigma_3 = \langle\sigma, \tau\rangle$ *"span an algebra which is closed under cross-bracket." More precisely, there is a commutative nonassociative algebra structure* $\times$ *and a nonsingular associative symmetric bilinear form* $\langle\cdot, \cdot\rangle$ *on* $\mathfrak{B}$ *such that* (V, π) *is a graded* (F_1, $\hat{\mathfrak{B}}$)*-module in the above sense, irreducible under* $\hat{\mathfrak{B}}$. *The bilinear maps* $\times$ *and* $\langle\cdot, \cdot\rangle$ *are uniquely determined by the requirement that* (V, π) *be a graded* $\hat{\mathfrak{B}}$*-module.*

From the definition of cross bracket, we now have:

COROLLARY. *The group* F_1 *preserves* $\times$ *and* $\langle\cdot, \cdot\rangle$ *on* $\mathfrak{B}$.

Precise formulas for $\times$ and $\langle\cdot, \cdot\rangle$ on $\mathfrak{B}$ may be computed in a straightforward way via the following theorem. As a result, we obtain the algebra and bilinear form designated $\mathfrak{B}$, $\times$, $\langle\cdot, \cdot\rangle$ in ref. 8, except that our present form $\langle\cdot, \cdot\rangle$ is ½ the form used in ref. 8.

THEOREM 10. *The space* V_{-1} *is the "adjoint representation" of* $\mathfrak{B}$ *in the sense that* $\pi(x \otimes t^0)\cdot\iota(y) = \iota(x \times y)$ *for all* x, y $\in \mathfrak{B}$. *In addition,* $\pi(x \otimes t^2)\cdot\iota(y) = 2\langle x, y\rangle 1 \otimes 1$ *for all* x, y $\in \mathfrak{B}$.

To compare $\mathfrak{B}$ with the algebra B of Griess (4), let $\mathfrak{B}_0$ be the orthogonal complement of $\mathbf{F}\Sigma_{k\in\Omega}\alpha_k^2$ in $\mathfrak{B}$ and let $\pi_0:\mathfrak{B} \to \mathfrak{B}_0$ be the orthogonal projection. Define a nonassociative algebra structure $\cdot$ on $\mathfrak{B}$ by $x\cdot y = \pi_0(\pi_0 x \times \pi_0 y)$ for all $x, y \in \mathfrak{B}$. Observe that F_1, and in particular C and σ, act as automorphisms of $(\mathfrak{B}, \cdot)$. The action of σ on $B = V_{-1}$ is explicitly described on $\mathfrak{l}$ (see Eqs. **20** and **21**) by *Theorem 7*, and on $p_1 \oplus h \otimes T$ (see Eq. **22**) by the identifications **23** and **24**.

THEOREM 11. (See ref. 8). *There is an algebra isomorphism between* $(\mathfrak{B}, \cdot)$ *and* B. *This isomorphism is an isometry up to a scalar multiple, and it transforms* C *to the group of automorphisms of* B *denoted* C *in ref. 4, and* σ *to the automorphism of* B *denoted* σ *in ref. 4.*

Since $\hat{\mathfrak{B}}$ acts irreducibly on V and the action of F_1 is compatible with the action of $\hat{\mathfrak{B}}$ on V as in formula **15** (see *Theorem 9*), F_1 acts faithfully on $\mathfrak{B}$. *Theorem 11* then shows that F_1 is the group studied by Griess. His result (4) thus gives:

THEOREM 12. *The group* F_1 *is a finite simple group of "Monster type."*

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