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Estimating Latent Variable Interactions With Non-Normal Observed Data: A Comparison of Four Approaches

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Abstract

A Monte Carlo simulation was conducted to investigate the robustness of four latent variable interaction modeling approaches (Constrained Product Indicator [CPI], Generalized Appended Product Indicator [GAPI], Unconstrained Product Indicator [UPI], and Latent Moderated Structural Equations [LMS]) under high degrees of non-normality of the observed exogenous variables. Results showed that the CPI and LMS approaches yielded biased estimates of the interaction effect when the exogenous variables were highly non-normal. When the violation of non-normality was not severe (normal; symmetric with excess kurtosis < 1), the LMS approach yielded the most efficient estimates of the latent interaction effect with the highest statistical power. In highly non-normal conditions, the GAPI and UPI approaches with ML estimation yielded unbiased latent interaction effect estimates, with acceptable actual Type-I error rates for both the Wald and likelihood ratio tests of interaction effect at $N \geq 500$. An empirical example illustrated the use of the four approaches in testing a latent variable interaction between academic self-efficacy and positive family role models in the prediction of academic performance.

Since Kenny and Judd's (1984) seminal contribution, researchers have developed numerous methods for estimating interactions between latent variables (e.g., Jöreskog & Yang, 1996; Klein & Moosbrugger, 2000; Marsh, Wen, & Hau, 2004; Wall & Amemiya, 2001, 2003; see Marsh, Wen, Nagengast, & Hau, 2012, for an overview). The following linear by linear latent interaction model has been the primary focus of attention in the literature:

$$\eta=\alpha+\gamma_1\xi_1+\gamma_2\xi_2+\gamma_3\xi_1\xi_2+\zeta \quad (1)$$

where η is the endogenous latent variable, ξ_1 and ξ_2 are the exogenous latent variables, $\xi_1\xi_2$ represents the interaction term, α is the intercept, γs are the structural coefficients, and ζ is the disturbance of η, with mean zero and variance ψ.

The measurement models for ξ_1 and ξ_2 and for η follow the traditional measurement structure of CFA models for exogenous and endogenous latent variables, respectively (e.g., Bollen, 1989, p. 18, equations 2.8 and 2.9):

$$\mathbf{X}=\boldsymbol{\tau}_X+\boldsymbol{\Lambda}_X\boldsymbol{\xi}+\boldsymbol{\delta} \quad (2)$$

$$\boldsymbol{Y}=\boldsymbol{\tau}_Y+\boldsymbol{\Lambda}_Y\boldsymbol{\eta}+\boldsymbol{\varepsilon} \quad (3)$$

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where $\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and $\boldsymbol{\eta} = (\eta)$. $\mathbf{X}$ and $\mathbf{Y}$ are the observed exogenous and endogenous variables, respectively. $\boldsymbol{\tau}_\mathbf{X}$ and $\boldsymbol{\tau}_\mathbf{Y}$ are the latent intercepts, $\boldsymbol{\Lambda}_\mathbf{X}$ and $\boldsymbol{\Lambda}_\mathbf{Y}$ are the factor loadings, and $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ are the unique factors for the exogenous and endogenous variables, respectively. In most latent variable interaction approaches, the elements in $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ are assumed to be uncorrelated. The chief advantage of latent variable approaches over the measured variable approach to estimating interaction effects is that the predictors are theoretically free of measurement error, minimizing bias in the estimates of γ_1, $\gamma_{2,}$ and γ_3 (Aiken & West, 1991; Bollen, 1989). Some authors (e.g., Marsh et al., 2012, p. 438) have claimed that latent variable approaches also raises statistical power; however, Fuller (1987) and Ledgerwood and Shrout (2011) have shown in other contexts that correction for measurement error may increase standard errors and hence decrease statistical power.

The present study focused on four approaches for estimating latent variable interactions that have been used in practice and that can be estimated using standard computer software. Three are variants of the Product Indicator (PI) approach: Constrained PI (CPI, Jöreskog & Yang, 1996), Generalized Appended PI (GAPI, Wall & Amemiya, 2001), and Unconstrained PI (UPI, Marsh et al., 2004). The final approach is a variant of the distribution analytic approach: Latent Moderated Structural Equations (LMS, Klein & Moosbrugger, 2000).

Product Indicator Approach

The PI approach was originally proposed by Kenny and Judd (1984), and several variants of the general approach have been developed since that time (see Marsh et al, 2012). In the PI approach, product terms of the variables in the vector $\mathbf{X}$ are computed to represent the indicators of $\xi_1\xi_2$. Now $\boldsymbol{\xi}$ becomes $\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_1\xi_2 \end{pmatrix}$ in equation (2). Following Jöreskog and Yang (1996), a mean structure is included in the model because the mean of $\xi_1\xi_2$ will only equal 0 when ξ_1 and ξ_2 are uncorrelated, even when all of the observed X variables are centered. The variants of the PI approach differ in the nonlinear constraints that they impose in the specification of the factor loadings ($\boldsymbol{\Lambda}_\mathbf{X}$), the exogenous latent variables covariance matrix ($\boldsymbol{\Phi}$), and the exogenous unique factors covariance matrix ($\boldsymbol{\Theta}_{\boldsymbol{\delta}}$). The constraints proposed in each variant of the PI approach are detailed in Table 1. Jöreskog and Yang's (1996) CPI approach imposes the largest number of constraints. These constraints are derived under the assumption of multivariate normality. Among the PI approaches, the CPI approach is theoretically expected to yield the most powerful tests of the latent variable interaction *if* the multivariate normality assumption is met. Wall and Amemiya (2001) noted that these constraints may not be correct when observed X variables are *not* multivariate normal. They proposed the GAPI approach in which the elements in the covariance matrix $\boldsymbol{\Phi}$ of the elements in $\boldsymbol{\xi}$ are freely estimated, but the constraints on $\boldsymbol{\Lambda}_\mathbf{X}$ and $\boldsymbol{\Theta}_{\boldsymbol{\delta}}$ are retained. Finally, to minimize distributional assumptions Marsh et al. (2004) suggested the UPI approach which eliminates all nonlinear model constraints with the exception that the mean of $\xi_1\xi_2$ is set equal to the covariance between ξ_1 and ξ_2. This approach is theoretically expected to be less powerful than the CPI approach when the assumption of multivariate normality of the observed X variables is met, but far more robust to violations of this assumption in terms of Type-I error rates.

Structural equation models are most commonly estimated using maximum likelihood (ML). ML estimation provides high efficiency and consistency in parameter estimation under conditions of multivariate normality of the observed variables (Bollen, 1989, p. 108). ML estimation is also fairly robust to violations of the assumption of multivariate normality (e.g., Marsh et al., 2004). Estimation of the $\xi_1\xi_2$ term is potentially problematic because this term is known to have a non-normal distribution (Jöreskog and Yang, 1996; Klein & Moosbrugger, 2000), even when each of the exogenous X variables is normally distributed. When the correlation between ξ_1 and ξ_2 is 0 and ξ_1 and ξ_2 are normally distributed, the skewness and kurtosis[1] of the $\xi_1\xi_2$ product term are 0 and 6, respectively; when the correlation between ξ_1 and ξ_2 is 0.8, the skewness and kurtosis of the $\xi_1\xi_2$ product term are approximately 2.8 and 11.7, respectively (Ma, 2010).

Latent Moderated Structural Equations Approach

An important alternative approach is the Latent Moderated Structural Equations approach (LMS) developed by Klein and Moosbrugger (2000) and Schermelleh-Engel, Klein, and Moosbrugger (1998). The LMS approach does *not* require the formation of product terms to represent $\xi_1\xi_2$. Instead, the LMS approach partitions the relationships between the exogenous and endogenous latent variables into linear and nonlinear components up to the second-order effects (here, linear by linear interaction). If the tested model including a latent variable interaction is correctly specified, the latent ξ_1 and ξ_2 variables are normally distributed, the unique factors of X and Y are normally distributed, the disturbance is normally distributed, and the *conditional* distribution of the endogenous latent variable on exogenous latent variables will be normal. The nonlinear component is approximated by a mixture model (see Kelava, Werner, Schermelleh-Engel, Moosbrugger, Zapf, Ma, Cham, Aiken, & West, 2011 for details). The LMS approach is currently implemented in M*plus* (Muthén & Muthén, 1998–2010).

The Satorra-Bentler Correction

Among procedures proposed to address non-normality, the Satorra-Bentler (SB) correction, which corrects the test statistic and the estimated standard errors of parameter estimates, is widely available and has been shown to have good performance in simulation studies of latent variable models *without* interactions (Chou, Bentler, & Satorra, 1991; Curran, West, & Finch, 1996). However, the SB correction appears to be potentially incompatible with the mixture modeling used in the LMS approach and the constraints used in the CPI approach -- these approaches would be expected to lead to inconsistent parameter estimates when the variables in the **X** vector are severely non-normal as both approaches are based on strong assumptions of normality. The GAPI and UPI approaches involve far less restrictive constraints. These approaches are expected to provide consistent parameter estimates under a non-normal **X** vector. The SB correction might improve the robustness of the GAPI and UPI approaches.

Significance Testing of the Interaction Effect for the PI and LMS Approaches

For the PI approach, researchers have traditionally used the Wald *z*-test of the latent interaction effect (e.g., Marsh et al., 2004). Although asymptotically equivalent to the Wald test under multivariate normality condition, the likelihood ratio (LR) test of the latent interaction effect of the PI approach was also investigated in the present study, as it might have advantages at smaller sample sizes or when the data are non-normal (Enders, 2010, pp. 79–80). In the LR test, the latent interaction effect is initially fixed to zero, and the increase in deviance that occurs when the latent interaction effect is freely estimated is tested. For the LMS approach, when the latent variables ξ_1 and $\xi_{2,}$ the unique factors, and the disturbance ζ of η are all multivariate normal, both the Wald and LR tests of the interaction effect

[1]Throughout this manuscript, kurtosis is defined as excess kurtosis having a value of 0 for a normally distributed variable.

yielded acceptable actual Type-I error rates (Klein & Moosbrugger, 2000). For non-normal **X**, Satorra and Bentler (2001, 2010) have proposed a method for implementing the LR test using the SB correction.

Overview of Previous Literature

Several simulation studies investigating the robustness of the PI and LMS approaches to non-normality have been previously conducted (Coenders, Batista-Foguet, & Saris, 2008; Klein & Moosbrugger, 2000; Klein & Muthén, 2007; Klein, Schermelleh-Engel, Moosbrugger, & Kelava, 2009; Marsh et al., 2004; Wall & Amemiya, 2001). To put these studies into a common metric for comparison of findings, we initially computed indices of skewness and kurtosis for the *observed exogenous variables* in previous simulation studies (see Table 2). We then report the results of the previous literature using the following criteria for acceptable performance, when reported, to evaluate the previous studies: (1) Relative bias of parameter estimate ≤ |10%| (Flora & Curran, 2004). (2) Standard error (*SE*) ratio (ratio of the mean of estimated *SE* to the empirical *SE*) within the range 0.9 to 1.1, which parallels the criterion for relative bias. (3) Coverage rate greater than 90% (Collins, Schafer, & Kam, 2001). (4) Actual Type-I error rate α falling within the binomial confidence interval for α, which is $\alpha \pm z_{(1-\alpha/2)}\sqrt{\alpha(1-\alpha)/k}$, where k is the number of replications (Savalei, 2010).

Univariate Skewness and Kurtosis

A key issue in the previous simulation studies is whether the univariate skewness and kurtosis of the exogenous X variables has been sufficiently large to provide a stringent test of the robustness of the four approaches for estimating latent variable interactions. We calculated the univariate skewness and kurtosis of the exogeneous X variables used in prior studies in two ways. First, Mattson's (1997) mathematical derivations were used to calculate the theoretical skewness and kurtosis of the variables, given the skewness and kurtosis of ξ and δ. Second, to verify these theoretical calculations, we simulated one large dataset (N = 1,000,000) for each condition represented in the literature and calculated the univariate skewness and excess kurtosis of the variables. The two methods of estimating the skewness and kurtosis of the studies closely agreed. Across the studies, the minimum and maximum values of the median skewness (0, 0.90) and kurtosis (−0.75, 1.56) of the observed exogenous variables were *not* extreme. This result implies that existing studies of the robustness of the PI and LMS approaches to non-normality have considered a range of values under which ML estimation would be unlikely to be problematic (Curran et al., 1996; West, Finch, & Curran, 1995). The effects of non-normality of observed variables on the *SE*s of the parameter estimates and test statistic have been shown analytically to have opposite effects under positively and negatively kurtotic conditions (Yuan, Bentler, & Zhang, 2005); we summarize these conditions of previous studies separately.

PI Approach

In the second and third conditions of Coenders et al. (2008), the median skewness and kurtosis of **X** were about 0.9 and 1.5, respectively. The CPI and UPI SB approaches yielded unbiased estimates of the latent interaction and lower-order effects. The SB *SE*s of these effects were also unbiased.

In the third distribution condition of study 4 of Marsh et al. (2004), the median skewness and kurtosis of **X** were about 0.7 and 0.8, respectively. Under these conditions, the estimate of latent interaction effect was unbiased using the CPI, GAPI, or UPI approaches. The ML *SE* of interaction effect was slightly underestimated across all three variants of the PI approach. The SB *SE* did *not* provide appropriate adjustment for the underestimation of the

ML *SE* of the latent variable interaction effect. Both the GAPI and UPI approaches yielded acceptable Type-I error rates for the Wald test of the interaction effect. Under some non-normal conditions, the CPI approach yielded an inflated Type-I error rate of the Wald test. The SB correction did *not* provide appropriate adjustment for the Type-I error rate inflation of the Wald test with the CPI approach; indeed in some cases the use of the SB correction led to Type-I error inflation of the Wald test of the interaction effect.

In Wall and Amemiya's (2001) χ_9^2 distribution condition, the median skewness and kurtosis of **X** were about 0.7 and 0.8, respectively. The GAPI approach yielded unbiased latent interaction effect estimate. In the normal condition, the GAPI approach did not yield an acceptable coverage rate for the interaction effect. The SB correction improved the coverage rate. However, in the χ_9^2 condition, the GAPI approach yielded an unacceptable coverage rate, whereas the SB correction improved the coverage rate to an acceptable level.

In the negatively kurtotic conditions of Marsh et al. (2004) and Wall and Amemiya (2001), the minimum median kurtosis of **X** was −0.75. The GAPI and UPI approaches yielded unbiased latent interaction effect estimate. The ML *SE* of the latent interaction effect was slightly underestimated, whereas the actual Type-I error rate and coverage rate were acceptable. The SB correction did not adjust for the underestimation of the ML *SE* under the negatively kurtotic conditions.

LMS Approach

The non-normal conditions of Klein and Moosbrugger (2000), Klein and Muthén (2007), and Klein et al. (2009) were identical (median skewness and kurtosis of **X** = 0.5 and 1.1, respectively). When the variables were multivariate normal, the LMS approach yielded unbiased estimates of the latent interaction and lower-order effects. The ML *SE* of the interaction effect estimated using the LMS approach was much smaller than the ML *SE* estimated using the CPI approach. The LMS approach was more efficient, yielding higher statistical power of the Wald test of the interaction effect.

In the non-normal conditions of Klein and Moosbrugger (2000) and Klein and Muthén (2007), the interaction effect estimates using the LMS approach were acceptable, but the ML *SE* of the interaction effect was underestimated. This resulted in an inflated Type-I error rates for both the Wald test (Klein et al., 2009) and the LR test of interaction effect using the LMS approach (Klein & Moosbrugger, 2000). In the non-normal conditions of Coenders et al. (2008), the LMS approach yielded biased latent interaction effect estimate.

Research Questions

The present study extends previous simulation research by examining the performance of estimators under substantially more extreme violations of multivariate normality to surface possible limitations of the four approaches. Specifically, we examined the performance of the four different approaches (CPI, GAPI, UPI, LMS) under five different distributional conditions of the observed exogenous variables (normal, uniform, symmetric and moderately leptokurtic, symmetric and highly leptokurtic, and skewed and moderately leptokurtic). The generality of the findings was probed by considering five different sample sizes ranging from 100 to 5000 and two different magnitudes of the interaction effect. The performance of the SB correction for non-normality was also investigated for the PI approaches. Finally, the performance of the Wald and the LR tests of the latent variable interaction was compared. Two primary research questions were addressed in the present study:

1. How robust is each of the PI and LMS approaches, when the observed exogenous variables represent a range of types of non-normality?
2. In cases in which the estimates of the latent interaction and lower-order effects are unbiased, how do the actual Type-I error rates and statistical power of the Wald and LR tests of the interaction effect compare with one another?

Method

The design of the simulation study was adapted from Ma (2010), Marsh et al. (2004), Wall and Amemiya (2001). Below are the details of the conditions manipulated in the simulation study.

(1) Sample size N: 100, 200, 500, 1000, 5000

The range of sample sizes covers the full range of those commonly seen in published research reports within psychology. $N = 200$ approximates the median sample size used in regression analysis (Jaccard & Wan, 1995). $N = 5000$ represents an upper bound sample size for psychological research at which the asymptotic properties of the estimators might be approximated. These sample sizes have been used in previous research (e.g. Curran et al., 1996; Hu & Bentler, 1998; Wall & Amemiya, 2001).

(2) Interaction Model

The latent interaction model in equation (1) was tested. The structural and measurement models were identical to the model used by Marsh et al. (2004), with three indicators of each latent variable. The population parameters values are summarized in Table 3. The population correlation between ξ_1 and ξ_2 was 0.5. We varied the effect size of the interaction by manipulating γ_3 and the disturbance variance ψ. Our theoretical calculations assumed ξ_1 and ξ_2 were bivariate normal, that ξ_1, ξ_2, and η could be measured *directly without measurement error* and therefore could be analyzed using OLS regression. Under these assumptions, the parameters for the interaction model yielded a population $R^2 = 0.4$ for the combined linear effects. The statistical power for the test of the interaction effect was manipulated to equal 0.7 or 0.9 (see Aiken & West, 1991, chapter 8). A condition in which γ_3 equaled zero was used to investigate the Type-I error rate for the tests of the latent variable interaction.

(3) Distributions of ξ_1, ξ_2, and δ

In practice, investigators can only estimate the skewness and kurtosis of *observed* variables. We sought to create conditions in which the observed exogenous variables were (a) normal, (b) uniform, (c) symmetric and moderately kurtotic, (d) symmetric and highly kurtotic, and (e) skewed and moderately kurtotic. ζ and **ε** were normally distributed in all conditions. The distributions of ξ_1, ξ_2, and **δ** were specified as follows to create observed exogenous variables with the desired properties (see also Wall & Ameniya, 2001).

a. Normal. ξ_1, ξ_2, and **δ** follow a normal distribution.

b. Uniform. ξ_1, ξ_2, and **δ** follow a uniform distribution.

c. Symmetric and Moderately Leptokurtic (K1). Adapted from Hu and Bentler (1998), a random t_5 variable and a random χ^2_5 variable were generated. The random K1 variable was defined as ($t_5/\sqrt{\chi^2_5/3}$). ξ_1, ξ_2, and **δ** followed this distribution. Given an extremely large sample size, this procedure results in **X** indicators that are symmetric with kurtosis ≈ 11 (Table 4).

d. Symmetric and Highly Leptokurtic (K2). A random K1 variable and a random χ_5^2 variable were generated. A random K2 variable is defined as ($\mathrm{K1}/\sqrt{\chi_5^2/3}$). ξ_1, ξ_2, and δ follow this distribution. Given an extremely large sample size, this procedure results in **X** that are symmetric with kurtosis ≈ 31 (Table 4).

e. Skewed and Moderately Leptokurtic χ_1^2. ξ_1, ξ_2, and δ followed a χ_1^2 distribution.

Table 4 shows the asymptotic univariate skewness and kurtosis of the observed indicators based on Mattson (1997) for the normal, uniform, and χ_1^2 conditions, and simulation results based on N = 1,000,000 for K1 and K2 conditions. The empirical univariate skewness and kurtotsis of **X** based on 1000 randomly generated data sets at N = 100, 200, 500, 1000, and 5000 are also shown. The empirical kurtosis of **X** for the K1, K2, and χ_1^2 conditions decreased as sample size decreased (see Reinartz, Echambadi, & Chin, 2002, for a discussion).

Figure 1 shows multivariate QQ plots of the **X** indicators (Friendly, 1991) that depict the degree of multivariate non-normality for each distributional contribution. In this plot the Y-axis is the observed quantile which represents the squared Mahalanobis distance of each point from the centroid based on the p observed X variables (here, 6 total, 3 indicators for each latent exogenous variable). The X-axis is the expected quantile from a χ^2 distribution with p degrees of freedom. The deviation of the line formed by the points from the 45-degree reference line indicates the degree to which multivariate normality does not hold. Figure 1 shows the multivariate QQ plots of the observed X variables from one simulated dataset, N = 1000, for each distributional condition. The normal condition has been extensively investigated in previous literature (e.g., Coenders et al., 2008; Klein & Moosbrugger, 2000; Ma, 2010; Marsh et al., 2004; Wall & Amemiya, 2001). The negatively kurtotic uniform condition was investigated by Marsh et al. (2004) and Wall and Amemiya (2001).

(4) Latent variable interaction estimators

Following Aiken and West (1991) and Marsh et al. (2004), the elements of **X** were mean-centered. For the PI approaches, the PI match procedure proposed by Marsh et al. (2004) was adopted, resulting in three product indicators for $\xi_1\xi_2$: X_1X_4, X_2X_5, X_3X_6. In total, seven approaches were investigated: (a) CPI ML, (b) CPI SB, (c) GAPI ML, (d) GAPI SB, (e) UPI ML, (f) UPI SB, and (g) LMS ML. M*plus* 6.12 (Muthén & Muthén, 1998–2010) was used for all the approaches. Default starting values were used. A maximum of 10,000 iterations were allowed for each replication (dataset) within each cell of the design. For the LMS approach, Hermite-Gaussian integration with 16 integration points was used, as suggested by Klein and Moosbrugger (2000).

Taken together, there were 525 conditions (5 sample sizes × 3 interaction effects × 5 distributions × 7 approaches to estimation). For each condition, 1500 replications were generated. Data were generated using SAS/IML Version 9.2.

Results

Model Convergence Rates

In each condition, a model with a freely estimated interaction effect γ_3 and a nested model with γ_3 fixed to zero were estimated. For each approach, a replication was defined as a converged case if the model converged using both the ML and SB procedures with no negative variance estimates, no estimated correlations that were out of range ($-1 < \phi$ or $\phi >$

1), and no estimated standard errors for parameters that were negative. The convergence rate was calculated based on the proportion of the 1500 replications that converged in each condition.

Most (85%) of the conditions had convergence rates greater than 95%. In the normal and uniform conditions, convergence rates were close to 1.0 except for $N = 100$, where convergence rates were lower (e.g., UPI mean convergence = 94%). In the leptokurtic K1, K2, and χ_1^2 conditions, convergence rates were substantially lower at $N = 100$: CPI (average 91%), GAPI (average 82%), and UPI (average 84%). As sample size increased, convergence rates increased for each of the three PI approaches, with all approaches exceeding a 95% rate by $N \geq 500$. In contrast, the convergence rates for LMS were high (> 98%) at $N = 100$ for all distributional conditions, but *decreased* as sample size increased in the leptokurtic K1 and K2 conditions. At $N = 5000$, the convergence rate was 91% for the K1 condition and 75% for the K2 condition. These results stem from the lack of consistency of LMS estimates with kurtotic **X**, particularly as kurtosis becomes more extreme.

Relative Bias of Latent Interaction Effect Estimate

Table 5 presents the relative bias (RB) of $\hat{\gamma}_3$, RB = $(\hat{\gamma}_3 - \gamma_3)/\gamma_3$, calculated when γ_3 was non-zero in the population. RB ≤ |10%| was considered acceptable (Flora & Curran, 2004).

In the normal distribution condition, all approaches provided acceptable estimates of $\hat{\gamma}_3$ at $N \geq 500$. CPI and LMS also produced unbiased estimates of $\hat{\gamma}_3$ at $N < 500$, whereas GAPI and UPI tended to overestimate $\hat{\gamma}_3$. In contrast, GAPI and UPI produced acceptable estimates of $\hat{\gamma}_3$ in the lepokurtoic K1, K2, and χ_1^2 conditions at $N \geq 500$, whereas CPI and LMS yielded substantial overestimates. These overestimates were the most severe in the χ_1^2 distribution condition, exceeding 200% at $N = 5000$. In the negatively kurtotic uniform condition, the CPI and LMS approaches tended to underestimate $\hat{\gamma}_3$, but were in the acceptable range. In contrast, GAPI and UPI tended to overestimate $\hat{\gamma}_3$, with these estimates no longer being acceptable at $N < 500$.

Mean Square Error, Standard Error Ratio, and Coverage Rate of Estimates of the Latent Interaction Effect

We considered four additional indices of the performance of the approaches for both the ML and SB procedures: the mean square error, the standard error ratio, the coverage rate, and the non-coverage rates. We used identical criteria to evaluate the performance of the different estimation approaches on these measures as in our earlier literature review. Few important differences were found as a function of the interaction effect size, ML versus SB estimation, or the use of the ML Wald versus likelihood ratio tests. Comparisons are only reported below when differences were found.

1. *Mean square error*. The mean square error (MSE) of the interaction effect γ_3 is defined as $\Sigma(\hat{\gamma}_3 - \gamma_3)^2/k$, where k is the number of replications (= 1000). The MSE represents the combination of the squared bias and the variance of $\hat{\gamma}_3$. MSE is used as a criterion when an estimator may be biased. Smaller values indicate higher precision of the parameter estimates when the estimate is unbiased.
2. *Standard error ratio*. The standard error ratio (*SE* ratio) is the mean of the ratio of the estimated standard error to the empirical standard error (standard deviation) of $\hat{\gamma}_3$. Paralleling the criterion of ≤ |10%| for the relative bias of parameter estimate, a *SE* ratio between 0.9 and 1.1 was considered to be acceptable.
3. *Coverage rate*. The coverage rate is the proportion of the 95% Wald confidence intervals of γ_3 across the replications that actually include the population value γ_3.

The coverage rate is jointly determined by the bias of $\hat{\gamma}_3$ and the value of its estimated *SE*. Coverage rate was considered to be acceptable when the coverage rate exceeded 90% (Collins et al., 2001).

4. *Non-coverage rates* > γ_3 *and* < γ_3. The non-coverage rate is the proportion of the confidence intervals both of whose limits are either greater than or less than the population value of the parameter (i.e., intervals > γ_3 or < γ_3, respectively). The non-coverage rates potentially provide valuable information when the empirical sampling distribution is asymmetric. A substantial discrepancy between the coverage failures of the Wald confidence intervals > γ_3 versus < γ_3 relative to $\alpha/2$ can indicate problematic asymmetry of the confidence intervals. As a criterion for acceptable non-coverages, we used the binomial confidence interval of $\alpha/2$ (= 0.025), which is $\alpha \pm z_{(1-\alpha/2)}\sqrt{\alpha(1-\alpha)/k}$, where k (= 1000) is the number of replications (Savalei, 2010).

Table 6 shows the MSE and *SE* ratio of $\hat{\gamma}_3$. Table 7 shows the coverage rate of γ_3. In the normal condition at $N \geq 500$, LMS had the lowest and UPI had the highest MSE. All the *SE* ratios were all close to the ideal value of 1.0, and the coverage rates for all four approaches (ML/SB) met the criterion. Non-coverage > γ_3 was slightly smaller than non-coverage < γ_3 across four approaches, although they typically met the criterion for acceptable rates of non-coverage. At $N = 100$ and 200, the *SE* ratio was acceptable only for LMS. The overall coverage rates for all four approaches were acceptable. The non-coverage < γ_3 for all approaches consistently exceeded the criterion. Finally, primarily reflecting smaller standard errors, the MSE was smaller for LMS than CPI, which, in turn, was smaller than GAPI and UPI.

In the negatively kurtotic uniform condition, the coverage rates were typically acceptable for all approaches. At $N \leq 500$, all approaches had smaller non-coverage > γ_3 than non-coverage < γ_3. In contrast, the non-coverages by both GAPI and UPI were close to balanced at $N \geq 1000$. The *SE* ratios of all approaches were acceptable at $N \geq 500$. The SB *SE* ratios were generally smaller than those of ML. The MSE was smaller for LMS than CPI, which, in turn, was smaller than the GAPI and UPI for all sample sizes.

In the symmetric and leptokurtic K1 and K2 conditions, coverage rates were influenced by interaction effect size. Both the GAPI and UPI approaches met the criterion for acceptable overall coverage rates for K1 and K2 in the power = 0.7 condition; acceptable coverage rates for K1 and K2 were not achieved until $N = 5000$ in the power = 0.9 condition. GAPI and UPI SB generally had lower coverage rates than those by ML. CPI and LMS had unacceptably low coverage rates at $N \geq 500$. Coverage rates for CPI and LMS decreased as sample size increased, providing an indication of the lack of consistency of these approaches with leptokurtic data. GAPI and UPI had close to balanced non-coverage rates for > γ_3 and < γ_3 at $N \geq 500$. At $N < 500$, GAPI and UPI had smaller non-coverage > γ_3 than non-coverage < γ_3. The *SE* ratios for GAPI and UPI were typically too low, reaching the criterion in the K1 condition only at $N = 1000$ in the power = 0.7 condition. At $N \geq 500$, MSE, reflecting the combination of squared bias and variance of γ_3 estimates, was consistently lower for both GAPI and UPI than CPI and LMS.

In the skewed and modestly leptokurtic χ^2_1 condition, at $N \geq 500$ both GAPI and UPI had acceptable overall coverage rates. Once again, the coverage rates of CPI and LMS decreased as sample size increased (failing miserably at the larger sample sizes), providing an indication of the lack of consistency of the CPI and LMS approaches with skewed and leptokurtic data. At $N \geq 500$, GAPI and UPI were close to balanced for non-coverage > γ_3 and < γ_3. The *SE* ratios did not meet the criterion until N=5000. At $N \geq 500$, MSE was substantially lower for both GAPI and UPI than for CPI, with LMS clearly having the worst performance.

Several conclusions are warranted. (1) In the normal and negatively kurtotic uniform conditions, the LMS and CPI approaches showed the best performance, with LMS being preferred due to the combination of lack of relative bias, acceptable coverage rates, and the smallest MSE. (2) In the symmetric and leptokurtic K1 and K2 conditions, both GAPI and UPI provided the best performance at larger sample sizes in terms of relative bias, MSE, and coverage rates, with their performance being very similar. (3) In the skewed and moderately leptokurtic χ_1^2 condition, GAPI and UPI clearly outperformed the CPI and LMS approaches at larger sample sizes in terms of bias and coverage rates. (4) When $\hat{\gamma}_3$ was unbiased, each approach tended to produce symmetric non-coverage rates at $N \geq 500$. (5) There was little evidence that the SB corrected standard errors improved the performance of any of the approaches.

Performance of Lower Order Effects

The same measures were calculated for the estimates of the lower order effects, α, γ_1, and γ_2. GAPI and UPI produced unbiased $\hat{\alpha}$, $\hat{\gamma}_1$, and $\hat{\gamma}_2$ at $N \geq 200$ for all distributions. Both GAPI and UPI had acceptable overall coverage rates for all lower order effects. Failures of coverage tended to be smaller than the population parameters across distributions and sample sizes. Paralleling their poor performance in estimating $\hat{\gamma}_3$ in the K1, K2, and χ_1^2 conditions, both CPI and LMS underestimated $\hat{\alpha}$, $\hat{\gamma}_1$, and $\hat{\gamma}_2$ and provided low coverage rates for these effects. In the normal and uniform conditions, the ascending order of the MSEs of lower order effect estimates was LMS < CPI < GAPI ≈ UPI. In the K1, K2, and χ_1^2 conditions when $N \geq 500$, GAPI and UPI had similar MSEs for lower order effect estimates that were substantially lower than those for either CPI or LMS.

Type-I Error Rate and Statistical Power

The actual Type-I error rate and statistical power of the interaction effect was examined. As a criterion for Type-I error rate, we used the binomial confidence interval for α (= .05), which is $\alpha \pm z_{(1-\alpha/2)}\sqrt{\alpha(1-\alpha)/k}$, where k (= 1000) is the number of replications (Savalei, 2010). Here, the criterion is [0.036, 0.064]. The actual Type-I error rates (Table 8) and statistical power (Table 9) of the Wald and LR tests of γ_3 are presented.

Actual Type-I Error Rate of Latent Interaction Effect—In the normal and uniform conditions at $N \geq 500$ all ML approaches had Type-I error rates at or below .05 for the Wald and LR tests of γ_3. Some of the approaches produced slightly conservative tests. At $N <$ 500, the ML approaches provided acceptable, sometimes conservative, Wald tests. The LR test and the SB correction often led to slightly increased Type-I error rates at small sample sizes (< 500). In the symmetric and leptokurtic K1 and K2 conditions, both GAPI and UPI ML had correct Wald and LR tests at $N \geq 500$, except that Type-I error rates were slightly inflated at that largest sample size ($N = 5000$) in the K2 condition. In the skewed and moderately leptokurtic χ_1^2 condition, surprisingly, all approaches typically had slightly inflated Type-I error rates for the Wald and LR tests at $N \geq 500$. In the K1, K2, and χ_1^2 conditions, the CPI and LMS approaches had *substantially* inflated Type-I error rates (.10 or more) for both the Wald and LR tests.

Actual Statistical Power of Latent Interaction Effect—Recall that the simulation was designed so that the theoretical statistical power of the test of γ_3 would equal 0.7 or 0.9. The theoretical calculations assumed that ξ_1 and ξ_2 are bivariate normal, and ξ_1, ξ_2, and η

could be measured *directly without measurement error* and analyzed using OLS regression (Aiken & West, 1991, chapter 8). Table 9 shows the actual statistical power of the tests of the interaction effect. In Table 9, conditions in which the Type-I error rate was too low (conservative) or too high (liberal) are identified by # and +, respectively. Perhaps the most striking feature of the table is the very low level of actual statistical power relative to the theoretically expected values. In the normal condition, actual power ranged from 0.12 to about 0.33 when the theoretical power was 0.7 and from 0.18 to 0.50 when the theoretical power was 0.9. These findings echo previous findings by Fuller (1987) and Ledgerwood and Shrout (2011) about the lack of power in latent variable models that correct for measurement error. In the normal condition, LMS had higher statistical power than CPI, which, in turn, had higher power than GAPI and UPI. In the uniform condition, a similar ordering of the approaches was found. In leptokurtic K1, K2, and χ^2_1 conditions at $N \geq 500$, UPI had slightly greater power than GAPI. The LR test usually had higher power than the Wald test.

Illustrative Example

Description

Our example compares the results of the four approaches with a real data example in which the non-normality of the observed X variables is modest. The example uses data from Proyeto: La Familia (The Family Project, Roosa, Liu, Torres, Gonzales, Knight, & Saenz, 2008), a longitudinal study of Mexican American families. We tested whether there is a possible interaction of child's reported academic self-efficacy and positive family role models on the child's academic performance (e.g., Roosa, O'Donnell, Cham, Gonzales, Zeiders, Tein, Knight, & Umaña-Taylor, 2012). Our analyses are based on 669 families with complete data.

The two exogenous scales were measured in 5th grade using different informants. For the measure of academic self-efficacy, self-reports were obtained from each children on 5 items assessing the extent to which they believed they could master schoolwork (items taken from the Patterns of Adaptive Learning Survey; Midgley, Maehr, Hruda, Anderman, Anderman, Freeman, et al., 2000). Items were measured on a five-point Likert scale, from 1 (none of them) to 5 (all of them). Mother's report of positive family role models was measured using a 5-item scale assessing the extent to which the adults in the family had experiences with academic engagement and success and full-time jobs. Items were measured on a five-point Likert scale, from 1 (not at all true) to 5 (very true). The outcome variable of academic performance was measured in 7th grade. English and math teachers reported the final grades (0.0 = F to 4.0 = A) they would give to the interviewed children in their courses up to the day of interview. The English and math grades were averaged.

Analysis

The 10 items from the two exogenous scales were mean centered. We checked the univariate skewness and kurtosis of each items. The academic self-efficacy items were negatively skewed (median skewness = −1.61) and positively kurtotic (median kurtosis = 2.67). The positive family role models items were symmetric (median skewness = 0.04) and slightly negatively kurtotic (median kurtosis = −0.49). A multivariate QQ plot of the exogenous indicators against a χ^2_{10} distribution which would be obtained if multivariate normality held reflected only a modest violation of multivariate normality (Figure 2). We report the results of each of the four approaches considered above (CPI, GAPI, UPI, and LMS) using both ML and SB estimation. For $N = 669$ and modestly non-normal data, our simulations suggested that the UPI approach would provide unbiased estimates, whereas the LMS approach would provide the greatest statistical power of the test of γ_3. However, the degree of non-normality was sufficient so that this increase in power might come at a cost of a modest increase in the Type-I error rate.

Following Marsh et al. (2004), each item from the academic self-efficacy scale was matched with an item from the positive family role models scale to form the product terms for the product indicator approaches. Items were matched according to the magnitude of their factor loadings on the underlying construct in a one factor measurement model. Table 10 shows the ML estimates of the interaction model, their ML and SB estimated standard errors, and ML and SB likelihood ratio tests of interaction effect by the four approaches. The estimates of the intercepts and the first order (average) effects (maximum difference = 3.4%), ML *SE*s (1.2%), and SB *SE*s (7.4%) differed only slightly. The differences in the estimates and standard errors for the interaction effect γ_3 across estimation approaches was more substantial: Differences in estimates of γ_3 ranged from 2.1% to 18.2%, differences in ML *SE*s for $\hat{\gamma}_3$ ranged from 0% to 44.0%, differences in SB *SE*s for $\hat{\gamma}_3$ ranged from 3.9% to 33.8%. The ML Wald tests of γ_3 varied from $z = -2.72$ ($p = 0.007$) for GAPI to $z = -1.64$ ($p = 0.10$) for UPI, and the ML LR tests of γ_3 varied from $\chi^2_1=7.63$, $p = 0.006$ for GAPI to $\chi^2_1=2.69$, $p = 0.10$ for UPI. In the present empirical example, of course, we do not know which of these results is correct.

Figure 3 probes the interaction effect using simple slope procedures (Aiken & West, 1991; Kelava et al., 2011) based on the results of the conservative UPI approach. The academic performance in 7th grade was regressed on the academic self-efficacy latent variable in 5th grade at three levels of positive family role models latent variable in 5th grade: 1 SD below the mean, at the mean, and 1 SD above the mean. At the mean of positive family role models, effect of academic self-efficacy on academic performance was positive. This positive effect tended to increase as positive family role models decreased, whereas this effect flattened at 1 SD above mean of positive family role models. The children's self-efficacy latent variable tended to have an increasing effect on the academic performance latent variable as positive family role models latent variable decreased.

Discussion

Previous research has been primarily focused on testing latent variable interactions using normal distribution or leptokurtic distributions with modest degrees of skewness and kurtosis of the observed elements in **X** (< 0.9 and <1.6, respectively). Under these modestly non-normal conditions, the three PI approaches have typically yielded unbiased estimates of the latent interaction effect. The LMS approach yielded unbiased latent interaction effect estimates, but with underestimated *SE*s when the observed elements in **X** had |skewness| < 0.5 and kurtosis < 1 (Klein & Moosbrugger, 2000). The modest values of skewness and kurtosis of the observed exogenous variables in previous simulation studies were the result of combining the non-normally distributed exogeneous latent variables ξ_1 and ξ_2 with normally distributed unique factors.

In practice, researchers have no method of estimating the distributions of latent variables and can only use sample values to estimate the population skewness and kurtosis values of the observed variables. Previous research (e.g., Curran et al., 1996; Hu, Bentler, & Kano., 1992) has detected problems (e.g., elevated Type-I error rate) in structural equation modeling *without* latent variable interactions when data are non-normally distributed. These prior studies have used transformation procedures (e.g., Fleishman, 1978; Vale & Maurelli, 1983) that produce more extreme degrees of non-normality in observed variables than those investigated previously in studies of latent variable interactions.

Wall and Amemiya (2001) pointed out that the latent interaction effect estimate could be biased and inconsistent using the CPI approach when the observed variables in **X** are nonnormal. In the present simulation study, the symmetric and moderately leptokurtic K1 (kurtosis ≈ 11), the symmetric and highly leptokurtic K2 (kurtosis ≈ 31), and the skewed and moderately leptokurtic χ^2_1 (skewness ≈ 2, kurtosis ≈ 6) distribution conditions generated exogenous observed variables with higher levels of kurtosis than those considered in previous research (Table 2). In these more non-normal conditions, the CPI and LMS approaches overestimated the latent interaction effect and produced inflated Type-I error rates, whereas the UPI and GAPI approaches showed generally unbiased estimates of both the latent interaction and lower-order effects with produced acceptable Type-I error rates once *N* reached 500.

The present study produced similar findings to those of previous research on the PI approaches with negatively kurtotic distribution conditions (Marsh et al., 2004; Wall & Amemiya, 2001). The results of the uniform condition (skewness = 0; kurtosis ≈ −0.6) generally replicated previous findings using negatively kurtotic distributions: Both the GAPI and UPI approaches yielded unbiased latent interaction and lower-order effects estimates at $N \geq 500$. The ML Wald test of CPI and LMS approaches showed acceptable performance in terms of Type-I error rates. The present study is the first to investigate the performance of LMS under negatively kurtotic condition.

In terms of the efficiency of the different approaches, the current results were also consistent with previous findings (e.g., Klein & Moosbrugger, 2000; Marsh et al., 2004). In the normal and uniform conditions, LMS was the most efficient approach, with the lowest mean squared error and smallest *SE* for the latent interaction effect. The GAPI and UPI approaches were the least efficient in the normal and uniform conditions. Given the substantial bias in the CPI and LMS approaches in the leptokurtic K1, K2, and skewed and leptokurtic χ^2_1 conditions, the GAPI and UPI approaches produced the lowest mean squared error at $N \geq 500$.

In the normal and uniform conditions, all approaches with ML yielded acceptable Type-I error rates for the Wald test of latent interaction effect. In the symmetric and leptokurtic K1 and K2 conditions, GAPI and UPI approaches yielded acceptable Type-I error rates for the Wald test at $N \geq 500$, except at $N = 5000$ in the K2 condition. In the skewed and moderately leptokurtic χ^2_1 condition, all approaches typically had slightly inflated Type-I error rates for the Wald test at $N \geq 500$. The SB correction was unnecessary in these distribution conditions. Indeed, the results showed that in some conditions, the SB correction yielded inflated Type-I error rates for the Wald and LR test, mitigating against its usage.

The present study was the first to consider the LR test of the latent interaction effect for the PI approaches; this test has previously been considered for the LMS approach (Klein & Moosbrugger, 2000). In linear SEM models, the LR test is asymptotically equivalent to the Wald test. However, the LR test has the advantage of scale independence over the Wald test (Gonzalez & Griffin, 2001), and can produce theoretically more accurate tests of parameters with small sample size (Enders, 2010, pp. 79–80). Similar to the findings for the Wald test, all approaches with ML yielded acceptable Type-I error rates for the LR test of the latent interaction effect at $N \geq 500$. Again, the SB correction was *unnecessary*. Indeed, inflated actual Type-I error rates for the SB LR test occurred in some conditions and estimation failures occurred in others. Comparing the results of the ML LR and Wald tests of the latent interaction effect in terms of statistical power under appropriate conditions (unbiased parameter estimates and acceptable actual Type-I error rates), the LR test had a slight advantage over the Wald test in terms of statistical power.

Contrary to the hope of many SEM researchers (e.g., Marsh et al., 2012), the statistical power of the significance tests of the latent variable interaction under multivariate normality was substantially lower across the four approaches than the theoretical statistical power based on OLS assuming that ξ_1, ξ_2 and η were measured without error. In other words, current SEM approaches to latent variable interaction modeling can correct for bias in the regression coefficients, but may often do so at a substantial cost in statistical power. This finding of a lack of statistical power replicates previous findings by Ma, Aiken, and West (2011) and is consistent with Fuller's (1987) and Ledgerwood and Shrout's (2011) observation that latent variable models that correct for measurement error minimize bias, but decrease statistical power. Both the PI and LMS approaches were *unable* to solve the problem of loss of statistical power due to measurement error (e.g., Aiken & West, 1991).

Conclusions and Limitations

Based on the current findings, we recommend the LMS approach when the observed exogenous variables are symmetric (skewness = 0), and with kurtosis < 1, because it had an advantage in terms of efficiency over the PI approach (see also Kelava et al., 2011). With modestly negatively kurtotic observed variables (kurtosis > −0.75), the LMS approach also outperforms the PI approaches in terms of efficiency, with acceptable bias at larger sample sizes. In distributions with higher levels of positive kurtosis, we recommend the UPI approach which had the best performance at sample sizes ≥ 500. Neither GAPI nor UPI performed well at smaller sample sizes with highly leptokurtic observed variables. Finally, no support for the use of the SB correction was found.

Like other simulation studies, the present study only investigated a limited set of conditions; the results can only be generalized to those conditions. The population effect sizes of the latent interaction effect at the larger sample sizes in the normal condition were small (e.g., $N = 1000$, $f^2 = .0062$; $N = 100$, $f^2 = .0630$ when power = 0.7). The effect size for each sample size was chosen to achieve a theoretical statistical power of 0.7 or 0.9 based on calculations for OLS regression with measurement error free (perfectly reliable) variables. The actual values of statistical power achieved by the SEM approaches were far less than these theoretical values of statistical power. In addition, at the larger sample sizes these values were smaller than the effect sizes that have characterized the empirical literature as summarized by Chaplin (1991), on average 4% of incremental variance accounted for by interaction above and beyond first order effects. Nevertheless, the present findings and conclusions should be generalizable to situations in which the effect sizes of the latent interaction effect are larger. The noncentrality parameter for the latent interaction effect is not available in standard computer software for the PI and LMS approaches, limiting the application of theoretical power analysis procedures (Mooijaart & Satorra, 2009). Mooijaart and Bentler (2010) have proposed an extended ML estimation procedure using third-order moments (i.e., skewness) of some observed variables to estimate the latent interaction effect and noncentrality parameter for power analysis.

A second limitation of current study resulted from a shortcoming of the available procedures for generating non-normal data (Reinartz et al., 2002). The smaller the sample size, the smaller was the univariate kurtosis of the observed variables of **X** in the K1, K2 and χ^2_1 conditions (Table 4). We investigated several methods of generating non-normal data; all methods shared this limitation. In addition, the population effect sizes of the interaction effect were different in the five sample size conditions. Our study investigated a wide range of different values of kurtosis. However, we did not investigate conditions in which kurtosis was < −0.6 or greater than 31. More negative values of kurtosis appear to be relatively rare in psychological data. Micceri's (1989) investigation of 440 large scale achievement and psychometric data sets found the most extreme negative value of kurtosis was −1.70 and the most extreme positive value was +37.7.

In summary, our simulation study investigated the performance of four approaches to estimating latent variable interactions over a greater range of distributions of observed exogenous variables than have previously been considered. These conditions included normally distributed, symmetric and negatively kurtotic (kurtosis = −0.60), symmetric and positively kurtotic (kurtosis ≈ 11 and ≈ 31), and moderately skewed and kurtotic (skewness ≈ 2, kurtosis ≈ 6). We recommend the LMS approach when the distribution is approximately normal as it has the greatest statistical power, but maintains acceptable Type-I error and coverage rates. Maximizing statistical power will often be an important consideration since interactions in practice are often small in magnitude and latent variable approaches do not appear to have high statistical power relative to OLS regression of observed scale scores (Ma, 2010; Ma et al., 2011). For distributions with appreciable kurtosis or both skewness and kurtosis, the UPI approach was clearly preferable at larger sample sizes ($N \geq 500$). Neither the use of the ML LR test nor the SB correction of the standard errors appreciably improved the performance of any of the approaches.

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

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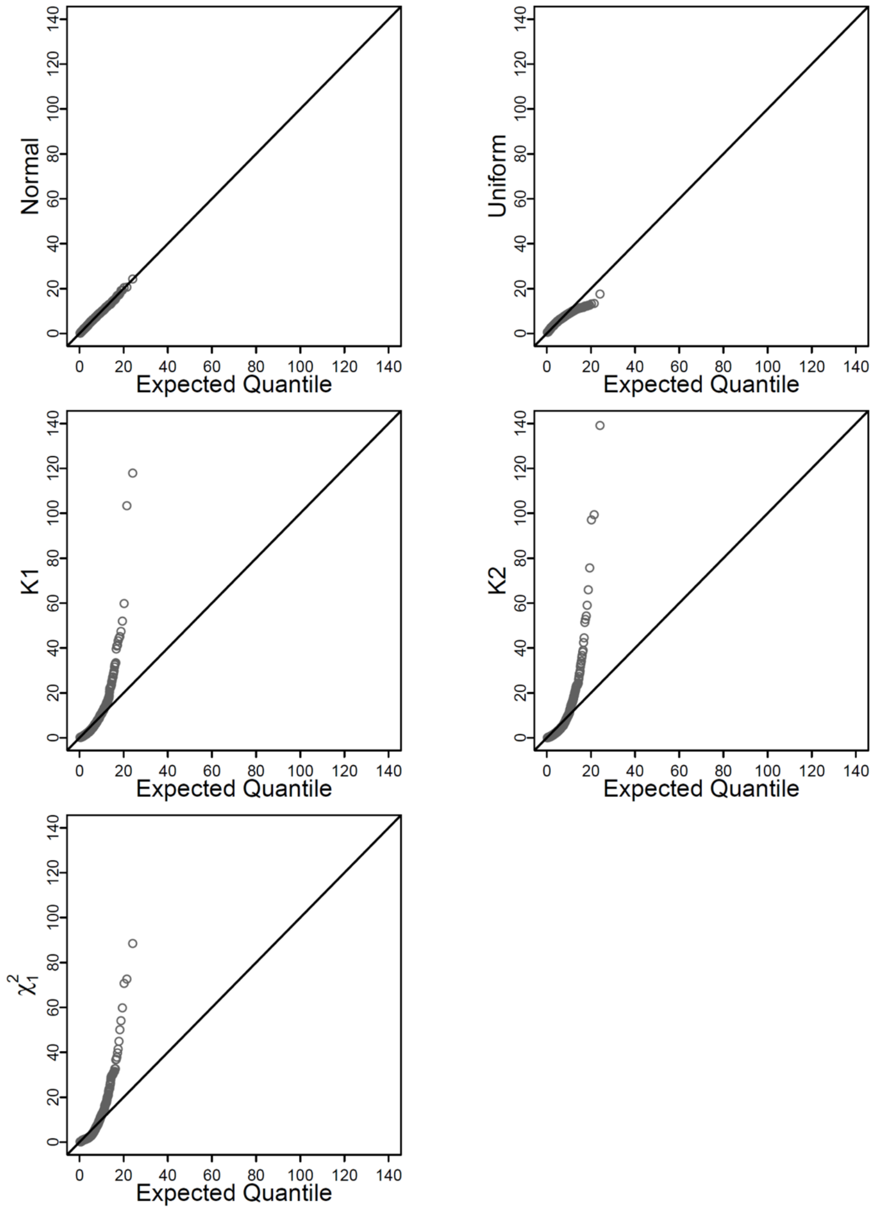

Figure 1.
Multivariate QQ Plots of Observed Exogenous Variables (Excluding Product Indictors) of Each Distribution Condition (From One Randomly Generated Dataset with Sample Size = 1000)
Note. Y-axis is the observed quantile (squared Mahalanobis distance).
X-axis is the expected quantile of χ^2_6 distribution.

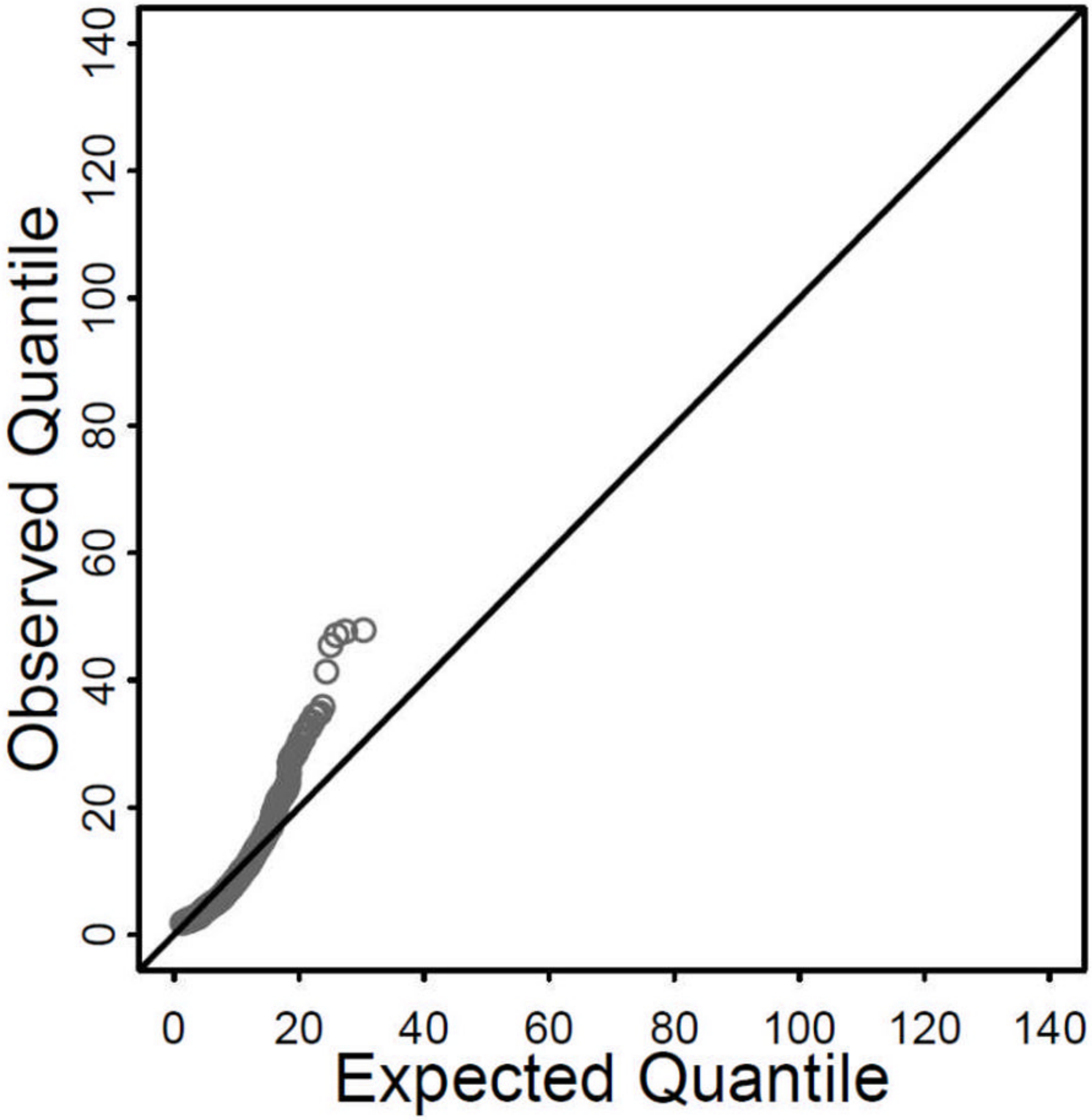

Figure 2.
Multivariate QQ Plot of Observed Exogenous Variables (Excluding Product Indictors) of the Illustrative Example
Note. Y-axis is the observed quantile (squared Mahalanobis distance).
X-axis is the expected quantile of χ^2_{10} distribution.

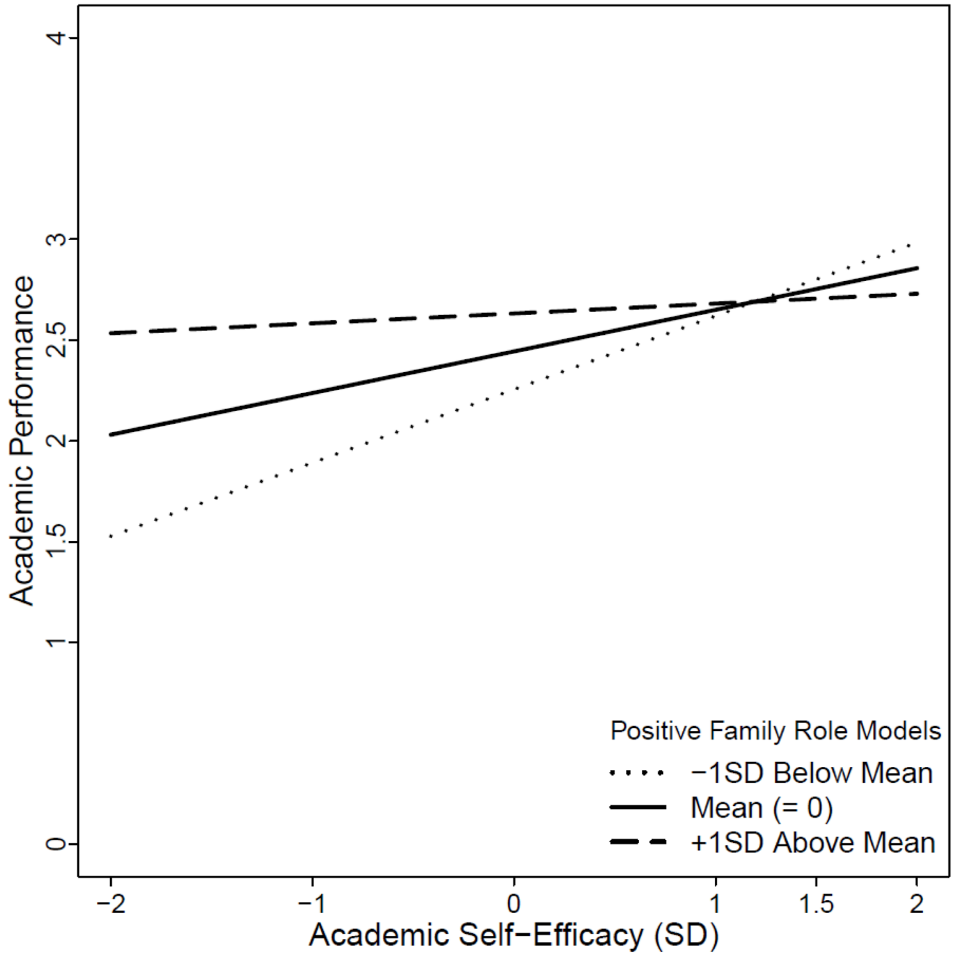

Figure 3.
Interaction Plot of Positive Family Role Models × Academic Self-Efficacy at 5th Grade on Academic Performance at 7th Grade

TABLE 1

Summary of Parameter Constraints and Assumptions of Various Product Indicator Approaches (Sources: Jöreskog & Yang, 1996; Kelava et al., 2011; Ma, 2010; Wall & Amemiya, 2001)

Model Specification	CPI	GAPI	UPI
1. Factor loadings of the product indicators on $\xi_1\xi_2$. For example, $\lambda_{X_1X_4} = \lambda_{X_1}\lambda_{X_4}$	Yes	Yes	No
2. $E(\xi_1\xi_2) = Cov(\xi_1, \xi_2)$	Yes	Yes	Yes
3. $Var(\xi_1\xi_2) = Var(\xi_1)Var(\xi_2) + Cov^2(\xi_1, \xi_2)$	Yes	No	No
4. $Cov(\xi_1, \xi_1\xi_2) = Cov(\xi_2, \xi_1\xi_2) = 0$	Yes	No	No
5. Variances of the unique factors of the product indicators. For example, $Var(\delta_{X_1X_4}) = \lambda_{X_1} Var(\xi_1)Var(\delta_{X_4}) + \lambda_{X_4} Var(\xi_2)Var(\delta_{X_1}) + Var(\delta_{X_1})Var(\delta_{X_4})$	Yes	Yes	No
6. Zero covariances between the unique factors of exogenous indicators and those of product indicators (assuming zero covariances among the unique factors of exogenous indicators)	Yes	Yes	Yes
7. Covariances between unique factors of product indicators that share the same exogenous indicators (assuming zero covariances among the unique factors of exogenous indicators). For example, $Cov(\delta_{X_1X_4}, \delta_{X_1X_5}) = \lambda_{X_4}\lambda_{X_5}Var(\xi_2)Var(\delta_{X_1})$	Yes	Yes	No
8. Normality assumptions of exogenous indicators from *model specification*	Yes	More Liberal	Most Liberal

Note. CPI means constrained product indicator approach. GAPI means generalized appended product indicator approach. UPI means unconstrained product indicator approach. As noted in 8, model constraints 3, 4, 5, 6, and 7 assume multivariate normality for the CPI approach. Constraint 7 is not a concern if the two exogenous latent variables have equal numbers of indicators. The distributional assumptions are relaxed for the GAPI and UPI approaches.

TABLE 2

Univariate Skewness and Kurtosis of Observed Exogenous Variables of Previous Studies

Coenders et al. (2008)		**Theoretical Values**		**Simulation Values**	
		Median |Skewness|	**Median Kurtosis**	**Median |Skewness|**	**Median Kurtosis**
ξ_1 and ξ_2 ~ Normal		0	0	0.000	0.005
$\xi_1 : \chi^2_9$	Low correlation	0.898	1.560	0.892	1.546
ξ_2: Skew.=1.1, Kurt.=1.9	High correlation	0.864	1.441	0.864	1.437
Marsh et al. (2004)					
ξ and **δ** ~ Normal	$r_{\xi_1, \xi_2} = 0.3$	0	0	−0.001	0.002
	$r_{\xi_1, \xi_2} = 0.7$	0	0	0.002	−0.001
ξ: Skew.=0, Kurt.= −0.69	$r_{\xi_1, \xi_2} = 0.3$	0	−0.479	0.001	−0.481
δ ~ Uniform	$r_{\xi_1, \xi_2} = 0.7$	0	−0.478	0.001	−0.479
ξ: Skew.=0.86, Kurt.=1.12	$r_{\xi_1, \xi_2} = 0.3$	0.713	0.790	0.717	0.801
$\boldsymbol{\delta} \sim \chi^2_6$	$r_{\xi_1, \xi_2} = 0.7$	0.715	0.791	0.716	0.801
Wall & Amemiya (2001)					
ξ and **δ** ~ Normal		0	0	−0.001	0.003
ξ and **δ** ~ Uniform		0	−0.750	−0.001	−0.751
$\boldsymbol{\delta} \sim \chi^2_9$		0.730	0.833	0.727	0.813
Klein & Moosbrugger (2000), Klein & Muthén (2007), and Klein et al. (2009)					
ξ and **δ** ~ Normal		0	0	−0.000	−0.002
ξ_1: Skew.= −2, Kurt.=6, ξ_2: Skew.=1.5, Kurt.=5, **δ** ~ Normal		0.519	1.083	0.515	1.070
ξ_1: Skew.= 2, Kurt.=6, ξ_2: Skew. = 1.5, Kurt. =5, **δ** ~ Normal		0.519	1.083	0.524	1.057

Note. |Skewness| means skewness taking absolute values. Skew. means skewness. Kurt. means excess kurtosis. Theoretical values were calculated based on Mattson (1997). Simulation values were calculated using one randomly generated dataset of size 1,000,000.

TABLE 3

Population Parameters of the Simulation Study

Parameter	
$E(\xi_1) = E(\xi_2)$	0
$Var(\xi_1) = Var(\xi_2)$	1
$Cov(\xi_1, \xi_2)$	0.5
$\tau_X = \tau_Y$	0
$\lambda_X = \lambda_Y$	1
$Var(\delta)$	1.286
$Var(\varepsilon)$	472.5
α	10
γ_1	7
γ_2	7

	Statistical power of γ_3					
	0		0.7		0.9	
Sample Size	γ_3	ψ	γ_3	ψ	γ_3	ψ
100	0	220.5	3.233	207.4	4.133	199.1
200	0	220.5	2.309	213.8	2.981	209.4
500	0	220.5	1.469	217.8	1.909	215.9
1000	0	220.5	1.041	219.1	1.356	218.2
5000	0	220.5	0.466	220.2	0.608	220.0

Note. Statistical power of γ_3 indicates the statistical power assuming that ξ_1 and ξ_2 are bivariate normal, and ξ_1, ξ_2, and η can be measured *directly without measurement error* and analyzed using OLS regression.

TABLE 4

Univariate Skewness and Kurtosis of Observed Exogenous Variables (Excluding Product Indicators) in Different Conditions

Distribution	Normal		Uniform		K1		K2		χ_1^2	
	Average Skewness	**Average Kurtosis**	**Average Skewness**	**Average Kurtosis**	**Average Skewness**	**Average Kurtosis**	**Average Skewness**	**Average Kurtosis**	**Average Skewness**	**Average Kurtosis**
Asymptotic Values	0[1]	0[1]	0[1]	−0.609[1]	−0.010[2]	10.967[2]	−0.030[2]	31.063[2]	2.012[1]	6.094[1]
Sample Size										
100	0.003	−0.001	0.000	−0.574	−0.007	3.345	0.030	5.684	1.821	4.492
200	0.000	0.001	0.000	−0.595	0.022	4.517	0.027	7.755	1.913	5.185
500	0.003	0.001	0.000	−0.600	−0.007	5.812	−0.038	10.502	1.967	5.654
1000	−0.001	0.001	0.000	−0.607	−0.033	7.218	−0.071	13.077	1.984	5.848
5000	0.000	0.000	0.000	−0.608	0.004	8.102	0.023	19.757	2.005	6.023

Note.

[1] means theoretical values calculated based on Mattson (1997).

[2] means simulation values calculated using one hundred randomly generated datasets of size 1,000,000.

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TABLE 5

Mean Estimate and Relative Bias of the Latent Interaction Effect (Statistical Power = 0.7)

			CPI		GAPI		UPI		LMS	
Sample Size	**Distribution**	γ_3	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$
100	Normal	3.233	3.337	3.2% *	3.909	20.9%	3.874	19.8%	3.220	−0.4% *
	Uniform	3.233	3.074	−4.9% *	4.106	27.0%	4.006	23.9%	2.987	−7.6% *
	K1	3.233	3.837	18.7%	3.845	18.9%	3.713	14.9%	3.981	23.2%
	K2	3.233	4.130	27.7%	3.816	18.1%	3.907	20.9%	4.237	31.1%
	χ_1^2	3.233	5.759	78.1%	3.974	22.9%	3.556	10.0%	6.147	90.1%
200	Normal	2.309	2.392	3.6% *	2.637	14.2%	2.533	9.7% *	2.426	5.1% *
	Uniform	2.309	2.121	−8.1% *	2.651	14.8%	2.714	17.6%	2.070	−10.3%
	K1	2.309	2.793	20.9%	2.492	7.9% *	2.420	4.8% *	2.833	22.7%
	K2	2.309	2.830	22.6%	2.440	5.7% *	2.469	6.9% *	3.069	32.9%
	χ_1^2	2.309	4.352	88.5%	2.532	9.7% *	2.586	12.0%	4.918	113.0%
500	Normal	1.469	1.511	2.8% *	1.554	5.8% *	1.554	5.7% *	1.510	2.7% *
	Uniform	1.469	1.315	−10.5%	1.548	5.3% *	1.545	5.1% *	1.298	−11.6%
	K1	1.469	1.790	21.8%	1.509	2.7% *	1.498	2.0% *	1.921	30.7%
	K2	1.469	1.849	25.9%	1.549	5.4% *	1.529	4.1% *	2.063	40.4%
	χ_1^2	1.469	3.244	120.8%	1.566	6.6% *	1.556	5.9% *	3.969	170.1%
1000	Normal	1.041	1.035	−0.6% *	1.048	0.7% *	1.044	0.3% *	1.051	0.9% *
	Uniform	1.041	0.929	−10.8%	1.071	2.8% *	1.067	2.5% *	0.920	−11.6%
	K1	1.041	1.267	21.7%	1.052	1.0% *	1.048	0.7% *	1.363	30.9%
	K2	1.041	1.283	23.2%	1.057	1.6% *	1.054	1.2% *	1.465	40.7%
	χ_1^2	1.041	2.577	147.5%	1.116	7.2% *	1.113	6.9% *	3.311	218.0%

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Mean Estimate and Relative Bias of the Latent Interaction Effect (Statistical Power = 0.7)

			CPI		GAPI		UPI		LMS	
Sample Size	**Distribution**	γ_3	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$
5000	Normal	0.466	0.487	4.4% *	0.490	5.0% *	0.489	4.8% *	0.489	4.7% *
	Uniform	0.466	0.426	−8.7% *	0.480	2.9% *	0.480	3.0% *	0.423	−9.4% *
	K1	0.466	0.567	21.5%	0.465	−0.2% *	0.465	−0.3% *	0.634	35.8%
	K2	0.466	0.552	18.3%	0.463	−0.8% *	0.463	−0.6% *	0.668	43.3%
	χ_1^2	0.466	1.770	279.6%	0.483	3.5% *	0.483	3.5% *	2.410	416.6%

Mean Estimate and Relative Bias of the Latent Interaction Effect (Statistical Power = 0.9)

			CPI		GAPI		UPI		LMS	
Sample Size	**Distribution**	γ_3	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$
100	Normal	4.133	4.277	3.5% *	5.220	26.3%	4.824	16.7%	4.175	1.0% *
	Uniform	4.133	3.730	−9.8% *	5.005	21.1%	4.924	19.1%	3.794	−8.2% *
	K1	4.133	5.084	23.0%	4.923	19.1%	4.784	15.7%	5.141	24.4%
	K2	4.133	5.241	26.8%	4.831	16.9%	4.720	14.2%	5.540	34.0%
	χ_1^2	4.133	6.549	58.5%	5.018	21.4%	4.588	11.0%	7.164	73.3%
200	Normal	2.981	3.063	2.7% *	3.393	13.8%	3.362	12.8%	2.991	0.3% *
	Uniform	2.981	2.772	−7.0% *	3.502	17.5%	3.561	19.4%	2.762	−7.4% *
	K1	2.981	3.740	25.5%	3.297	10.6%	3.257	9.2% *	3.831	28.5%
	K2	2.981	3.837	28.7%	3.364	12.8%	3.291	10.4%	4.087	37.1%
	χ_1^2	2.981	5.260	76.4%	3.403	14.2%	3.402	14.1%	5.821	95.3%
500	Normal	1.909	1.995	4.5% *	2.058	7.8% *	2.051	7.4% *	1.934	1.3% *
	Uniform	1.909	1.652	−13.5%	1.914	0.2% *	1.908	−0.1% *	1.670	−12.5%
	K1	1.909	2.362	23.7%	1.990	4.2% *	1.964	2.9% *	2.536	32.8%
	K2	1.909	2.395	25.5%	1.990	4.2% *	1.985	4.0% *	2.643	38.4%

Mean Estimate and Relative Bias of the Latent Interaction Effect (Statistical Power = 0.9)

			CPI		GAPI		UPI		LMS	
Sample Size	**Distribution**	γ_3	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$	$E(\hat{\gamma}_3)$	$RB(\hat{\gamma}_3)$
	χ_1^2	1.909	3.684	93.0%	2.044	7.1% *	2.027	6.2% *	4.529	137.3%
1000	Normal	1.356	1.362	0.4% *	1.384	2.1% *	1.383	2.0% *	1.371	1.2% *
	Uniform	1.356	1.232	−9.1% *	1.414	4.3% *	1.409	3.9% *	1.247	−8.0% *
	K1	1.356	1.693	24.9%	1.405	3.7% *	1.402	3.4% *	1.836	35.5%
	K2	1.356	1.681	24.0%	1.405	3.6% *	1.391	2.6% *	2.198	62.2%
	χ_1^2	1.356	2.883	112.7%	1.385	2.2% *	1.388	2.4% *	3.709	173.6%
5000	Normal	0.608	0.613	0.7% *	0.616	1.2% *	0.616	1.2% *	0.625	2.7% *
	Uniform	0.608	0.546	−10.2%	0.622	2.3% *	0.623	2.4% *	0.552	−9.2% *
	K1	0.608	0.739	21.4%	0.610	0.3% *	0.609	0.1% *	0.820	34.7%
	K2	0.608	0.730	19.9%	0.614	0.9% *	0.613	0.8% *	0.876	44.1%
	χ_1^2	0.608	1.909	213.8%	0.612	0.5% *	0.613	0.7% *	2.597	326.9%

Note. RB means relative bias.

* means relative bias ≤ |10%| criterion met.

Note. RB means relative bias.

* means relative bias ≤ |10%| criterion met.

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TABLE 6

Mean Squared Error and Standard Error Ratio of Latent Interaction Effect Estimate (Statistical Power = 0.7)

		Mean Squared Error				Standard Error (*SE*) Ratio						
Sample Size	**Dist.**	**CPI**	**GAPI**	**UPI**	**LMS**	**CPI (ML)**	**CPI (SB)**	**GAPI (ML)**	**GAPI (SB)**	**UPI (ML)**	**UPI (SB)**	**LMS**
100	Normal	13.13	29.23	31.50	8.19	0.85	0.74	0.87	0.79	0.88	0.79	0.95 *
	Uniform	14.60	43.25	43.11	10.69	0.89	0.74	1.03 *	0.93 *	0.95 *	0.86	0.92 *
	K1	11.13	17.79	51.92	9.69	0.76	0.70	0.84	0.81	1.65	1.27	0.83
	K2	13.26	20.76	76.52	13.91	0.68	0.63	0.94 *	0.99 *	0.69	0.69	0.81
	χ_1^2	29.34	41.42	51.90	29.70	0.63	0.55	0.97 *	0.94 *	1.27	1.33	0.73
200	Normal	4.21	7.35	6.77	3.30	0.99 *	0.92 *	0.92 *	0.90 *	0.93 *	0.90 *	0.98 *
	Uniform	5.85	11.66	12.90	4.50	0.93 *	0.83	0.91 *	0.88	0.89	0.85	0.95 *
	K1	3.91	5.66	4.24	3.41	0.76	0.81	0.63	0.62	0.69	0.68	0.82
	K2	3.69	4.35	8.22	3.87	0.73	0.79	0.71	0.70	0.54	0.56	0.81
	χ_1^2	8.66	8.41	10.72	11.09	0.75	0.75	0.75	0.75	0.76	0.82	0.90 *
500	Normal	1.56	1.84	1.91	1.17	0.99 *	0.96 *	0.97 *	0.98 *	0.96 *	0.97 *	0.99 *
	Uniform	1.88	2.85	2.93	1.52	0.99 *	0.90 *	0.96 *	0.96 *	0.95 *	0.95 *	0.98 *
	K1	0.93	0.57	0.55	0.99	0.79	0.99 *	0.84	0.84	0.84	0.84	0.84
	K2	0.90	0.47	0.48	1.09	0.70	0.91 *	0.78	0.77	0.76	0.75	0.80
	χ_1^2	4.47	1.18	1.04	7.59	0.69	0.79	0.82	0.84	0.86	0.87	0.89
1000	Normal	0.67	0.75	0.76	0.52	1.04 *	1.02 *	1.01 *	1.03 *	1.01 *	1.02 *	1.03 *
	Uniform	0.92	1.27	1.29	0.74	1.00 *	0.91 *	0.99 *	1.00 *	0.98 *	1.00 *	1.00 *
	K1	0.34	0.16	0.17	0.39	0.82	1.14	0.92 *	0.93 *	0.90 *	0.91 *	0.86
	K2	0.34	0.13	0.13	0.53	0.67	0.98 *	0.83	0.81	0.83	0.82	0.71
	χ_1^2	2.92	0.41	0.37	5.75	0.65	0.81	0.84	0.88	0.87	0.90 *	0.86

Mean Squared Error and Standard Error Ratio of Latent Interaction Effect Estimate (Statistical Power = 0.7)

		Mean Squared Error				Standard Error (*SE*) Ratio						
Sample Size	**Dist.**	**CPI**	**GAPI**	**UPI**	**LMS**	**CPI (ML)**	**CPI (SB)**	**GAPI (ML)**	**GAPI (SB)**	**UPI (ML)**	**UPI (SB)**	**LMS**
5000	Normal	0.15	0.16	0.16	0.11	0.97 *	0.96 *	0.96 *	0.98 *	0.95 *	0.98 *	0.97 *
	Uniform	0.16	0.22	0.22	0.14	1.04 *	0.96 *	1.03 *	1.05 *	1.02 *	1.05 *	1.02 *
	K1	0.06	0.02	0.02	0.09	0.68	1.16	0.96 *	0.96 *	0.95 *	0.96 *	0.69
	K2	0.05	0.01	0.01	0.13	0.54	1.08 *	0.89	0.90 *	0.87	0.89	0.49
	χ_1^2	1.79	0.05	0.05	3.88	0.66	0.95 *	0.90 *	0.97 *	0.93 *	0.97 *	0.83

Mean Squared Error and Standard Error Ratio of Latent Interaction Effect Estimate (Statistical Power = 0.9)

		Mean Squared Error				Standard Error (*SE*) Ratio						
Sample Size	**Dist.**	**CPI**	**GAPI**	**UPI**	**LMS**	**CPI (ML)**	**CPI (SB)**	**GAPI (ML)**	**GAPI (SB)**	**UPI (ML)**	**UPI (SB)**	**LMS**
100	Normal	14.67	47.28	33.66	10.11	0.84	0.74	1.20	1.17	0.99 *	0.92 *	0.89
	Uniform	12.73	40.56	51.77	10.75	0.96 *	0.80	1.02 *	0.93 *	1.06 *	0.93 *	0.93 *
	K1	14.37	23.95	53.64	13.48	0.70	0.66	0.82	0.78	0.75	0.44	0.77
	K2	13.19	25.00	23.96	16.99	0.71	0.67	0.80	0.75	0.80	0.79	0.76
	χ_1^2	23.73	39.03	32.05	29.84	0.64	0.58	1.02 *	0.98 *	0.94 *	0.87	0.75
200	Normal	5.43	9.12	9.96	3.62	0.88	0.82	0.83	0.80	0.82	0.79	0.94 *
	Uniform	5.59	11.69	14.47	4.27	0.95 *	0.84	0.91 *	0.88	0.86	0.83	0.97 *
	K1	4.73	3.35	4.07	3.99	0.72	0.80	0.78	0.77	0.71	0.71	0.85
	K2	5.17	4.99	6.31	5.55	0.67	0.74	0.72	0.71	0.61	0.61	0.75
	χ_1^2	12.66	12.60	9.73	13.32	0.58	0.60	0.66	0.68	0.68	0.67	0.84
500	Normal	1.59	1.96	2.05	1.15	0.98 *	0.95 *	0.95 *	0.96 *	0.94 *	0.94 *	1.00 *
	Uniform	1.97	2.92	2.99	1.57	0.98 *	0.89	0.95 *	0.96 *	0.95 *	0.95 *	0.98 *
	K1	1.15	0.62	0.62	1.27	0.76	0.95 *	0.83	0.83	0.82	0.83	0.84
	K2	1.12	0.62	0.71	1.31	0.68	0.90 *	0.72	0.74	0.66	0.70	0.85

Mean Squared Error and Standard Error Ratio of Latent Interaction Effect Estimate (Statistical Power = 0.9)

		Mean Squared Error				Standard Error (*SE*) Ratio						
Sample Size	**Dist.**	**CPI**	**GAPI**	**UPI**	**LMS**	**CPI (ML)**	**CPI (SB)**	**GAPI (ML)**	**GAPI (SB)**	**UPI (ML)**	**UPI (SB)**	**LMS**
	χ_1^2	4.59	1.15	1.06	8.37	0.66	0.78	0.82	0.87	0.84	0.87	0.86
1000	Normal	0.77	0.85	0.86	0.55	0.97 *	0.96 *	0.96 *	0.98 *	0.96 *	0.98 *	1.00 *
	Uniform	0.95	1.33	1.35	0.73	0.98 *	0.90 *	0.97 *	0.99 *	0.97 *	0.98 *	1.00 *
	K1	0.47	0.22	0.23	0.63	0.74	1.02 *	0.80	0.82	0.79	0.81	0.76
	K2	0.41	0.15	0.14	68.28	0.65	0.99 *	0.80	0.82	0.80	0.82	0.31
	χ_1^2	2.87	0.41	0.37	6.14	0.67	0.84	0.83	0.89	0.87	0.90 *	0.88
5000	Normal	0.14	0.15	0.15	0.11	0.99 *	0.99 *	0.97 *	1.00 *	0.97 *	1.00 *	1.00 *
	Uniform	0.18	0.25	0.25	0.15	0.99 *	0.92 *	0.97 *	1.00 *	0.97 *	1.00 *	0.97 *
	K1	0.07	0.02	0.02	0.12	0.68	1.15	0.86	0.87	0.86	0.87	0.64
	K2	0.05	0.01	0.01	0.17	0.58	1.17	0.88	0.90 *	0.87	0.88	0.47
	χ_1^2	1.78	0.05	0.05	4.07	0.63	0.91 *	0.90 *	0.97 *	0.93 *	0.97 *	0.80

Note. ML means maximum likelihood. SB means Satorra-Bentler correction.

* means (0.9 ≤ *SE* Ratio ≤ 1.1) criterion met.

Note. ML means maximum likelihood. SB means Satorra-Bentler correction.

* means (0.9 ≤ *SE* Ratio ≤ 1.1) criterion met.

TABLE 7

Coverage Rate of Latent Interaction Effect (Statistical Power = 0.7)

		CPI		GAPI		UPI		LMS
Sample Size	**Distribution**	**ML**	**SB**	**ML**	**SB**	**ML**	**SB**	
100	Normal	0.947 *	0.903 *	0.954 *	0.923 *	0.932 *	0.904 *	0.948 *
	Uniform	0.923 *	0.880	0.960 *	0.931 *	0.948 *	0.915 *	0.927 *
	K1	0.910 *	0.856	0.925 *	0.891	0.914 *	0.869	0.956 *
	K2	0.888	0.807	0.929 *	0.876	0.918 *	0.871	0.962 *
	χ^2_1	0.822	0.724	0.933 *	0.904 *	0.931 *	0.898	0.924 *
200	Normal	0.956 *	0.941 *	0.956 *	0.949 *	0.958 *	0.941 *	0.958 *
	Uniform	0.940 *	0.902 *	0.955 *	0.946 *	0.959 *	0.947 *	0.932 *
	K1	0.892	0.891	0.918 *	0.899	0.917 *	0.893	0.931 *
	K2	0.879	0.864	0.930 *	0.887	0.926 *	0.875	0.913 *
	χ^2_1	0.691	0.672	0.924 *	0.928 *	0.933 *	0.922 *	0.755
500	Normal	0.955 *	0.947 *	0.952 *	0.955 *	0.954 *	0.952 *	0.950 *
	Uniform	0.956 *	0.916 *	0.958 *	0.956 *	0.963 *	0.955 *	0.943 *
	K1	0.849	0.901 *	0.931 *	0.909 *	0.935 *	0.908 *	0.870
	K2	0.811	0.874	0.899	0.868	0.906 *	0.869	0.813
	χ^2_1	0.375	0.479	0.916 *	0.916 *	0.926 *	0.921 *	0.269
1000	Normal	0.954 *	0.953 *	0.953 *	0.960 *	0.947 *	0.951 *	0.964 *
	Uniform	0.953 *	0.941 *	0.958 *	0.962 *	0.956 *	0.958 *	0.946 *
	K1	0.855	0.935 *	0.932 *	0.920 *	0.932 *	0.921 *	0.841
	K2	0.741	0.879	0.923 *	0.896	0.924 *	0.900	0.735
	χ^2_1	0.149	0.266	0.913 *	0.918 *	0.926 *	0.922 *	0.049

Coverage Rate of Latent Interaction Effect (Statistical Power = 0.7)

		CPI		GAPI		UPI		LMS
Sample Size	**Distribution**	**ML**	**SB**	**ML**	**SB**	**ML**	**SB**	
5000	Normal	0.947 *	0.943 *	0.944 *	0.951 *	0.945 *	0.950 *	0.943 *
	Uniform	0.955 *	0.930 *	0.958 *	0.960 *	0.957 *	0.960 *	0.950 *
	K1	0.776	0.946 *	0.940 *	0.930 *	0.944 *	0.931 *	0.763
	K2	0.667	0.907 *	0.937 *	0.923 *	0.935 *	0.921 *	0.649
	χ_1^2	0.000	0.003	0.927 *	0.946 *	0.936 *	0.945 *	0.000

Coverage Rate of Latent Interaction Effect (Statistical Power = 0.9)

		CPI		GAPI		UPI		LMS
Sample Size	**Distribution**	**ML**	**SB**	**ML**	**SB**	**ML**	**SB**	
100	Normal	0.928 *	0.891	0.949 *	0.919 *	0.937 *	0.905 *	0.931 *
	Uniform	0.941 *	0.879	0.970 *	0.945 *	0.949 *	0.922 *	0.939 *
	K1	0.906 *	0.844	0.921 *	0.876	0.914 *	0.865	0.954 *
	K2	0.870	0.815	0.912 *	0.877	0.888	0.838	0.952 *
	χ_1^2	0.785	0.706	0.904 *	0.873	0.890	0.866	0.915 *
200	Normal	0.933 *	0.908 *	0.950 *	0.942 *	0.951 *	0.931 *	0.956 *
	Uniform	0.942 *	0.908 *	0.959 *	0.950 *	0.956 *	0.946 *	0.944 *
	K1	0.857	0.881	0.886	0.884	0.892	0.876	0.920 *
	K2	0.839	0.851	0.909 *	0.892	0.904 *	0.877	0.899
	χ_1^2	0.667	0.671	0.879	0.863	0.882	0.865	0.732
500	Normal	0.957 *	0.943 *	0.943 *	0.957 *	0.945 *	0.950 *	0.951 *
	Uniform	0.938 *	0.921 *	0.949 *	0.948 *	0.948 *	0.952 *	0.937 *
	K1	0.836	0.900	0.899	0.891	0.892	0.884	0.843
	K2	0.779	0.858	0.898	0.903 *	0.891	0.888	0.803

Coverage Rate of Latent Interaction Effect (Statistical Power = 0.9)

		CPI		GAPI		UPI		LMS
Sample Size	**Distribution**	**ML**	**SB**	**ML**	**SB**	**ML**	**SB**	
	χ^2_1	0.392	0.511	0.914 *	0.920 *	0.921 *	0.923 *	0.238
1000	Normal	0.945 *	0.941 *	0.945 *	0.949 *	0.948 *	0.948 *	0.952 *
	Uniform	0.936 *	0.921 *	0.945 *	0.952 *	0.946 *	0.952 *	0.955 *
	K1	0.794	0.902 *	0.900	0.894	0.898	0.895	0.776
	K2	0.739	0.886	0.895	0.887	0.895	0.888	0.681
	χ^2_1	0.157	0.295	0.905 *	0.912 *	0.906 *	0.913 *	0.043
5000	Normal	0.952 *	0.951 *	0.946 *	0.951 *	0.945 *	0.953 *	0.960 *
	Uniform	0.945 *	0.925 *	0.938 *	0.943 *	0.936 *	0.940 *	0.940 *
	K1	0.728	0.934 *	0.921 *	0.907 *	0.916 *	0.898	0.700
	K2	0.639	0.910 *	0.931 *	0.921 *	0.931 *	0.916 *	0.572
	χ^2_1	0.000	0.006	0.917 *	0.935 *	0.932 *	0.938 *	0.000

Note. ML means maximum likelihood. SB means Satorra-Bentler correction.

* means (Coverage Rate > 0.9) criterion met.

Note. ML means maximum likelihood. SB means Satorra-Bentler correction.

* means (Coverage Rate > 0.9) criterion met.

TABLE 8

Actual Type-I Error Rate of Interaction Effect

		CPI				GAPI				UPI				LMS	
		ML		SB		ML		SB		ML		SB			
Sample Size	Dist.	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT
100	Normal	0.053	0.080 +	0.099 +	0.092 +	0.025 #	0.083 +	0.050	0.079 +	0.026 #	0.084 +	0.050	0.080 +	0.033 #	0.066 +
100	Uniform	0.044	0.072 +	0.108 +	0.075 +	0.010 #	0.064 +	0.024 #	0.078 +	0.015 #	0.075 +	0.028 #	0.088 +	0.037	0.067 +
100	K1	0.057	0.074 +	0.125 +	0.149 +	0.036 #	0.055	0.081 +	0.079 +	0.039	0.067 +	0.088 +	0.081 +	0.032 #	0.072 +
100	K2	0.062	0.084 +	0.155 +	0.188 +	0.029 #	0.058	0.070 +	0.061	0.037	0.068 +	0.077 +	0.071 +	0.050	0.093 +
100	χ_1^2	0.156 +	0.184 +	0.309 +	0.349 +	0.036 #	0.064 +	0.059	0.090 +	0.047	0.077 +	0.074 +	0.101 +	0.104 +	0.194 +
200	Normal	0.046	0.056	0.069 +	0.067 +	0.038	0.057	0.043	0.058	0.033 #	0.060	0.042	0.064 +	0.041	0.057
200	Uniform	0.055	0.059	0.091 +	0.069 +	0.035 #	0.070 +	0.055	0.066 +	0.033 #	0.068 +	0.049	0.064 +	0.047	0.064 +
200	K1	0.083 +	0.091 +	0.100 +	0.151 +	0.045	0.059	0.083 +	0.048	0.040	0.058	0.086 +	0.059	0.058	0.087 +
200	K2	0.085 +	0.096 +	0.125 +	0.178 +	0.045	0.059	0.082 +	0.054	0.038	0.052	0.086 +	0.065 +	0.076 +	0.104 +
200	χ_1^2	0.241 +	0.255 +	0.324 +	0.384 +	0.045	0.061	0.060	0.061	0.041	0.059	0.064 +	0.064 +	0.213 +	0.299 +
500	Normal	0.031 #	0.034 #	0.034 #	0.039	0.032 #	0.044	0.035 #	0.037	0.032 #	0.042	0.032 #	0.039	0.034 #	0.036 #
500	Uniform	0.041	0.047	0.073 +	0.052	0.045	0.051	0.048	0.053	0.044	0.057	0.045	0.050	0.046	0.051
500	K1	0.105 +	0.110 +	0.060	0.137 +	0.053	0.054	0.077 +	0.046	0.052	0.052	0.073 +	0.054	0.099 +	0.108 +
500	K2	0.119 +	0.126 +	0.104 +	0.174 +	0.058	0.059	0.093 +	0.041	0.057	0.057	0.092 +	0.047	0.109 +	0.136 +
500	χ_1^2	0.522 +	0.531 +	0.465 +	0.602 +	0.065 +	0.070 +	0.071 +	0.072 +	0.058	0.064 +	0.069 +	0.060	0.585 +	0.640 +
1000	Normal	0.030 #	0.030 #	0.037	0.036 #	0.040	0.042	0.039	0.034 #	0.040	0.043	0.039	0.033 #	0.043	0.046
1000	Uniform	0.047	0.049	0.074 +	0.050	0.051	0.054	0.046	0.048	0.053	0.055	0.048	0.051	0.050	0.051
1000	K1	0.132 +	0.133 +	0.075 +	0.136 +	0.057	0.057	0.087 +	0.043	0.055	0.059	0.092 +	0.050	0.102 +	0.117 +
1000	K2	0.154 +	0.156 +	0.082 +	0.162 +	0.049	0.051	0.079 +	0.036 #	0.049	0.052	0.083 +	0.038	0.132 +	0.142 +

		CPI				GAPI				UPI				LMS	
		ML		SB		ML		SB		ML		SB			
Sample Size	Dist.	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT
1000	χ_1^2	0.828 +	0.829 +	0.696 +	0.831 +	0.058	0.060	0.054	0.059	0.055	0.054	0.053	0.051	0.906 +	0.924 +
5000	Normal	0.041	0.042	0.047	0.044	0.052	0.053	0.043	0.044	0.052	0.053	0.044	0.045	0.046	0.046
5000	Uniform	0.058	0.058	0.075 +	0.058	0.057	0.062	0.049	0.050	0.059	0.060	0.049	0.050	0.039	0.039
5000	K1	0.193 +	0.192 +	0.051	0.128 +	0.055	0.056	0.068 +	0.035 #	0.058	0.058	0.068 +	0.038	0.150 +	0.152 +
5000	K2	0.270 +	0.267 +	0.083 +	0.186 +	0.073 +	0.073 +	0.092 +	0.049	0.069 +	0.069 +	0.092 +	0.047	0.213 +	0.192 +
5000	χ_1^2	1.000 +	1.000 +	0.991 +	0.999 +	0.079 +	0.080 +	0.066 +	0.055	0.075 +	0.073 +	0.067 +	0.057	1.000 +	1.000 +

Note. Two-tailed Wald test with nominal Type-I error rate = 0.05 ($z_{critical} = \pm 1.96$) was used to calculate the actual Type-I error rate. One-tailed likelihood ratio test with nominal Type-I error rate = 0.05 ($\chi^2_{critical} = 3.84$) was used to calculate the actual Type-I error rate. Negative likelihood ratio test statistic by SB correction was treated as missing values.

means the actual Type-I error rate is smaller than the lower limit criterion 0.036.

+ means the actual Type-I error rate is larger than the upper limit criterion 0.064.

TABLE 9

Actual Statistical Power of Interaction Effect (Statistical Power = 0.7)

		CPI				GAPI				UPI				LMS	
		ML		SB		ML		SB		ML		SB			
Sample Size	**Dist.**	**Wald**	**LRT**	**Wald**	**LRT**	**Wald**	**LRT**	**Wald**	**LRT**	**Wald**	**LRT**	**Wald**	**LRT**	**Wald**	**LRT**
100	Normal	0.202	0.254 $^{+}$	0.285 $^{+}$	0.248 $^{+}$	0.131 $^{\#}$	0.237 $^{+}$	0.187	0.211 $^{+}$	0.118 $^{\#}$	0.242 $^{+}$	0.166	0.213 $^{+}$	0.202 $^{\#}$	0.283 $^{+}$
100	Uniform	0.124	0.164 $^{+}$	0.199 $^{+}$	0.180 $^{+}$	0.052 $^{\#}$	0.147 $^{+}$	0.081 $^{\#}$	0.155 $^{+}$	0.040 $^{\#}$	0.155 $^{+}$	0.075 $^{\#}$	0.163 $^{+}$	0.140	0.200 $^{+}$
100	K1	0.492	0.534 $^{+}$	0.538 $^{+}$	0.517 $^{+}$	0.411 $^{\#}$	0.464	0.471 $^{+}$	0.332 $^{+}$	0.415	0.476 $^{+}$	0.484 $^{+}$	0.336 $^{+}$	0.458 $^{\#}$	0.553 $^{+}$
100	K2	0.572	0.625 $^{+}$	0.609 $^{+}$	0.602 $^{+}$	0.468 $^{\#}$	0.519	0.530 $^{+}$	0.400	0.458	0.504 $^{+}$	0.534 $^{+}$	0.401 $^{+}$	0.496	0.615 $^{+}$
100	χ_1^2	0.642 $^{+}$	0.682 $^{+}$	0.694 $^{+}$	0.691 $^{+}$	0.344	0.356 $^{+}$	0.385	0.419 $^{+}$	0.343	0.356 $^{+}$	0.407 $^{+}$	0.420 $^{+}$	0.564 $^{+}$	0.710 $^{+}$
200	Normal	0.205	0.233	0.248 $^{+}$	0.230 $^{+}$	0.175	0.229	0.196	0.187	0.159 $^{\#}$	0.235	0.186	0.183 $^{+}$	0.256	0.308
200	Uniform	0.146	0.174	0.205 $^{+}$	0.174 $^{+}$	0.105 $^{\#}$	0.169 $^{+}$	0.123	0.162 $^{+}$	0.089 $^{\#}$	0.178 $^{+}$	0.112	0.166 $^{+}$	0.165	0.186 $^{+}$
200	K1	0.582 $^{+}$	0.600 $^{+}$	0.536 $^{+}$	0.559 $^{+}$	0.552	0.564	0.576 $^{+}$	0.360	0.540	0.570	0.552 $^{+}$	0.371	0.571	0.620 $^{+}$
200	K2	0.655 $^{+}$	0.675 $^{+}$	0.594 $^{+}$	0.647 $^{+}$	0.605	0.634	0.617 $^{+}$	0.430	0.602	0.633	0.619 $^{+}$	0.438 $^{+}$	0.635 $^{+}$	0.683 $^{+}$
200	χ_1^2	0.798 $^{+}$	0.815 $^{+}$	0.795 $^{+}$	0.802 $^{+}$	0.403	0.405	0.428	0.460	0.420	0.429	0.447 $^{+}$	0.453 $^{+}$	0.819 $^{+}$	0.874 $^{+}$
500	Normal	0.245 $^{\#}$	0.249 $^{\#}$	0.255 $^{\#}$	0.239	0.225 $^{\#}$	0.242	0.226 $^{\#}$	0.219	0.221 $^{\#}$	0.248	0.215 $^{\#}$	0.227	0.288 $^{\#}$	0.301 $^{\#}$
500	Uniform	0.154	0.166	0.221 $^{+}$	0.177	0.153	0.173	0.153	0.166	0.142	0.173	0.148	0.163	0.192	0.202
500	K1	0.707 $^{+}$	0.709 $^{+}$	0.544	0.643 $^{+}$	0.693	0.698	0.693 $^{+}$	0.429	0.696	0.706	0.690 $^{+}$	0.424	0.740 $^{+}$	0.747 $^{+}$
500	K2	0.803 $^{+}$	0.807 $^{+}$	0.697 $^{+}$	0.766 $^{+}$	0.805	0.810	0.808 $^{+}$	0.505	0.813	0.819	0.805 $^{+}$	0.492	0.832 $^{+}$	0.846 $^{+}$
500	χ_1^2	0.963 $^{+}$	0.964 $^{+}$	0.924 $^{+}$	0.950 $^{+}$	0.532 $^{+}$	0.529 $^{+}$	0.523 $^{+}$	0.539 $^{+}$	0.546	0.538 $^{+}$	0.537 $^{+}$	0.497	0.987 $^{+}$	0.992 $^{+}$
1000	Normal	0.209 $^{\#}$	0.213 $^{\#}$	0.220	0.211 $^{\#}$	0.218	0.221	0.206	0.197 $^{\#}$	0.209	0.218	0.202	0.191 $^{\#}$	0.284	0.295
1000	Uniform	0.167	0.174	0.209 $^{+}$	0.178	0.167	0.173	0.158	0.163	0.164	0.177	0.155	0.166	0.203	0.207
1000	K1	0.778 $^{+}$	0.779 $^{+}$	0.570 $^{+}$	0.695 $^{+}$	0.786	0.787	0.769 $^{+}$	0.486	0.783	0.784	0.769 $^{+}$	0.480	0.790 $^{+}$	0.797 $^{+}$
1000	K2	0.871 $^{+}$	0.870 $^{+}$	0.706 $^{+}$	0.830 $^{+}$	0.878	0.878	0.873 $^{+}$	0.565 $^{\#}$	0.881	0.881	0.881 $^{+}$	0.565	0.876 $^{+}$	0.879 $^{+}$

Actual Statistical Power of Interaction Effect (Statistical Power = 0.7)

		CPI				GAPI				UPI				LMS	
		ML		SB		ML		SB		ML		SB			
Sample Size	Dist.	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT
1000	χ_1^2	0.998 $^{+}$	0.998 $^{+}$	0.983 $^{+}$	0.993 $^{+}$	0.601	0.594	0.560	0.580	0.615	0.610	0.588	0.547	1.000 $^{+}$	1.000 $^{+}$
5000	Normal	0.263	0.265	0.267	0.263	0.263	0.265	0.250	0.245	0.265	0.265	0.249	0.245	0.325	0.326
5000	Uniform	0.160	0.160	0.192 $^{+}$	0.164	0.157	0.161	0.151	0.152	0.158	0.162	0.146	0.148	0.194	0.195
5000	K1	0.878 $^{+}$	0.877 $^{+}$	0.574	0.810 $^{+}$	0.917	0.918	0.894 $^{+}$	0.694 $^{\#}$	0.916	0.916	0.896 $^{+}$	0.679	0.890 $^{+}$	0.882 $^{+}$
5000	K2	0.926 $^{+}$	0.922 $^{+}$	0.662 $^{+}$	0.875 $^{+}$	0.969 $^{+}$	0.969 $^{+}$	0.959 $^{+}$	0.734	0.971 $^{+}$	0.971 $^{+}$	0.958 $^{+}$	0.719	0.937 $^{+}$	0.918 $^{+}$
5000	χ_1^2	1.000 $^{+}$	1.000 $^{+}$	1.000 $^{+}$	1.000 $^{+}$	0.643 $^{+}$	0.641 $^{+}$	0.580 $^{+}$	0.606	0.652 $^{+}$	0.651 $^{+}$	0.611 $^{+}$	0.601	1.000 $^{+}$	1.000 $^{+}$

Actual Statistical Power of Interaction Effect (Statistical Power = 0.9)

		CPI				GAPI				UPI				LMS	
		ML		SB		ML		SB		ML		SB			
Sample Size	Dist.	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT
100	Normal	0.295	0.352 $^{+}$	0.374 $^{+}$	0.323 $^{+}$	0.206 $^{\#}$	0.314 $^{+}$	0.248	0.262 $^{+}$	0.184 $^{\#}$	0.319 $^{+}$	0.227	0.262 $^{+}$	0.346 $^{\#}$	0.437 $^{+}$
100	Uniform	0.185	0.241 $^{+}$	0.292 $^{+}$	0.251 $^{+}$	0.104 $^{\#}$	0.228 $^{+}$	0.133 $^{\#}$	0.228 $^{+}$	0.068 $^{\#}$	0.224 $^{+}$	0.118 $^{\#}$	0.231 $^{+}$	0.217	0.310 $^{+}$
100	K1	0.607	0.651 $^{+}$	0.634 $^{+}$	0.612 $^{+}$	0.525 $^{\#}$	0.583	0.566 $^{+}$	0.447 $^{+}$	0.514	0.576 $^{+}$	0.549 $^{+}$	0.447 $^{+}$	0.576 $^{\#}$	0.686 $^{+}$
100	K2	0.680	0.714 $^{+}$	0.688 $^{+}$	0.674 $^{+}$	0.564 $^{\#}$	0.615	0.621 $^{+}$	0.477	0.567	0.613 $^{+}$	0.620 $^{+}$	0.466 $^{+}$	0.629	0.720 $^{+}$
100	χ_1^2	0.750 $^{+}$	0.780 $^{+}$	0.777 $^{+}$	0.762 $^{+}$	0.461	0.465 $^{+}$	0.485	0.518 $^{+}$	0.454	0.458 $^{+}$	0.504 $^{+}$	0.493 $^{+}$	0.700 $^{+}$	0.822 $^{+}$
200	Normal	0.331	0.356	0.389 $^{+}$	0.337 $^{+}$	0.286	0.356	0.319	0.313	0.265 $^{\#}$	0.347	0.286	0.300 $^{+}$	0.387	0.440
200	Uniform	0.227	0.251	0.309 $^{+}$	0.250 $^{+}$	0.163 $^{\#}$	0.236 $^{+}$	0.179	0.227 $^{+}$	0.131 $^{\#}$	0.247 $^{+}$	0.161	0.227 $^{+}$	0.259	0.297 $^{+}$
200	K1	0.734 $^{+}$	0.756 $^{+}$	0.679 $^{+}$	0.676 $^{+}$	0.702	0.733	0.708 $^{+}$	0.507	0.687	0.733	0.695 $^{+}$	0.513	0.752	0.787 $^{+}$
200	K2	0.787 $^{+}$	0.809 $^{+}$	0.720 $^{+}$	0.754 $^{+}$	0.727	0.753	0.736 $^{+}$	0.517	0.728	0.746	0.741 $^{+}$	0.545 $^{+}$	0.793 $^{+}$	0.826 $^{+}$

Actual Statistical Power of Interaction Effect (Statistical Power = 0.9)

		CPI				GAPI				UPI				LMS	
		ML		SB		ML		SB		ML		SB			
Sample Size	Dist.	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT	Wald	LRT
200	χ_1^2	0.890 +	0.902 +	0.866 +	0.860 +	0.537	0.529	0.525	0.569	0.547	0.556	0.547 +	0.572 +	0.907 +	0.941 +
500	Normal	0.376 #	0.386 #	0.383 #	0.358	0.356 #	0.373	0.349 #	0.333	0.343 #	0.376	0.342 #	0.338	0.453 #	0.476 #
500	Uniform	0.234	0.246	0.304 +	0.253	0.232	0.257	0.227	0.235	0.221	0.259	0.217	0.233	0.288	0.308
500	K1	0.840 +	0.848 +	0.740	0.789 +	0.827	0.834	0.824 +	0.582	0.833	0.840	0.817 +	0.575	0.873 +	0.875 +
500	K2	0.890 +	0.893 +	0.785 +	0.858 +	0.877	0.879	0.869 +	0.640	0.883	0.891	0.874 +	0.635	0.907 +	0.912 +
500	χ_1^2	0.986 +	0.989 +	0.963 +	0.978 +	0.693 +	0.679 +	0.656 +	0.671 +	0.703	0.696 +	0.687 +	0.652	0.994 +	0.997 +
1000	Normal	0.366 #	0.369 #	0.371	0.359 #	0.350	0.362	0.339	0.328 #	0.347	0.364	0.337	0.326 #	0.454	0.470
1000	Uniform	0.261	0.262	0.312 +	0.273	0.253	0.260	0.239	0.250	0.247	0.262	0.237	0.251	0.314	0.320
1000	K1	0.908 +	0.908 +	0.797 +	0.874 +	0.909	0.909	0.888 +	0.651	0.909	0.909	0.891 +	0.636	0.924 +	0.928 +
1000	K2	0.956 +	0.955 +	0.866 +	0.932 +	0.950	0.951	0.942 +	0.696 #	0.958	0.958	0.945 +	0.687	0.957 +	0.958 +
1000	χ_1^2	1.000 +	1.000 +	0.996 +	0.997 +	0.717	0.708	0.683	0.688	0.727	0.725	0.702	0.672	1.000 +	1.000 +
5000	Normal	0.381	0.382	0.379	0.376	0.369	0.371	0.357	0.354	0.368	0.370	0.356	0.350	0.495	0.494
5000	Uniform	0.248	0.250	0.311 +	0.263	0.257	0.261	0.244	0.244	0.257	0.262	0.245	0.245	0.317	0.316
5000	K1	0.954 +	0.954 +	0.788	0.927 +	0.969	0.969	0.960 +	0.786 #	0.968	0.968	0.961 +	0.784	0.963 +	0.960 +
5000	K2	0.989 +	0.985 +	0.888 +	0.974 +	0.995 +	0.995 +	0.992 +	0.816	0.996 +	0.996 +	0.993 +	0.803	0.988 +	0.978 +
5000	χ_1^2	1.000 +	1.000 +	1.000 +	1.000 +	0.813 +	0.810 +	0.765 +	0.783	0.822 +	0.819 +	0.792 +	0.779	1.000 +	1.000 +

Note. Two-tailed Wald test with nominal Type-I error rate = 0.05 ($z_{critical}$ = ±1.96) was used to calculate the actual Type-I error rate. One-tailed likelihood ratio test with nominal Type-I error rate = 0.05 ($\chi^2_{critical}$ = 3.84) was used to calculate the actual Type-I error rate. Negative likelihood ratio test statistic by SB correction was treated as missing values.

[#] means the actual Type-I error rate (Table 8) is smaller than the lower limit criterion 0.036.

[+] means the actual Type-I error rate (Table 8) is larger than the upper limit criterion 0.064.

Note. Two-tailed Wald test with nominal Type-I error rate = 0.05 ($z_{critical}$ = ±1.96) was used to calculate the actual Type-I error rate. One-tailed likelihood ratio test with nominal Type-I error rate = 0.05 ($\chi^2_{critical}$ = 3.84) was used to calculate the actual Type-I error rate. Negative likelihood ratio test statistic by SB correction was treated as missing values.

[#] means the actual Type-I error rate (Table 8) is smaller than the lower limit criterion 0.036.

[+] means the actual Type-I error rate (Table 8) is larger than the upper limit criterion 0.064.

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TABLE 10

Illustrative Example Results Using the Four Approaches

	CPI			GAPI			UPI			LMS	
	Est.	SE_{ML}	SE_{SB}	**Est.**	SE_{ML}	SE_{SB}	**Est.**	SE_{ML}	SE_{SB}	**Est.**	SE_{ML}
Intercept	2.448	0.043 **	0.043 **	2.448	0.043 **	0.043 **	2.444	0.043 **	0.043 **	2.444	0.043 **
Positive Family Role Model	0.203	0.053 **	0.050 **	0.205	0.053 **	0.051 **	0.203	0.053 **	0.051 **	0.204	0.053 **
Academic Self-Efficacy	0.343	0.085 **	0.075 **	0.355	0.086 **	0.081 **	0.349	0.086 **	0.079 **	0.347	0.086 **
Interaction Effect	−0.293	0.109 **	0.098 **	−0.302	0.111 **	0.102 **	−0.287	0.175	0.148 +	−0.247	0.098 *
LRT of Interaction Effect		7.274 **	8.639 **		7.634 **	8.147 **		2.685	2.875		6.556 *

Note. Est. means parameter estimate. SE_{ML} means the ML estimated standard error. SE_{SB} means the Satorra-Bentler correction estimated standard error. LRT means likelihood ratio test.

+ $p < 0.10$,

* $p < 0.05$,

** $p < 0.01$.