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Micropolar continuum modelling of bi-dimensional tetrachiral lattices

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The in-plane behaviour of tetrachiral lattices should be characterized by bi-dimensional orthotropic material owing to the existence of two orthogonal axes of rotational symmetry. Moreover, the constitutive model must also represent the chirality inherent in the lattices. To this end, a bi-dimensional orthotropic chiral micropolar model is developed based on the theory of irreducible orthogonal tensor decomposition. The obtained constitutive tensors display a hierarchy structure depending on the symmetry of the underlying microstructure. Eight additional material constants, in addition to five for the hemitropic case, are introduced to characterize the anisotropy under Z_2 invariance. The developed continuum model is then applied to a tetrachiral lattice, and the material constants of the continuum model are analytically derived by a homogenization process. By comparing with numerical simulations for the discrete lattice, it is found that the proposed continuum model can correctly characterize the static and wave properties of the tetrachiral lattice.

1. Introduction

Lightweight lattices are usually homogenized as a micropolar medium either to explain the physical foundation of micropolar continuum [1] or to reduce the computational cost of lattice structures [2]. More recently,

representations of lattices with various topologies as micropolar media [3–8] or couple-stress media [9,10] are investigated to characterize size effects or non-local behaviours owing to the discrete microstructure. The presence of chirality in lattice structures, as first proposed by Prall & Lakes [11] for a trichiral lattice, couples internal rotation and tension in the materials and leads to many interesting in-plane properties, such as the negative Poisson's ratio [11], negative dynamic bulk modulus [12], etc. The static and dynamic properties of trichiral, hexachiral and tetrachiral lattices are analysed by many authors with targeted applications [13–16]. However, in these studies the lattices are modelled with classical Cauchy material or achiral micropolar material, hence the chiral nature of the lattice cannot be taken into account [17]. This is probably owing to the fact that no explicit bi-dimensional chiral micropolar constitutive model is available, although the necessary rationale already exists implicitly in theories of group representation of tensors. Recently, Liu *et al.* [18] proposed a micropolar model for bi-dimensional isotropic chiral (bi-dimensional hemitropic) solids, which can be applied to trichiral or hexachiral lattices possessing Z_3 and Z_6 invariance. For a tetrachiral lattice with Z_2 (or Z_4 if properties along the two symmetrical axes are the same) invariance, the continuum model for its in-plane behaviour should be bi-dimensional orthotropic. To characterize such lattices, the corresponding bi-dimensional orthotropic chiral micropolar model is necessary.

To include chiral effects, generalized continuum models are usually used, as the classical Cauchy theory cannot admit chirality [17]. In general, there are two types of generalized theories, namely the high-order theory and the high-gradient theory, can admit chirality. The high-order theory [19] endows extra degrees of freedom (DOFs) to material particles and can be classified as micropolar, microstretch and micromorphic theories with regard to the complexity of micro-DOFs introduced. To describe the chirality, the micropolar theory [20] is mostly used owing to its simplicity. Compared to the achiral micropolar theory, three additional material constants are introduced for a tri-dimensional isotropic chiral material, and these constants alternate their signs according to the handedness of the microstructure [17]. The developed theory provides an efficient tool for modelling chiral effects presented in materials and structures, e.g. nanotubes and chiral rods [21,22], wave propagation in chiral solids [23,24], etc. However, a straightforward reasoning [16] shows that, for bi-dimensional hemitropic solids, the constitutive relation cannot be directly obtained from the reduction in the tri-dimensional counterpart. From the viewpoint of symmetry group of tensors, the reason is that a bi-dimensional hemitropic tensor is $SO(2)$ invariant, whereas a tensor of tri-dimensional hemitropic material is $SO(3)$ invariant [25]. Therefore, a bi-dimensional hemitropic micropolar model should be reduced from the corresponding tri-dimensional version with $SO(2)$ invariance, i.e. transverse hemitropic model, which, however, is not readily found in the literature. High-gradient elasticity [26] can also describe chirality. Papanicolopoulos [27] investigated the chiral effect by a tri-dimensional isotropic strain gradient theory. Auffray *et al.* [28,29] examined the elastic tensors of strain gradient theory in bi- and tri-dimensional context for all material symmetry groups, including chiral ones. However, as in the following, the most pronounced feature of bi-dimensional chiral lattices investigated in this paper is the coupling between the bulk deformation and independent rotation of material particles, which is actually the reason for many unusual behaviours, such as auxetics, high compressibility, etc. To this end, the micropolar theory can provide a simpler and clear physical picture of such media, therefore it is adopted in our following analysis.

In this work, we will develop a bi-dimensional orthotropic chiral micropolar constitutive model, and will apply it to a tetrachiral lattice to derive material constants from its microstructure. The study on general forms of elastic tensor under all possible symmetry groups has a long history and attracts interest of many researchers from mechanics, material science, physics and mathematics communities [30–33]. The technique of irreducible decomposition [25,34,35] separates a tensor into isotropic and various anisotropic parts and determines the rotationally invariant quantities individually according to different symmetry groups, thus it provides more intuitive understanding of the considered tensor. Although most of the studies focus on the traditional elasticity, the mathematical rationale in the group representation and irreducible decomposition can also offer an efficient tool for the analysis of elastic tensors of micropolar

media. Here in this paper, the form of decomposition for a general bi-dimensional tensor given by Zou *et al.* [35] will be employed as a basis for our derivation.

This paper is organized as follows. In §2, a general constitutive relation for a bi-dimensional orthotropic chiral micropolar solid is presented. In §3, a tetrachiral lattice is homogenized in context of the proposed theory and the effective material constants are derived. In §4, the theory is illustrated by examining the dependence of the effective material constants and plane wave propagation on the lattice microstructure. The main result of this work is concluded in §5.

2. Bi-dimensional orthotropic chiral micropolar elasticity

Micropolar theory endows every material point three rotational DOFs ϕ_i in addition to displacements u_i . As a consequence, higher order strain and stress measures, say the curvature $k_{ji} = \phi_{i,j}$ and the couple stress m_{ji} , are introduced in addition to the ordinary strain $\varepsilon_{kl} = u_{l,k} + e_{lkm}\phi_m$ and stress σ_{kl} , and all of them are asymmetric. In our following analysis, body forces and body couples are neglected, and the momentum conservations are given by

$$\sigma_{ji,j} = \frac{\rho \partial^2 u_i}{\partial t^2} \quad (2.1a)$$

and

$$m_{ji,j} + e_{ikl}\sigma_{kl} = \frac{J \partial^2 \phi_i}{\partial t^2}, \quad (2.1b)$$

where e_{ikl} is the Levi-Civita tensor, ρ and J are the density and micro-inertia, respectively. Here and henceforth, the Einstein's summation convention and the comma in subscript denoting partial differentiation with respect to spatial coordinates are understood. Moreover, subscripts of Latin letters range over 1–3, and subscripts of Greek letters range over 1–2. A micropolar elastic medium is characterized by the following constitutive equations:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + H_{ijkl}\phi_{l,k} \quad (2.2a)$$

and

$$m_{ij} = H_{klji}\varepsilon_{kl} + D_{ijkl}\phi_{l,k}, \quad (2.2b)$$

where \mathbf{C} , \mathbf{D} and \mathbf{H} are elastic tensors of rank four. Note that a medium is called non-centrosymmetric when $\mathbf{H} \neq 0$, and for a tri-dimensional hemitropic material the \mathbf{H} tensor contains the information of chirality.

As we are interested in characterizing the in-plane anisotropy of tetrachiral lattices, a planar micropolar problem in the $x_1 - x_2$ plane defined by $u_3 = \phi_1 = \phi_2 = \partial/\partial x_3 = 0$ will be considered, and the non-zero in-plane quantities are u_α , ϕ_3 , $\phi_{3,\alpha}$, $\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$ and $m_{\alpha 3}$, respectively. The geometric, momentum balance and constitutive equations are therefore reduced to

$$\varepsilon_{\alpha\beta} = u_{\beta,\alpha} + e_{\beta\alpha}\phi, \quad k_\alpha = \phi_{,\alpha}, \quad (2.3a)$$

$$\sigma_{\beta\alpha,\beta} = \frac{\rho \partial^2 u_\alpha}{\partial t^2}, \quad m_{\alpha,\alpha} + e_{\alpha\beta}\sigma_{\alpha\beta} = \frac{J \partial^2 \phi}{\partial t^2} \quad (2.3b)$$

and

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\rho}\varepsilon_{\gamma\rho} + H_{\alpha\beta\rho}\kappa_\rho, \quad m_\rho = H_{\alpha\beta\rho}\varepsilon_{\alpha\beta} + D_{\rho\gamma}\kappa_\gamma, \quad (2.3c)$$

where $\phi_3 \equiv \phi$, $\kappa_{\alpha 3} \equiv \kappa_\alpha$, $m_{\alpha 3} \equiv m_\alpha$ are used for brevity, and $e_{\alpha\beta} \equiv e_{3\alpha\beta}$ can be considered as a bi-dimensional Levi-Civita tensor. Note that the elastic tensors \mathbf{H} and \mathbf{D} are reduced to the third and second order, respectively, in the considered planar micropolar elasticity. For the tensors \mathbf{C} and \mathbf{D} , the major symmetry requires that

$$C_{\alpha\beta\gamma\rho} = C_{\gamma\rho\alpha\beta} \quad \text{and} \quad D_{\alpha\beta} = D_{\beta\alpha}. \quad (2.4)$$

The strain energy reads

$$W = \frac{1}{2}C_{\alpha\beta\gamma\rho}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\rho} + \frac{1}{2}D_{\alpha\beta}\kappa_\alpha\kappa_\beta + \varepsilon_{\alpha\beta}H_{\alpha\beta\gamma}\kappa_\gamma. \quad (2.5)$$

Now we will determine the form of elastic tensors corresponding to the Z_2 or Z_4 invariance for a tetrachiral lattice by employing the irreducible orthogonal decomposition [35] of bi-dimensional general micropolar tensors. The main ingredients of the decomposition necessary to understand the derivation are summarized in appendix A. In general, starting from equation (A 2), and applying the major symmetry requirement equation (2.4), the bi-dimensional micropolar elastic tensors in equation (2.3c) can be decomposed as

$$C_{\alpha\beta\gamma\rho} = [a^1\delta_{\alpha\beta}\delta_{\gamma\rho} + a^2(e_{\alpha\beta}\delta_{\gamma\rho} + e_{\gamma\rho}\delta_{\alpha\beta}) + a^3e_{\alpha\beta}e_{\gamma\rho} + a^4L_{\beta\gamma\rho\alpha}] \\ + [(\delta_{\alpha\beta}D_{\gamma\rho}^1 + \delta_{\gamma\rho}D_{\alpha\beta}^1) + (e_{\alpha\beta}D_{\gamma\rho}^2 + e_{\gamma\rho}D_{\alpha\beta}^2)] + D_{\alpha\beta\gamma\rho}, \quad (2.6a)$$

$$H_{\alpha\beta\gamma} = (\delta_{\beta\gamma}v_\alpha^1 + e_{\beta\gamma}v_\alpha^2 + L_{\alpha\beta\gamma\rho}v_\rho^3) + D_{\alpha\beta\gamma} \quad (2.6b)$$

and
$$D_{\alpha\beta} = a^5\delta_{\alpha\beta} + D_{\alpha\beta}^3, \quad (2.6c)$$

where a^i , v_α^i , $D_{\alpha\beta}^i$, $D_{\alpha\beta\gamma}$ and $D_{\alpha\beta\gamma\rho}$ are the zeroth-, first-, second-, third- and fourth-order independent deviatoric tensors, respectively, and $L_{\alpha\beta\gamma\rho} = \delta_{\alpha\beta}\delta_{\gamma\rho} + \varepsilon_{\alpha\beta}\varepsilon_{\gamma\rho}$. Note that for the tensors of even order ($C_{\alpha\beta\gamma\rho}$ and $D_{\alpha\beta}$) the entries containing scalar variables constitute the in-plane hemitropic parts possessing the $SO(2)$ invariance, while the anisotropy is represented by the parts with deviatoric tensors of orders greater than zero. For the tensors of odd order, e.g. $H_{\alpha\beta\gamma}$, there is no hemitropic part, hence they have to vanish for a bi-dimensional hemitropic micropolar material. Note that this is only true for bi-dimensional tensors because the chiral-sensitive Levi-Civita tensor $e_{\alpha\beta}$ is of rank two, and no longer correct for tri-dimensional tensors. As a consequence, the elastic tensors for a bi-dimensional hemitropic micropolar medium simply read

$$\left. \begin{aligned} C_{\alpha\beta\gamma\rho}^{\text{chemi}} &= \lambda\delta_{\alpha\beta}\delta_{\gamma\rho} + A(e_{\alpha\beta}\delta_{\gamma\rho} + e_{\gamma\rho}\delta_{\alpha\beta}) + (\mu + \kappa)\delta_{\alpha\gamma}\delta_{\beta\rho} + (\mu - \kappa)\delta_{\alpha\rho}\delta_{\beta\gamma} \\ \text{and } D_{\alpha\beta}^{\text{chemi}} &= \gamma\delta_{\alpha\beta}, \end{aligned} \right\} \quad (2.7)$$

where the material constants λ , μ and κ ($a^1 = \lambda + \mu$, $a^4 = \mu$, $a^3 = \kappa$) are used in line with the traditional micropolar theory, and $a^2 = A$ is a chiral parameter reversing its sign according to the handedness of material. Equation (2.7) is exactly the same as that in Ref. [18] derived by a different method. For clarity, the hemitropic parts of the constitutive equation given by equation (2.7) are written in a matrix form as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & A & -A \\ \lambda & \lambda + 2\mu & A & -A \\ A & A & \mu + \kappa & \mu - \kappa \\ -A & -A & \mu - \kappa & \mu + \kappa \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{pmatrix} \quad \text{or } \boldsymbol{\sigma} = \mathbf{C}^{\text{chemi}} \boldsymbol{\varepsilon} \quad (2.8a)$$

and

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{or } \mathbf{m} = \mathbf{C}^{\text{chemi}} \boldsymbol{\kappa}. \quad (2.8b)$$

Now consider a medium with Z_4 invariance, i.e. the elastic tensors are invariant under $n\pi/2$ rotation of coordinate. An example of such material is a square tetrachiral lattice. By examining the general expression in equations (2.6) and (A 3)–(A 6), it is easy to verify that the item

$$C_{\alpha\beta\gamma\rho}^{Z_4} = D_{\alpha\beta\gamma\rho} \quad (2.9)$$

should be considered in addition to $C_{\alpha\beta\gamma\rho}^{\text{chemi}}$, while $H_{\alpha\beta\gamma} = 0$ still holds and no extra item will be applied to $D_{\alpha\beta}^{\text{chemi}}$. With the help of equation (A 3), a fourth-order deviatoric tensor is constructed through

$$D_{\alpha\beta\gamma\rho} = \alpha P_{\alpha\beta\gamma\rho} + B Q_{\alpha\beta\gamma\rho}, \quad (2.10)$$

where two independent material constants α and B are introduced additionally owing to the anisotropy of Z_4 invariance, and the fourth-order bi-dimensional deviatoric tensor basis $P_{\alpha\beta\gamma\rho}$

and $Q_{\alpha\beta\gamma\rho}$ are calculated from equation (A 4). Both α and B depend on the considered direction and vary with the coordinate rotation of θ by

$$\begin{pmatrix} \alpha' \\ B' \end{pmatrix} = \begin{pmatrix} \cos 4\theta & \sin 4\theta \\ -\sin 4\theta & \cos 4\theta \end{pmatrix} \begin{pmatrix} \alpha \\ B \end{pmatrix}. \quad (2.11)$$

Although B alternates sign under mirror reflection, it is relevant to the anisotropy but not chirality, since from equation (2.11), we can always find a direction angle $\theta = \arctan(B/\alpha)/4$ making $B' = 0$. Note that even B can be zero at some special directions for materials with Z_4 invariance, the number of independent parameters is two, because this special direction is not known *a priori*. If the chirality vanishes, i.e. the material possesses D_4 invariance, the privileged axes of mirror symmetry can be found, which, according to equation (A 7), leads to zero value of the parameter B in a reference frame aligned with the symmetric axes, thus the number of independent parameters can be reduced to one. In short, unlike the chiral parameter A , the parameters α and B characterize the anisotropy of the deviatoric part of a fourth-order bi-dimensional tensor whether they are zero or not is not able to indicate the chirality in a material. However, the relationship between these two parameters is indeed related to the chirality. In a matrix form, equation (2.10) can be written as

$$\mathbf{C}^{Z_4} = \begin{pmatrix} \alpha & -\alpha & B & B \\ -\alpha & \alpha & -B & -B \\ B & -B & -\alpha & -\alpha \\ B & -B & -\alpha & -\alpha \end{pmatrix}. \quad (2.12)$$

If a material possesses Z_2 invariance, i.e. the elastic tensors are invariant under $n\pi$ rotation of coordinate (taking a rectangular tetrachiral lattice as an example), the terms including the second-order deviatoric tensors in equation (2.6),

$$C_{\alpha\beta\gamma\rho}^{Z_2} = (\delta_{\alpha\beta} D_{\gamma\rho}^1 + \delta_{\gamma\rho} D_{\alpha\beta}^1) + (e_{\alpha\beta} D_{\gamma\rho}^2 + e_{\gamma\rho} D_{\alpha\beta}^2) \quad (2.13)$$

and

$$D_{\alpha\beta}^{Z_2} = D_{\alpha\beta}^3, \quad (2.14)$$

should be taken into account in addition to the hemitropic and Z_4 invariant parts. With the help of equation (A 3), the three second-order deviatoric tensors can be constructed by

$$\left. \begin{aligned} D_{\alpha\beta}^1 &= \frac{\beta_1 + \beta_2}{2} P_{\alpha\beta} + \frac{C_1 + C_2}{2} Q_{\alpha\beta}, \\ D_{\alpha\beta}^2 &= \frac{C_1 - C_2}{2} P_{\alpha\beta} + \frac{\beta_1 - \beta_2}{2} Q_{\alpha\beta} \\ D_{\alpha\beta}^3 &= \gamma_1 P_{\alpha\beta} + \gamma_2 Q_{\alpha\beta}, \end{aligned} \right\} \quad (2.15)$$

and

where six independent material constants β_1 , β_2 , C_1 , C_2 , γ_1 and γ_2 characterizing the anisotropy are additionally introduced owing to the presence of Z_2 invariance. If the chirality is not present and the material possesses D_2 invariance, C_1 , C_2 and γ_2 will be zero under the reference frame of the mirror symmetry, thus the number of independent constants reduces to three. These material constants of Z_2 invariance vary under the coordinate rotation as

$$\left. \begin{aligned} \begin{pmatrix} \beta_2' \\ C_1' \end{pmatrix} &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \beta_2 \\ C_1 \end{pmatrix}, \\ \begin{pmatrix} \beta_1' \\ C_2' \end{pmatrix} &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \beta_1 \\ C_2 \end{pmatrix}, \\ \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \end{aligned} \right\} \quad (2.16)$$

and

Table 1. Summary of material constants for bi-dimensional orthotropic chiral micropolar materials.

	ordinary	higher order
hemitropic	λ, μ, κ, A	γ
Z_4	α, B	
Z_2	$\beta_1, \beta_2, C_1, C_2$	γ_1, γ_2

In a matrix form, equations (2.13) and (2.14) can be written as

$$\mathbf{C}^{Z_2} = \begin{pmatrix} \beta_1 + \beta_2 & 0 & C_1 & C_2 \\ 0 & -(\beta_1 + \beta_2) & C_2 & C_1 \\ C_1 & C_2 & \beta_1 - \beta_2 & 0 \\ C_2 & C_1 & 0 & -(\beta_1 - \beta_2) \end{pmatrix} \quad \text{and} \quad \mathbf{D}^{Z_2} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}. \quad (2.17)$$

In summary, for a general bi-dimensional orthotropic chiral micropolar medium, the elastic tensors have the following form

$$\left. \begin{aligned} \mathbf{C} &= \mathbf{C}^{\text{hemi}} + \mathbf{C}^{Z_4} + \mathbf{C}^{Z_2}, \\ \mathbf{H} &= 0 \\ \mathbf{D} &= \mathbf{D}^{\text{hemi}} + \mathbf{D}^{Z_2}. \end{aligned} \right\} \quad (2.18)$$

and

where totally 13 independent material constants are introduced and listed in table 1. Note that if the chirality is absent we have $A = B = C_1 = C_2 = \gamma_2 = 0$ in the principal axes, then bi-dimensional achiral orthotropic micropolar elastic tensors are recovered and the number of constants reduces to eight. For simplicity, the elastodynamic equations for material of Z_4 invariance are listed below by using equations (2.3), (2.8) and (2.12)

$$\left. \begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + 2\mu + \alpha)u_{,xx} + 2(-A + B)u_{,xy} + (\mu + \kappa - \alpha)u_{,yy} + (A + B)v_{,xx} \\ &\quad + (\lambda + \mu - \kappa - 2\alpha)v_{,xy} - (A + B)v_{,yy} - 2A\phi_{,x} + 2\kappa\phi_{,y}, \\ \rho \frac{\partial^2 v}{\partial t^2} &= (\lambda + 2\mu + \alpha)v_{,yy} + 2(A - B)v_{,xy} + (\mu + \kappa - \alpha)v_{,xx} + (A + B)u_{,xx} \\ &\quad + (\lambda + \mu - \kappa - 2\alpha)u_{,xy} - (A + B)u_{,yy} - 2A\phi_{,y} - 2\kappa\phi_{,x} \\ \text{and} \quad J \frac{\partial^2 \phi}{\partial t^2} &= \gamma(\phi_{,xx} + \phi_{,yy}) - 4\kappa\phi - 2\kappa(v_{,x} - u_{,y}) + 2A(u_{,x} + v_{,y}). \end{aligned} \right\} \quad (2.19)$$

To gain some physical insights on the new material constants owing to anisotropy, it is helpful to decompose the second-order asymmetric strain tensor based on equation (A 2a)

$$\varepsilon_{\alpha\beta} = \bar{\varepsilon}\delta_{\alpha\beta} + \hat{\varepsilon}e_{\alpha\beta} + \varepsilon'P_{\alpha\beta} + \varepsilon''Q_{\alpha\beta}, \quad (2.20)$$

or more clearly in a matrix form

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \bar{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hat{\varepsilon} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \varepsilon' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \varepsilon'' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.21)$$

where $\bar{\varepsilon} = (\varepsilon_{11} + \varepsilon_{22})/2$ is the in-plane hydrostatic strain, and $\hat{\varepsilon} = (\varepsilon_{12} - \varepsilon_{21})/2 = \phi - e_{\alpha\beta}u_{\beta,\alpha}$ represents a pure rotation of a material particle, $\varepsilon' = (\varepsilon_{11} - \varepsilon_{22})/2$ and $\varepsilon'' = (\varepsilon_{12} + \varepsilon_{21})/2$ are the symmetric part of the strain. Applying equations (2.20) and (2.18) to equation (2.5), the density of

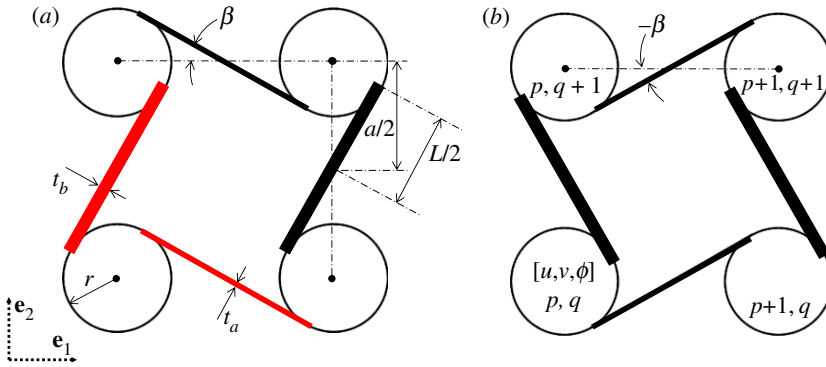


Figure 1. Geometry of a tetrachiral lattice, shown with (a) $\beta > 0$ and (b) its reversion $\beta < 0$. (Online version in colour.)

strain energy can be expressed as

$$W = 2(\lambda + \mu)\bar{\varepsilon}^2 + 4A\bar{\varepsilon}\hat{\varepsilon} + 2\kappa\hat{\varepsilon}^2 + 2(\mu + \alpha)\varepsilon'^2 + 2(\mu - \alpha)\varepsilon''^2 + 4B\varepsilon'\varepsilon'' + \frac{\gamma\phi_k\phi_k}{2}. \quad (2.22)$$

It is seen from equation (2.22) that the chiral parameter A is responsible for the coupling between bulk strain and pure rotation, while the parameter B is responsible for the coupling between the two orthogonal parts of deviatoric strains ($P_{\alpha\beta}$ and $Q_{\alpha\beta}$), respectively. It is also straightforward from equation (2.22) to give the thermodynamic constraint on the effective material constants

$$\mu^2 - \alpha^2 - B^2 \geq 0, \quad (2.23)$$

while the other conditions are consistent with those given in Ref. [18]. For a material with Z_2 invariance, a similar procedure can also be employed but the relations are more lengthy and not presented here.

3. Application to tetrachiral lattices

In this section, we will apply the proposed model to a tetrachiral lattice and derive analytically the new material constants proposed in §2 by a homogenization method.

(a) Geometry of a tetrachiral lattice

The geometry of a tetrachiral lattice is shown in figure 1. It consists of circles of radius r linked by straight ligaments of equal length L , and the ligaments are required to be tangential to the circles. The circles are arranged into a square lattice with a lattice constant a , a unit cell with the area $A_{\text{cell}} = a^2$ is highlighted by the dashed lines in figure 1. The angle between the line connecting the circle centres and the ligament is defined as β . The geometry parameters are connected by the following relations:

$$\cos \beta = \frac{L}{a} \quad \text{and} \quad \sin \beta = \frac{2r}{a}. \quad (3.1)$$

The topology parameter $\beta \in [-\pi/2, \pi/2]$ plays an important role on the layout and the mechanical behaviour of the lattice [11,16]. When $\beta \rightarrow 0$, the circles shrink to dots and a traditional square lattice can be obtained. When $\beta \rightarrow \pm\pi/2$, the ligaments vanish and a lattice of packed circles will be recovered. It should be noted that chirality disappears in these two extreme cases. As shown in figure 1, we adopt a sign convention of β according to relative orientation between the ligament and the link of circle centres. When β reverses its sign, the handedness of the lattice is reversed.

To make the lattice as a general bi-dimensional orthotropic material and at the same time to keep the formulation as simple as possible, the thicknesses of the ligaments along the horizontal and vertical directions, say t_a and t_b , respectively, are assumed to be different instead of using

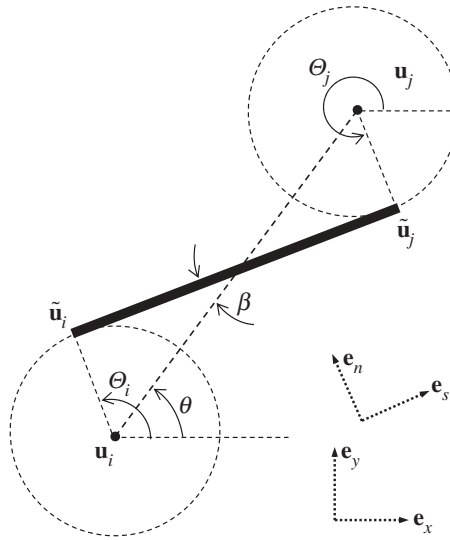


Figure 2. Geometry and symbol definitions for the analysis of a single ligament.

a more general rectangular lattice. Young's modulus of the ligament material is denoted in the following by E_s .

(b) Determination of effective material constants

To formulate the problem in an analytical manner, the circles are assumed to be rigid with mass and rotational inertia being m and \bar{J} , respectively. Furthermore, the ligament is assumed to be massless.

Let vector $\mathbf{u}_i = \{u_i \ v_i \ \phi_i\}^T$ denote the displacement and rotation DOFs at the centre of the rigid circle i . Motion of the rigid circle is constrained by the ends of deformable ligaments on its circumference $\tilde{\mathbf{u}}_i = \{\tilde{u}_i \ \tilde{v}_i \ \tilde{\phi}_i\}^T$, which is related to \mathbf{u}_i by $\tilde{\mathbf{u}}_i = \mathbf{T}(\Theta_i)\mathbf{u}_i$. The transformation matrix is defined by

$$\mathbf{T}(\Theta_i) = \begin{pmatrix} 1 & 0 & -r \sin \Theta_i \\ 0 & 1 & r \cos \Theta_i \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.2)$$

where Θ_i is the azimuthal angle of the beam end on the circle i (figure 2). Giving the topological parameter β and the direction angle θ of the circles i and j linked by the beam, the relations $\Theta_i = \pi/2 + \theta - \beta$ and $\Theta_j = 3\pi/2 + \theta - \beta$ are implied. DOFs of the beam ends $\tilde{\mathbf{u}}'$ in the local system ($\mathbf{e}_s - \mathbf{e}_n$ in figure 2) are then linked to that $\mathbf{u} = \{\mathbf{u}_i \ \mathbf{u}_j\}^T$ at the circle centres as

$$\tilde{\mathbf{u}}' = \mathbf{R}(\theta)\mathbf{T}(\Theta_i, \Theta_j)\mathbf{u}, \quad (3.3)$$

where

$$\mathbf{T}(\Theta_i, \Theta_j) = \begin{pmatrix} \mathbf{T}(\Theta_i) & 0 \\ 0 & \mathbf{T}(\Theta_j) \end{pmatrix}, \quad (3.4)$$

$$\mathbf{R}(\theta) = \begin{pmatrix} \mathbf{R}_{3 \times 3}(\theta) & 0 \\ 0 & \mathbf{R}_{3 \times 3}(\theta) \end{pmatrix} \quad (3.5)$$

and

$$\mathbf{R}_{3 \times 3}(\theta) = \begin{pmatrix} \cos(\theta - \beta) & \sin(\theta - \beta) & 0 \\ -\sin(\theta - \beta) & \cos(\theta - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.6)$$

With the help of Euler–Bernoulli beam theory, the stiffness of the ligament in its local system is expressed as

$$\mathbf{K}' = \frac{E_s t}{6L} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ & 6t^2/L^2 & 3t^2/L & 0 & -6t^2/L^2 & 3t^2/L \\ & & 2t^2 & 0 & -3t^2/L & t^2 \\ & & & 1 & 0 & 0 \\ \text{sym} & & & & 6t^2/L^2 & -3t^2/L \\ & & & & & 2t^2 \end{pmatrix}, \quad (3.7)$$

where $t = t_a$ or t_b . In the global system ($\mathbf{e}_x - \mathbf{e}_y$ in figure 2), the stiffness matrix relating the general displacement and force vectors to the centres of the circles can be obtained as

$$\mathbf{K} = \mathbf{T}^T(\Theta_i, \Theta_j) \mathbf{R}^T(\theta) \mathbf{K}' \mathbf{R}(\theta) \mathbf{T}(\Theta_i, \Theta_j). \quad (3.8)$$

Referring to figure 1a, the strain energies of the two ligaments in a unit cell are written as

$$W_1 = \frac{1}{2} \left\{ \mathbf{u}_{p,q} \quad \mathbf{u}_{p+1,q} \right\}^T \mathbf{K}|_{\theta=0, t=t_a} \left\{ \mathbf{u}_{p,q} \quad \mathbf{u}_{p+1,q} \right\} \quad (3.9a)$$

and

$$W_2 = \frac{1}{2} \left\{ \mathbf{u}_{p,q} \quad \mathbf{u}_{p,q+1} \right\}^T \mathbf{K}|_{\theta=\pi/2, t=t_b} \left\{ \mathbf{u}_{p,q} \quad \mathbf{u}_{p,q+1} \right\}, \quad (3.9b)$$

respectively, and therefore the density of deformation energy of the lattice reads

$$w = \frac{W_1 + W_2}{A_{\text{cell}}}. \quad (3.10)$$

In order to express the energy density in terms of the macroscopic strain and curvature, the following expansions are made based on Taylor's expansion

$$\left. \begin{aligned} u_{p+1,q} &= u_{p,q} + \varepsilon_{11} \Delta x, & u_{p,q+1} &= u_{p,q} + (\varepsilon_{21} - \phi) \Delta y, \\ v_{p+1,q} &= v_{p,q} + (\varepsilon_{12} + \phi) \Delta x, & v_{p,q+1} &= v_{p,q} + \varepsilon_{22} \Delta y \\ \phi_{p+1,q} &= \phi_{p,q} + k_1 \Delta x, & \phi_{p,q+1} &= \phi_{p,q} + k_2 \Delta y, \end{aligned} \right\} \quad (3.11)$$

and

where $\Delta x = \Delta y = a$, and the terms of the order $O(\Delta x^2, \Delta y^2)$ are neglected. The micropolar elastic tensors are then obtained by

$$C_{\alpha\beta\gamma\rho} = \frac{\partial^2 w}{\partial \varepsilon_{\alpha\beta} \partial \varepsilon_{\gamma\rho}} \quad \text{and} \quad D_{\alpha\beta} = \frac{\partial^2 w}{\partial k_{\alpha} \partial k_{\beta}}. \quad (3.12)$$

Finally, the homogenized material constants of the tetrachiral lattice, corresponding to different parts of symmetry groups in equation (2.18), are obtained in the principal direction as

$$\text{Hemitropic : } \begin{cases} \lambda = \frac{E_s}{16} (\eta_1 + \eta_2) [1 - 2(\eta_1^2 - \eta_1 \eta_2 + \eta_2^2) + \cos 2\beta] \cos 2\beta \sec^3 \beta, \\ \mu = \frac{E_s}{16} (\eta_1 + \eta_2) [1 + 2(\eta_1^2 - \eta_1 \eta_2 + \eta_2^2) + \cos 2\beta] \sec^3 \beta, \\ \kappa = \frac{E_s}{8} (\eta_1 + \eta_2) [1 + 2(\eta_1^2 - \eta_1 \eta_2 + \eta_2^2) - \cos 2\beta] \sec \beta, \\ A = -\frac{E_s}{8} (\eta_1 + \eta_2) [1 - 2(\eta_1^2 - \eta_1 \eta_2 + \eta_2^2) + \cos 2\beta] \sec \beta \tan \beta, \\ \gamma = \frac{E_s a^2}{48} (\eta_1 + \eta_2) [3 + 8(\eta_1^2 - \eta_1 \eta_2 + \eta_2^2) - 3 \cos 2\beta]. \end{cases} \quad (3.13a)$$

$$Z_4: \begin{cases} \alpha = \lambda, \\ B = A, \end{cases} \quad (3.13b)$$

$$Z_2: \begin{cases} \beta_1 = \frac{E_s}{8}(\eta_1 - \eta_2)\eta[1 + 2(\eta_1^2 + \eta_1\eta_2 + \eta_2^2) + \cos 2\beta] \sec^3 \beta, \\ \beta_2 = \frac{E_s}{8}(\eta_1 - \eta_2)\eta[1 - 2(\eta_1^2 + \eta_1\eta_2 + \eta_2^2) + \cos 2\beta] \cos 2\beta \sec^3 \beta, \\ C_1 = \frac{E_s}{4}(\eta_2 - \eta_1)\eta[1 - 2(\eta_1^2 + \eta_1\eta_2 + \eta_2^2) + \cos 2\beta] \sec \beta \tan \beta, \\ C_2 = 0, \\ \gamma_1 = \frac{E_s a^2}{48}(\eta_1 - \eta_2)[3 + 8(\eta_1^2 + \eta_1\eta_2 + \eta_2^2) - 3 \cos 2\beta], \\ \gamma_2 = 0. \end{cases} \quad (3.13c)$$

where two dimensionless parameters

$$\eta_1 = \frac{t_a}{a} \quad \text{and} \quad \eta_2 = \frac{t_b}{a} \quad (3.14)$$

are defined. It can be verified that the chiral parameter A as well as the non-zero anisotropic parameters B and C_1 are odd functions of β , which implies the fact that, in the principal coordinates, reverse of the handedness directly leads to the mirror reflection of the material with respect to a certain axis, as illustrated in figure 1. However, in the non-principal coordinates, the hemitropic parameters are fixed while the Z_4 and Z_2 ones transform according to equations (2.11) and (2.16), thus the mirror reflection cannot be achieved by changing the sign of β . This will be clarified in §4. In the case of $\eta_1 = \eta_2$, all the constants of the Z_2 invariance vanish accordingly and the results of a square tetrachiral lattice will be recovered. Finally, when $\beta = 0$, the lattice reduces to an achiral rectangular lattice with the same results found in the literature [3,7].

4. Numerical examples and discussions

Numerical examples are provided in this section to examine the properties of the tetrachiral lattice with the proposed model. Only the case of Z_4 invariance ($\eta_1 = \eta_2 = \eta$) is considered here.

(a) Properties of effective micropolar constants

From equation (3.13), it is found that although more material constants are introduced owing to the anisotropy, the dependence of the effective constants on the microstructure in the principal system as well as corresponding discussions are similar to those of the trichiral lattice [18]. In figure 3, the hemitropic and Z_4 invariant elastic constants, i.e. $\lambda = \alpha$, $A = B$, μ and κ normalized by E_s , are plotted versus β , where two ligament slenderness ratios $\eta = 1/20$ and $\eta = 1/50$ are used for comparison. It is seen that A and B are odd functions of β and increase with the increase in $|\beta|$ in the region of roughly $|\beta| < 75^\circ$. Lattice with thicker ligaments is observed to give greater A , B , μ and κ , while for λ this dependence is not monotonic.

Note that in a coordinate system other than the principal one, equation (2.11) should be applied to equation (3.13a) to evaluate α' and B' in the new coordinate system. However, as B alternates its sign according to β while α does not, α' and B' have no longer a simple sign dependence on the topological parameter β . This seems contrary to the conclusion in §2, stating that if the material is mirror reflected with respect to the coordinate axis, B' should reverse its sign while α' remains unchanged, no matter which coordinate system is used. This issue is clarified in figure 4. In a coordinate system different from the principal one by a rotation θ , consider that the material pattern is mirror reflected with respect to y' -axis. However, as shown in figure 4a, the sign change of β would not induce the mirror image of the material in this case. The correct operation to circumvent this problem is illustrated in figure 4b, a compensatory negative θ should be applied

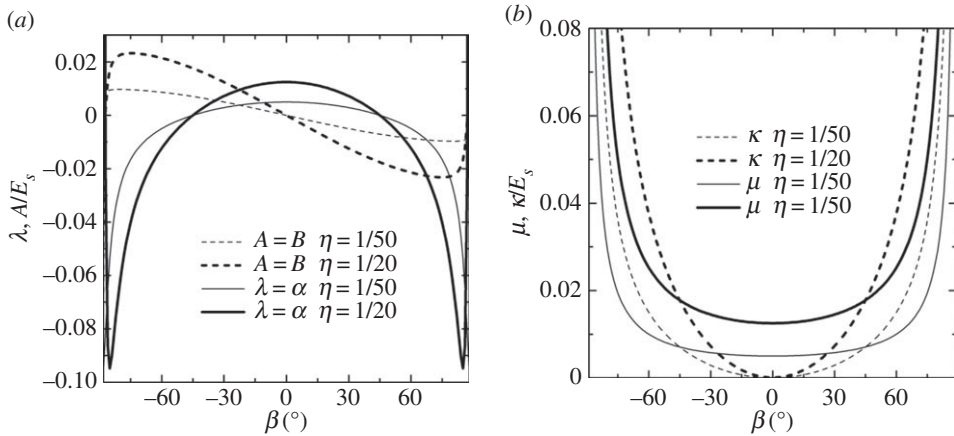


Figure 3. Variation of effective material constants (a) $\lambda = \alpha, A = B$ and (b) μ, κ in principal system as function of β .

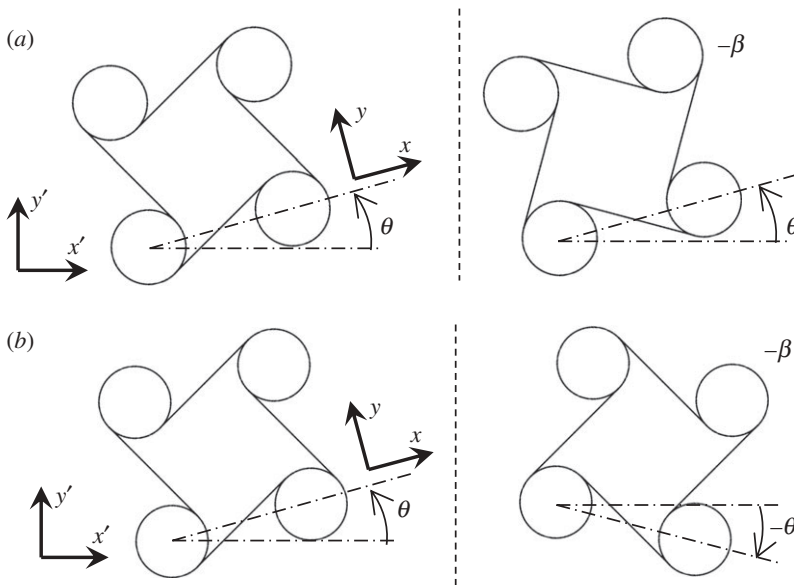


Figure 4. Schematic of the transformation of material constants under inversion of non-principal coordinate system: (a) sign reversion of β and (b) sign reversion of β plus a compensatory rotation.

before the sign change of β , that is

$$\begin{pmatrix} \alpha'' \\ B'' \end{pmatrix} = \begin{pmatrix} \cos 4\theta & -\sin 4\theta \\ \sin 4\theta & \cos 4\theta \end{pmatrix} \begin{pmatrix} \alpha(-\beta) \\ B(-\beta) \end{pmatrix},$$

which, together with equations (2.11) and (3.13), maintains the correct sign rule under inversion of an arbitrary coordinate system.

An interesting property of a bi-dimensional chiral lattice is the negative Poisson's ratio. Young's modulus and Poisson's ratio should be redefined in the framework of the new constitutive equations and they are for a material with Z_4 invariance

$$E = \frac{4(B^2 + \alpha^2 - \mu^2)(-A^2 + \kappa(\lambda + \mu))}{A^2(-\alpha + \mu) + \kappa(B^2 + (\alpha - \mu)(\alpha + \lambda + 2\mu))} \quad (4.1a)$$

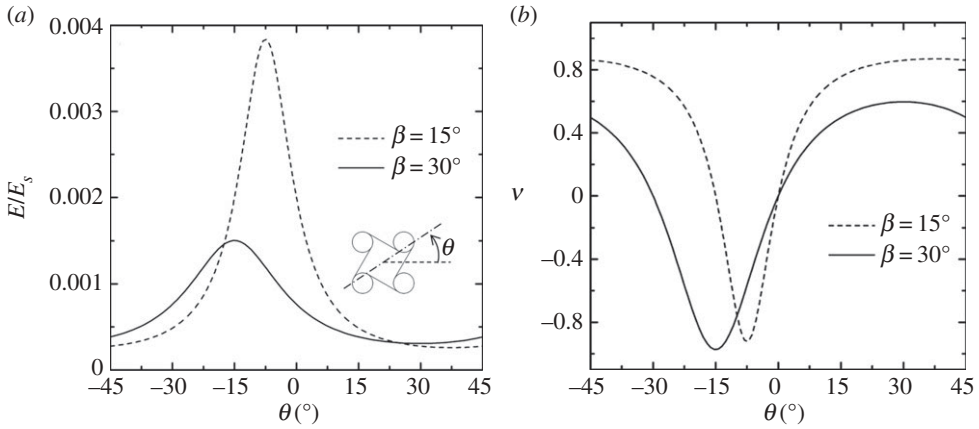


Figure 5. Variation of effective (a) Young's modulus and (b) Poisson's ratio along different directions.

and

$$\nu = -\frac{\kappa(B^2 + (\alpha - \lambda)(\alpha - \mu)) + A^2(\alpha - \mu)}{A^2(-\alpha + \mu) + \kappa(B^2 + (\alpha - \mu)(\alpha + \lambda + 2\mu))}. \quad (4.1b)$$

Substituting equation (3.13) into equation (4.1) yields $E = (E_s \eta^3 \sec^3 \beta) / (\sin^2 \beta + \eta^2)$ and $\nu = 0$ in the principal direction. Note that Poisson's ratio in the principal direction vanishes independent of β and this is different from that of trichiral lattices which gives $\nu = -1$ in any direction. Actually, E and ν vary in accordance with equation (2.11) in other directions. The variations of E and ν as the function of the direction angle $\theta \in [-45^\circ, 45^\circ]$ are illustrated in figure 5a, b, respectively, where $\eta_1 = \eta_2 = 1/20$ and two topological parameters $\beta = 15^\circ$ and $\beta = 30^\circ$ are examined. It is found that there are extreme values for both E and ν at the directions of $\theta = 7.5^\circ$ for $\beta = 15^\circ$ and $\theta = 15^\circ$ for $\beta = 30^\circ$, respectively. In fact, this special direction of $\theta = \beta/2$ can be strictly anticipated from equations (4.1), (3.13) and (2.11). Along this direction, Young's modulus of the lattice is much larger than that in other directions, and Poisson's ratio approaches -1 . Larger β induces a weaker peak of Young's modulus but a wider range of negative Poisson's ratio. However, in contrast with a trichiral lattice, Poisson's ratio of a tetrachiral lattice is positive along most directions.

(b) Plane wave propagation

Consider an infinite planar orthotropic chiral micropolar medium under a plane wave in the principal (+x) direction, the displacement and micro-rotation are assumed to be of the following form:

$$(u, v, \phi) = (\hat{u}, \hat{v}, \hat{\phi}) \exp(iqx - i\omega t), \quad (4.2)$$

where q and ω denote the wavenumber and circular frequency, respectively, $(\hat{u}, \hat{v}, \hat{\phi})$ are the (complex) amplitudes and $i = \sqrt{-1}$. Substituting equation (4.2) into equation (2.9) yields the secular equation

$$\begin{pmatrix} (\lambda + 2\mu + \alpha)q^2 - \rho\omega^2 & (A + B)q^2 & 2iAq \\ (A + B)q^2 & (\mu + \kappa - \alpha)q^2 - \rho\omega^2 & 2i\kappa q \\ -2iAq & -2i\kappa q & \gamma q^2 + 4\kappa - J\omega^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\phi} \end{pmatrix} = 0, \quad (4.3)$$

where the effective density $\rho = m/A_{\text{cell}}$ and the micro-rotational inertia $J = \bar{J}/A_{\text{cell}}$ is assumed. With the help of equation (2.11), the wave propagation along a predefined direction different from the principal one by an angle θ (see the inset in figure 6a) can be readily obtained by

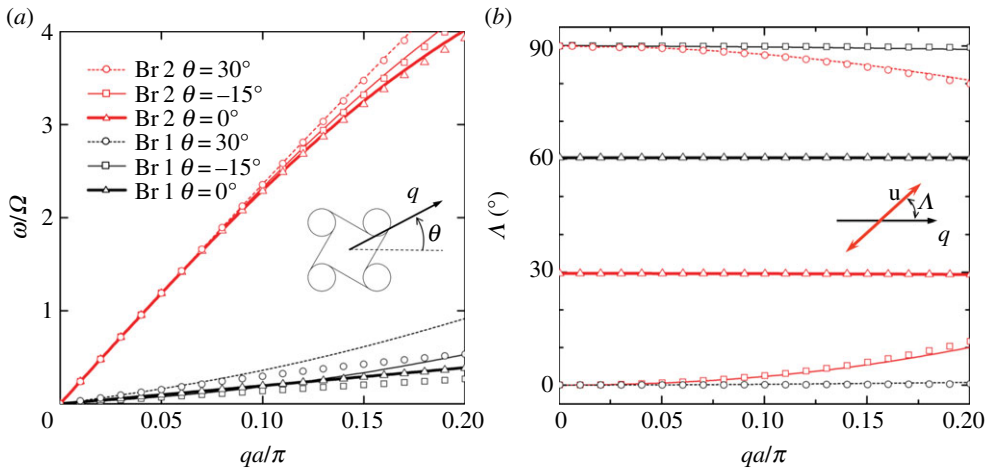


Figure 6. (a) Dispersion curves and (b) corresponding polarization angles of the first two wave branches of plane wave propagation along the principal, -15° and 30° directions, calculated from homogenized (lines) and discrete (dots) model. (Online version in colour.)

replacing α and B with α' and B' . In the following, equation (4.3) is numerically solved to obtain the dispersion relation and wave mode information. Tetrachiral lattices of the parameters $a = 1$, $\eta_1 = \eta_2 = 1/20$, $m = \bar{J} = 1$ with various values of β are examined in numerical examples. The exact solutions using the corresponding discrete lattice model are also calculated for comparison.

Figure 6a shows the dispersion curves of the first two branches (in black and red, respectively) of the lattice with the topological parameter $\beta = 30^\circ$, whose wave modes are displacement dominated. The third-rotation-dominated branch is not shown here, as it is similar to that of a traditional micropolar medium. The wave propagating along the $\theta = 0^\circ$ (principal), $\theta = -15^\circ$ and $\theta = 30^\circ$ directions are calculated and shown in the figure with different line types, respectively. The wave frequency is normalized by $\Omega = 2\sqrt{(E_s t^3)/(ma^3)}$. The results obtained from the homogenized model agree well with those of the discrete lattice model (shown by dots) in the long wave approximation. Similar to the case of trichiral lattices, all branches are coupled together and dispersive, and there are in general no longer pure longitudinal (P) and transverse (S) waves. However, there is a large difference between the wave speeds of the first two branches. This is different from that of trichiral lattices, where dispersion curves of the first two branches are almost superposed at the long wave limit.

As the wave is always linearly polarized, a polarization angle Λ measured with respect to the wave propagating direction, $\tan \Lambda = |\hat{v}/\hat{u}|$, is defined to characterize the wave mode shape. In figure 6b, the polarization angles corresponding to each dispersive branch in figure 6a are plotted as functions of the wavenumber, where the same line types are used for a one-to-one matching. It is seen that the polarization angle depends on the wavenumber (as well as the frequency and wave speed). When the wave propagates in the principal direction, S-dominated ($\Lambda \sim 60^\circ$) and P-dominated ($\Lambda \sim 30^\circ$) waves are observed for the first and second branches, respectively. In the direction of $\theta = -15^\circ$, almost pure S- and P-waves exist, while in the direction of $\theta = 30^\circ$, the two branches are exchanged, i.e. the wave with a lower phase speed is P-dominated and the wave with a higher phase speed is S-dominated. This feature is not found in a traditional achiral medium.

To give a full view of wave directionality, we plot in figure 7 the polarization angle of the first branch as the function of the wave propagation direction θ , where a certain wavenumber $q = 0.1$ is fixed. In the figure, three lattice topologies ($\beta = 0$, $\beta = 5^\circ$ and $\beta = 30^\circ$) are compared. For the achiral case ($\beta = 0$), the polarization also deviates from a pure S-wave owing to the anisotropy, however, there is no P-S exchange when the propagation direction varies. Note that a pure S-wave is recovered along $\theta = \pm 45^\circ$ because the lattice is symmetric in this direction for

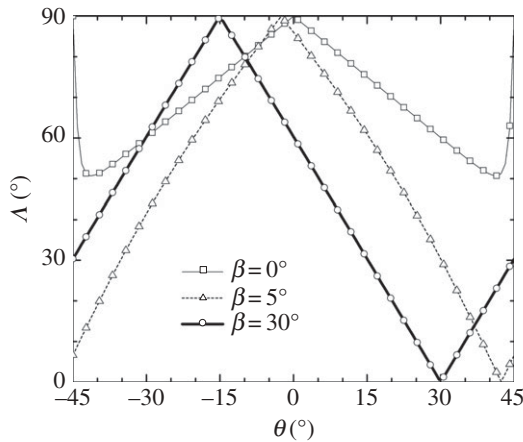


Figure 7. Variation of polarization angle of the first wave branch against the propagation direction.

$\beta = 0$. If the chirality is present, as shown in the figure, the variation of the polarization angle covers the full 0° – 90° range, implying the existence of P–S-wave exchange. Moreover, it is found that the variation of the polarization angle with respect to the propagation direction is almost linear. Again, the results based on the homogenized model agree well with those of the accurate discrete lattice model. Interestingly, it is also found from the figure that the wave is of pure S-type just along the direction $\theta = -\beta/2$, and of pure P-type along the direction $\theta = -\beta/2 + 45^\circ$. Further numerical examinations show that this relation holds for all β values.

5. Conclusion

We propose a micropolar constitutive model for bi-dimensional orthotropic chiral solids. The form of micropolar elastic tensor is derived based on the theory of group representation and irreducible decomposition of tensors. For a general bi-dimensional orthotropic chiral material, totally 13 independent material constants are necessary to represent the normal and higher order elastic tensors, which possess a hierarchy structure, i.e. can be grouped into a hemitropic part, Z_4 invariant part and Z_2 invariant part, respectively. The developed theory is then assessed by a homogenization procedure for a tetrachiral lattice material, and the effective material constants are derived analytically. Compared to trichiral lattices, the auxetic behaviour of square lattices displays strong directionality, i.e. its Poisson's ratio approaches to -1 only at a special orientation of loading. For the plane wave propagation, continuous transition of the P- and S-wave modes for the first two branches is observed depending on the wave propagating direction. The proposed theory provides a valuable tool for the modelling and analysis of structures made of such orthotropic chiral materials.

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Appendix A. Orthogonal irreducible decomposition

Some main ingredients of the orthogonal irreducible decomposition of tensors critical to understand this paper are summarized here, the details can be found in [35].

In general, a generic tensor of order n can be decomposed into a sum of deviatoric tensors of orders not higher than n and such a decomposition is irreducible and orthogonal. A tensor $\mathcal{D}_{i_1 i_2 \dots i_n}$

is *deviatoric* if it is completely symmetric and traceless, i.e. if it satisfies

$$\mathcal{D}_{i_1 i_2 i_3 \dots i_n} = \mathcal{D}_{i_2 i_1 i_3 \dots i_n} = \mathcal{D}_{i_3 i_2 i_1 \dots i_n} = \dots = \mathcal{D}_{i_n i_2 i_3 \dots i_1}, \quad \mathcal{D}_{kk i_3 \dots i_n} = 0. \quad (\text{A } 1)$$

By definition, scalars and vectors can be considered as the zeroth- and first-order deviatoric tensors. In comparison with the generic one, a deviatoric tensor is easier to understand and use. It is proved that the n th-order deviatoric tensor has always two independent variables in the bi-dimensional case and $2n + 1$ independent variables in the tri-dimensional case. For the second-, third- and fourth-order bi-dimensional tensors involved in this paper, the corresponding decompositions are

$$T_{\alpha\beta} = a^1 \delta_{\alpha\beta} + a^2 e_{\alpha\beta} + \mathcal{D}_{\alpha\beta}, \quad (\text{A } 2a)$$

$$T_{\alpha\beta\gamma} = (\delta_{\beta\gamma} v'_\alpha + e_{\beta\gamma} v'_\alpha + L_{\alpha\beta\gamma\rho} v'_\rho) + \mathcal{D}_{\alpha\beta\gamma} \quad (\text{A } 2b)$$

and

$$T_{\alpha\beta\gamma\rho} = [a^1 \delta_{\alpha\beta} \delta_{\gamma\rho} + a^2 \varepsilon_{\alpha\beta} \delta_{\gamma\rho} + a^3 \varepsilon_{\gamma\rho} \delta_{\alpha\beta} + a^4 \varepsilon_{\alpha\beta} \varepsilon_{\gamma\rho} + L_{\beta\gamma\rho\zeta} (a^5 \delta_{\zeta\alpha} + a^6 \varepsilon_{\zeta\alpha}) + (\delta_{\gamma\rho} \mathcal{D}_{\alpha\beta}^1 + \varepsilon_{\gamma\rho} \mathcal{D}_{\alpha\beta}^2 + L_{\beta\gamma\rho\zeta} \mathcal{D}_{\zeta\alpha}^3 + L_{\beta\gamma\rho\zeta} \mathcal{D}_{\zeta\alpha}^4) + \mathcal{D}_{\alpha\beta\gamma\rho}, \quad (\text{A } 2c)$$

respectively, where $L_{\alpha\beta\gamma\rho} = \delta_{\alpha\beta} \delta_{\gamma\rho} + \varepsilon_{\alpha\beta} \varepsilon_{\gamma\rho}$, a' and v'_α are independent scalar and vector variables, $\mathcal{D}_{\alpha\beta}$, $\mathcal{D}_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta\gamma\rho}$ are the second-, third- and fourth-order deviatoric tensors, respectively, which all have two independent variables.

A bi-dimensional generic n th-order deviatoric tensor can be expressed by

$$\mathcal{D}^{(n)} = c_1 \mathbf{P}^{(n)} + c_2 \mathbf{Q}^{(n)}, \quad (\text{A } 3)$$

where c_1 and c_2 are the two independent parameters, and $(\mathbf{P}^{(n)}, \mathbf{Q}^{(n)})$ are the bases of $\mathcal{D}^{(n)}$, which can be constructed as the following. Denoting $(\mathbf{e}_1, \mathbf{e}_2)$ as the orthonormal basis of the bi-dimensional Euclidean space and defining complex vector $\boldsymbol{\omega} = \mathbf{e}_1 + i\mathbf{e}_2$, the two bases of an n th-order deviatoric tensor can be defined as

$$\mathbf{P}^{(n)} = \text{Re}[\boldsymbol{\omega}^{\otimes n}] \quad \text{and} \quad \mathbf{Q}^{(n)} = \text{Im}[\boldsymbol{\omega}^{\otimes n}], \quad (\text{A } 4)$$

where

$$\boldsymbol{\omega}^{\otimes n} = \underbrace{\boldsymbol{\omega} \boldsymbol{\omega} \dots \boldsymbol{\omega}}_n \quad (\text{A } 5)$$

represents tensor multiplication of n times. Now consider a new coordinate frame, which is rotated by θ , characterized by

$$\mathbf{e}'_1 = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$$

and

$$\mathbf{e}'_2 = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2.$$

From equations (A 4) and (A 5), we have

$$\mathbf{P}^{(n)} + i\mathbf{Q}^{(n)} = (\mathbf{e}_1 + i\mathbf{e}_2)^{\otimes n} = e^{in\theta} (\mathbf{e}'_1 + i\mathbf{e}'_2)^{\otimes n} = e^{in\theta} (\mathbf{P}'^{(n)} + i\mathbf{Q}'^{(n)}),$$

which turns out that

$$\left. \begin{aligned} \mathbf{P}'^{(n)} &= \cos n\theta \mathbf{P}^{(n)} + \sin n\theta \mathbf{Q}^{(n)} \\ \mathbf{Q}'^{(n)} &= -\sin n\theta \mathbf{P}^{(n)} + \cos n\theta \mathbf{Q}^{(n)}. \end{aligned} \right\} \quad (\text{A } 6)$$

It is clear that $\mathbf{P}^{(n)}$ and $\mathbf{Q}^{(n)}$ are invariant under the coordinate rotation of $\theta = 2\pi/n$, and so does $\mathcal{D}^{(n)}$. It is helpful to check out the appropriate form of material tensor possessing a certain rotational symmetry. If the coordinate frame is mirror reflected, e.g. $(\mathbf{e}'_1, \mathbf{e}'_2) = (-\mathbf{e}_1, \mathbf{e}_2)$, similarly we have

$$\left. \begin{aligned} \mathbf{P}'^{(n)} &= (-1)^n \mathbf{P}^{(n)} \\ \mathbf{Q}'^{(n)} &= (-1)^{n-1} \mathbf{Q}^{(n)}, \end{aligned} \right\} \quad (\text{A } 7)$$

which can be used to get rid of some material parameters if there exist axes of mirror symmetry.

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