Conceptual dynamical models for turbulence

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Understanding the complexity of anisotropic turbulent processes in engineering and environmental fluid flows is a formidable challenge with practical significance because energy often flows intermittently from the smaller scales to impact the largest scales in these flows. Conceptual dynamical models for anisotropic turbulence are introduced and developed here which, despite their simplicity, capture key features of vastly more complicated turbulent systems. These conceptual models involve a large-scale mean flow and turbulent fluctuations on a variety of spatial scales with energy-conserving wave–mean-flow interactions as well as stochastic forcing of the fluctuations. Numerical experiments with a six-dimensional conceptual dynamical model confirm that these models capture key statistical features of vastly more complex anisotropic turbulent systems in a qualitative fashion. These features include chaotic statistical behavior of the mean flow with a sub-Gaussian probability distribution function (pdf) for its fluctuations whereas the turbulent fluctuations have decreasing energy and correlation times at smaller scales, with nearly Gaussian pdfs for the large-scale fluctuations and fat-tailed non-Gaussian pdfs for the smaller-scale fluctuations. This last feature is a manifestation of intermittency of the small-scale fluctuations where turbulent modes with small variance have relatively frequent extreme events which directly impact the mean flow. The dynamical models introduced here potentially provide a useful test bed for algorithms for prediction, uncertainty quantification, and data assimilation for anisotropic turbulent systems.

wave–mean interaction | stochastic model

Understanding the complexity of anisotropic turbulence processes over a wide range of spatiotemporal scales in engineering shear turbulence (1–3) as well as climate atmosphere ocean science (4–6) is a grand challenge of contemporary science. This is especially important from a practical viewpoint because energy often flows intermittently from the smaller scales to affect the largest scales in such anisotropic turbulent flows. The typical features of such anisotropic turbulent flows are the following (2–4):

(A) The large-scale mean flow is usually chaotic but more predictable than the smaller-scale fluctuations. The overall single point probability distribution function (pdf) of the flow field is nearly Gaussian whereas the mean flow pdf is sub-Gaussian, in other words, with less extreme variability than a Gaussian random variable.

(B) There are nontrivial nonlinear interactions between the large-scale mean flow and the smaller-scale fluctuations which conserve energy.

(C) There is a wide range of spatial scales for the fluctuations with features where the large-scale components of the fluctuations contain more energy than the smaller-scale components. Furthermore, these large-scale fluctuating components decorrelate faster in time than the mean-flow fluctuations on the largest scales, whereas the smaller-scale fluctuating components decorrelate faster in time than the larger-scale fluctuating components.

(D) The pdfs of the larger-scale fluctuating components of the turbulent field are nearly Gaussian, whereas the smaller-scale fluctuating components are intermittent and have fat-tailed pdfs, in other words, a much higher probability of extreme events than a Gaussian distribution (see figures 8.4 and 8.5 from ref. 3 for such experimental features in a turbulent jet).

The goal here is to develop the simplest conceptual dynamical model for anisotropic turbulence that captures all of the features in (A)–(D) in a transparent qualitative fashion. In contrast with deterministic models of turbulence which are derived by Galerkin truncation of the Navier–Stokes equation (7) and do not display all of the features in (A)–(D), the conceptual models developed here are low-dimensional stochastic dynamical systems; the nonlinear interactions between the large-scale mean-flow component and the smaller-scale fluctuating components are completely deterministic but the potential direct nonlinear interactions between the smaller-scale fluctuating components are modeled stochastically by damping and stochastic forcing (6, 8). The conceptual models developed here are not derived quantitatively from the Navier–Stokes equations but are developed to capture the key features in anisotropic turbulent flows listed in (A)–(D) by mimicking key physical processes. Besides aiding the understanding of anisotropic turbulent flows, such conceptual models are useful for designing and testing numerical algorithms for prediction and data assimilation in such complex turbulent systems.

Conceptual Model

The model has a mean scalar variable $\bar{u}$ representing the largest scales and a family of small-scale variables $\vec{u}' = (u'_1, u'_2, \ldots, u'_K)^T \in \mathbb{R}^K$ so that there are $\mathbb{R}^{K+1}$ variables in the system $\vec{u} = \begin{pmatrix} \bar{u} \\ \vec{u}' \end{pmatrix}$. The variables $u'_k, 1 \le k \le K$ represent contributions

Significance

Understanding the complexity of anisotropic turbulent processes in engineering and environmental fluid flows is a formidable challenge with practical significance because energy often flows intermittently from the smaller scales to impact the largest scales in these flows. These complex features strongly impact practical prediction, uncertainty quantification, and data assimilation strategies in such anisotropic turbulent systems. The large scales in turbulence are chaotic whereas the small scales with low variance have relatively frequent extreme events—intermittency—which can impact the large scales. Here conceptual dynamical models of turbulence are developed which, despite their simplicity, capture many of these key features of anisotropic turbulent systems in a qualitative fashion. The paper is a self-contained treatment of these conceptual models and their properties.

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to the turbulent fluctuations from increasingly smaller scales as k increases with

$$u' = \sum_{k=1}^{K} u'_k \qquad [1]$$

the turbulent fluctuations. One can think of $\bar{u}$ as the large-scale spatial average of the turbulent dynamics at a single grid point in a more complex system and u' as the turbulent fluctuations at the grid point with

$$u(t) = \bar{u}(t) + \sum_{k=1}^{K} u'_k(t) \qquad [2]$$

the total turbulent field. To add a sense of spatial scale, one can also regard u'_k as the amplitude of the kth Fourier cosine mode evaluated at a grid point but such an interpretation is not necessary here. Note that the large-scale mean $\bar{u}$ can have fluctuating, chaotic dynamics in time through interactions with turbulence and its own intrinsic dynamics. The nonlinear interactions in turbulence conserve the total energy of the mean and fluctuations and a key feature of the conceptual model is to use nonlinear interactions which conserve the energy E, which we take as given by

$$E(\bar{u}, \vec{u}') = \frac{1}{2}\left(\bar{u}^2 + \sum_{k=1}^{K} \left(u'_k\right)^2\right). \qquad [3]$$

A hallmark of turbulence is that the large scales can destabilize the smaller scales in the turbulent fluctuations intermittently and this increased small-scale energy can impact the large scales; this key feature is captured in the conceptual models. With the above discussion, here are the simplest models with all these features, the conceptual dynamical models for turbulence:

$$\frac{d\bar{u}}{dt} = -\bar{d}\,\bar{u} + \gamma \sum_{k=1}^{K} \left(u'_k\right)^2 - \bar{\alpha}\,\bar{u}^3 + \bar{F},$$
$$\frac{du'_k}{dt} = -d_k u'_k - \gamma \bar{u} u'_k + \sigma_k \dot{W}_k, \; 1 \le k \le K. \qquad [4]$$

The reader can think of u'_k as the amplitude of the kth Fourier cosine mode to aid the interpretation of the model but this is not necessary here. The system of $K+1$ dimensional stochastic differential equations in **4** is written in physicist's notation with $\dot{W}_k$ independent white noises for each k but the system in **4** is always interpreted in the Ito sense below. The reader easily verifies that the nonlinear interactions in **4** conserve the energy E in **3**, which can be modified by the linear terms, the external forcing $\bar{F}$, nonlinearity of the large scales, and the random forcing of the small scales. The turbulence dissipation coefficients d_k for $k = 1, 2, \ldots, K$ are positive, $d_k > 0$, in order for the turbulence to have a statistical steady state but the coefficient $\bar{d}$ for the large scales can be either positive or negative reflecting large-scale instability. When $\bar{d}$ is negative so there is instability on the large scales we add the stabilizing cubic term with $\bar{\alpha} > 0$ whereas for positive $\bar{d}$ we assume $\bar{\alpha} = 0$; both cases are studied below. The external force $\bar{F}$ is a constant which is varied below to mimic fully turbulent regimes with (A)–(D). For a fixed coefficient of nonlinear interaction $\gamma > 0$, there is local growth and instability in time for the kth turbulent scale provided that

$$-d_k - \gamma \bar{u} > 0, \quad \text{i.e., } \bar{u} < \frac{-d_k}{\gamma}, \qquad [5]$$

and chaotic fluctuations of $\bar{u}$ will create intermittency in u'_k through this mechanism. Thus, the overall system can have a statistical steady state whereas there is intermittent instability on the small scales which increases their energy and impacts the large scales, creating non-Gaussian intermittent behavior in the system. With $\bar{u} \equiv 0$, the equation for the kth turbulent scale u'_k is a simple Langevin process with Gaussian statistical steady state with zero mean and variance $\sigma_k^2 / 2d_k = E_k$; it is natural to pick these energy densities to have power-law behavior for this energy spectrum, i.e.,

$$\frac{\sigma_k^2}{2d_k} = E_k = E_0 |k|^{-\alpha}, \qquad [6]$$

with $E_0 > 0$ and $\alpha \ge 0$ fixed constants (8). For example, $\alpha = 5/3$ corresponds to the Kolmogorov spectrum (3, 8). Note that we could allow coefficient γ in **4** to vary with k for $k = 1, 2, \ldots, K$ but we refrain from discussing this generalization here. On the other hand, it is natural to have the damping d_k vary with k to represent various dissipative processes such as viscosity or Ekman friction (8). This completes the description of the conceptual models.

Mathematical Properties

Note that the equation for the large-scale mean $\bar{u}$ is deterministic and without any direct stochastic forcing; this deterministic structure mimics that at the large scales for realistic turbulent flows. Nevertheless, the large-scale mean $\bar{u}$ interacts with the fluctuations u'_k which are stochastically forced. We claim that even with the above degenerate noise, the conceptual models in **4** are geometrically ergodic (9); in other words, for any value of $\bar{F}$, a unique smooth ergodic invariant measure exists with exponential convergence of suitable statistics from time averages in the long time limit. To prove this, we apply the main theorem in ref. 9 with the Lyapunov function given by the total energy in **3**. Two things need to be checked; the first is the coercivity of the generator applied to the Lyapunov function which is immediately satisfied given our hypotheses; the second condition is the hypoellipticity of the generator of **4**. To check hypoellipticity we consider the K-vector fields

$$X_k = (\sigma_k \delta_{ik}), \quad 0 \le i \le k, 1 \le k \le K, X_k \in \mathbb{R}^{K+1}, 1 \le k \le K$$

and

$$Y = \begin{pmatrix} -\bar{d}\,\bar{u} + \gamma \sum_k \left(u'_k\right)^2 - \bar{\alpha}\,\bar{u}^3 + \bar{F} \\ -d_k u'_k - \gamma \bar{u} u'_k \end{pmatrix}, \quad Y \in \mathbb{R}^{K+1}.$$

We only need to show that $X_k, [X_k, Y], [X_k, [X_k, Y]]$ span all of $\mathbb{R}^{K+1}$ where $[X, Y] = X \cdot \nabla Y - Y \cdot \nabla X$ is the Lie bracket. Because $[X_k, [X_k, Y]] = \begin{pmatrix} 2\gamma\sigma_k^2 \\ 0 \end{pmatrix}$ and the $X_k, 1 \le k' \le K$ span the orthogonal complement, hypoellipticity is satisfied.

Phase Plane Analysis

Here we develop intuition regarding the parameters of the conceptual models which provide important guidelines to demonstrate below that these models with $K \ge 2$ can capture all of the features of anisotropic turbulence listed in (A)–(D) above. For such intuition, there is a revealing phase plane analysis of the 2D system for $(\bar{u}, u')$ which is the special case of the model in **4** where $K = 1$ and without noise. This system is given by

$$\frac{du'}{dt} = -(d + \gamma \bar{u}) u',$$
$$\frac{d\bar{u}}{dt} = -\bar{d}\,\bar{u} + \gamma \left(u'\right)^2 - \bar{\alpha}\,\bar{u}^3 + \bar{F}. \qquad [7]$$

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The linear subspace $(\overline{u}, 0)$ is invariant for the dynamics which reduces on this subspace to the scalar equation

$$\frac{d\overline{u}}{dt} = -\overline{d}\,\overline{u} - \overline{\alpha}\,\overline{u}^3 + \overline{F}, \qquad [8]$$

whereas the general dynamics of **7** is invariant under the flip symmetry $(\overline{u}, u') \to (\overline{u}, -u')$. Thus, there are between one and three critical points of **7** with the form $(\overline{u}_{CR}, 0)$ as $\overline{F}$ varies, provided that $\overline{d} < 0$ and $\overline{\alpha} > 0$ and only a single critical point of the form $(\overline{u}_{CR}, 0)$ with $\overline{u}_{CR} = \overline{F}/\overline{d}$ for $\overline{d} > 0, \overline{\alpha} = 0$; regardless of these stability properties along the $\overline{u}$ axis, such critical points are unstable to u' perturbations if and only if $d + \gamma\overline{u}_{CR} < 0$, i.e., the instability condition in **5** is satisfied. For suitable values of $\overline{F}$, there is another family of critical points for **7** with the form $(\overline{u}_*, \pm u'_{CR})$ where

$$\overline{u}_* = -\frac{d}{\gamma} \qquad [9]$$
$$\gamma u'^2_{CR} = \overline{d}\,\overline{u} + \overline{\alpha}\,\overline{u}^3 - \overline{F}.$$

Note that $\overline{u}_*$ is exactly the critical value of neutral stability from **5** for the conceptual model. The linear stability matrix at these critical points for **7** has the form

$$\begin{pmatrix} -\overline{d} - 3\overline{\alpha}\,\overline{u}_*^2 & 2\omega \\ -\omega & 0 \end{pmatrix}, \qquad [10]$$

with $\omega = \gamma u'_{CR}$ so these critical points are stable (unstable) if and only if $-\overline{d} - 3\overline{\alpha}\,\overline{u}_*^2 < 0$ (>0).

To develop guidelines in choosing parameters for the numerical experiments for $K \geq 2$ with the conceptual model in **4**, we consider the phase plane analysis in two scenarios with positive and negative large-scale damping. In both cases, the parameters $\gamma = 1.5$ and $d = d_1 \equiv 1$ are fixed below, whereas for

$$\begin{array}{lll} \text{positive large scale damping,} & \overline{d} = 0.01 \text{ and } \overline{\alpha} = 0 \\ \text{negative large scale damping,} & \overline{d} = -0.1 \text{ and } \overline{\alpha} = 0.05. \end{array} \qquad [11]$$

First consider positive large-scale damping; the two critical points $(\overline{u}_*, \pm u'_{CR})$ occur for $\overline{F} < \overline{F}_{CR} = -\overline{d}d/\gamma = -0.0067$ and are both stable by the criterion in **10**, whereas the critical point $(\overline{F}/\overline{d}, 0)$ along the $\overline{u}$ axis is unstable to u' perturbation provided $\overline{F} < \overline{F}_{CR}$. Because the energy is a Lyapunov function for **7**, trajectories off the $\overline{u}$ axis converge to either of the critical points $(\overline{u}_*, \pm u'_{CR})$ with $\overline{u}_*$ the marginally stable value; thus we can expect more turbulent behavior in the conceptual stochastic models with $K \geq 2$ as the forcing $\overline{F}$ increases in magnitude through negative values, $\overline{F}$ with $\overline{F} \leq \overline{F}_{CR} = -0.0067$. A similar scenario occurs for the case with negative damping in **7** for $\overline{F} \leq -0.0545$ with a single critical point along the $\overline{u}$ axis which is unstable to perturbations in u' with two critical points $(\overline{u}_*, \pm u'_{CR})$, $u'_{CR} \neq 0$, which are also unstable because $-\overline{d} - 3\overline{\alpha}\,\overline{u}^2 > 0$; in this case, with all three equilibrium points unstable, trajectories off the $\overline{u}$ axis necessarily converge to periodic orbits encircling the critical points $(\overline{u}_*, \pm u'_{CR})$ and frequently visit values of $\overline{u}$ with instability in the u' dynamics. We also anticipate different behavior for $\overline{F} > -0.0545$ because a stable critical point appears at $\overline{u} = 0.8329$ for this and larger values of $\overline{F}$. See the tables in *SI Appendix*.

Numerical Experiments for $K = 5$ in the Conceptual Model

Here we use simple numerical experiments to demonstrate that the six-dimensional conceptual model in **4** with $K = 5$ has all of the statistical features listed in (A)–(D) including intermittency of the small scales. The parameters $\overline{d}, \overline{\alpha}$, and $\gamma = 1.5$ have already been discussed in **11**. The damping coefficients d_k are a mixture of uniform and scale-selective damping with $d_k = 1 + 0.02k^2$ for $k = 1, 2, \ldots, 5$ so that the smaller scales are damped more rapidly; the noise level set by σ_k for the kth mode is determined by

$$\frac{\sigma_k^2}{d_k} = \frac{0.004}{(1+k)^{5/3}}, \; k = 1, 2, \ldots, 5 \qquad [12]$$

so that a −5/3 spectrum is calibrated to occur for these modes provided $\overline{u} \equiv 0$ in the equations for u'_k (8). This specifies all parameters in the conceptual model for turbulence used here. For all numerical simulations below and in *SI Appendix*, the Euler–Maruyama method is used with a time step $\Delta t = 5 \times 10^{-3}$ and the system is integrated for a long time $T = 2 \times 10^5$ with the first $t = 2 \times 10^3$ time data ignored for postprocessing the equilibrium statistics. In all simulations the initial value is $\overline{u} = 1.5$ with $u'_k = 0$ for $k = 1, 2, \ldots, 5$.

First we consider the case with large-scale instability for $\overline{u}$ with negative damping, $\overline{d} = -0.1$ and $\overline{\alpha} = 0.05$ with the forcing value $\overline{F} = -0.055$ motivated by the phase portrait analysis above. Fig. 1 depicts the pdfs for the total turbulent field u, the large-scale mean $\overline{u}$, and the turbulent fluctuations u'_k, $k = 1, 2, \ldots, 5$ as well as a sample of the time series of each variable in the conceptual model; the pdfs are plotted with a logarithmic vertical coordinate to highlight fat tails of intermittency whereas the Gaussian distribution with the same variance is the parabola in the figure. The pdf for the overall turbulent field u in **2** is nearly Gaussian whereas the pdfs for the mean $\overline{u}$ and the largest scale fluctuating mode u'_1 are both slightly sub-Gaussian. The variable u'_2 has a Gaussian tail whereas the variables u'_3, u'_4, u'_5 all have significant fat tails, which are a hallmark of intermittency. The time series for u'_3, u'_4, u'_5 in Fig. 1 clearly display highly intermittent behavior of extreme values, with the amplitude of u'_3 occasionally spiking to the typical amplitude of u'_1 even though the statistical equilibrium variance of u'_3 is nearly eight times smaller than that for u'_1 (see *SI Appendix*, Table S2). The statistical equilibrium mean value for $\overline{u}$ is $-0.6733 = \langle\overline{u}\rangle$ and $\langle\overline{u}\rangle$ is very close to the marginal stability value $\overline{u}_* = -0.6667 = -(d/\gamma)$ motivated from **7** whereas the standard deviation of $\overline{u}$ is 0.1993, indicating that the instability mechanism elucidated in **5** is operating on all modes and creating intermittency. The total energy of the mean flow $\overline{u}$ exceeds that of the fluctuations u'_k. The variables $u'_k, k = 1, 2, \ldots, 5$ have essentially zero means with variances 0.0446, 0.0174, 0.0049, 0.0014, and 0.0005, respectively, with the correlation time for $\overline{u}$ ∼34, whereas those for $u'_k, k = 1, 2, \ldots, 5$ are decreasing with k and ∼29, 16, 6, 4, and 3, respectively. These are all of the features of anisotropic turbulence required from (A)–(D) and demonstrated in the conceptual dynamical models; furthermore, all of these conditions occur in a robust fashion for $\overline{F}$ increasing in magnitude with $\overline{F} \leq -0.055$ and $0.055 \leq |\overline{F}| \leq 0.1$. All of the detailed data discussed above can be found in *SI Appendix*, Tables S1–S3. There is an evident role for the unstable damping of the large scales $\overline{d} = -0.1$ to increase the variance of $\overline{u}$ with its mean near the marginally critical value $\overline{u}_*$ so that the instability mechanism from **5** operates vigorously in the model and creates more variance in $u'_k, k = 1, 2, \ldots, 5$. Thus, we expect the system with stable damping and the same values of $\overline{F}$ with $\overline{F} = -0.055$ to have less variance.

We consider the case with positive large-scale damping, $\overline{d} = 0.01$, for $\overline{F} = -0.080$; in Fig. 2 we show the pdfs of all variables as well as a piece of the time series of the turbulent signal $u, \overline{u}$, and $u'_k, k = 1, 2, \ldots, 5$. The intermittency of the small-scale modes with less variance is evident in Fig. 2. The mean-flow variable $\overline{u}$ has the largest total energy with equilibrium statistical mean $\langle\overline{u}\rangle = -0.6853$, which is very close to the marginal critical values $\overline{u}_* = -0.6667$ so the intermittent instability mechanism in **5** is operating once again. Both the variances and correlation times

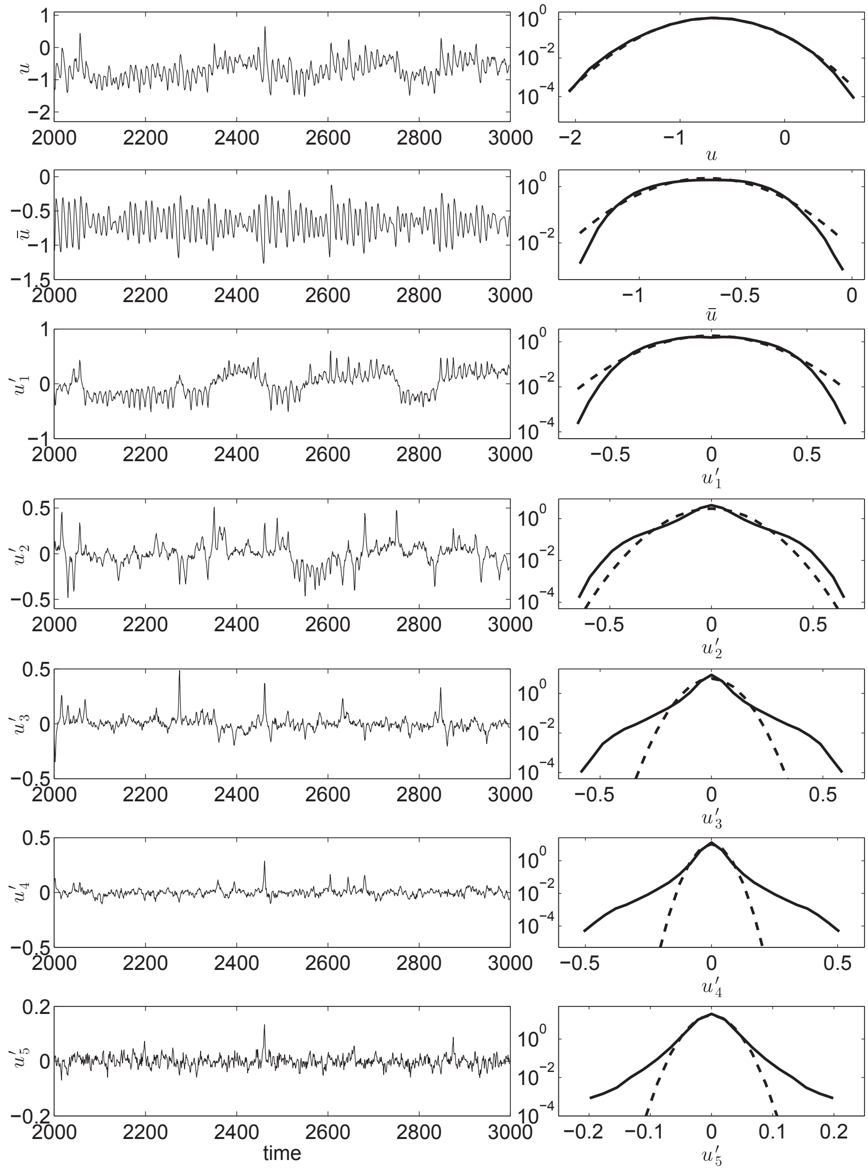

Fig. 1. Negative large-scale damping: time series (*Left*) and pdfs (*Right*) of the turbulent signal u, $\bar{u}$ and u'_k, $k=1,2,\ldots,5$ with $\overline{F}=-0.055$. Note the logarithmic scale of pdfs in the y axis. Dashed lines are Gaussian distributions with the same mean and variance.

behave in a similar fashion as for the negative large-scale damping case discussed above and as required in (A)–(D) so the conceptual model with positive large-scale damping also is a qualitative dynamical model for anisotropic turbulence with all of the features in (A)–(D). Furthermore, all of these features persist for $\overline{F}$ with $-0.055 \le \overline{F} \le -0.1$; the pdfs are all Gaussian with no fat tails for $\overline{F}$ with sufficiently small absolute value such as $\overline{F}=-0.01$, as shown in *SI Appendix*. As expected from our discussion of the unstable case, for fixed forcing with $\overline{F} \le -0.055$ there is between a factor of 2 and 3 less variance in all variables in the positive large-scale damping case compared with the negative large-scale damping case. Documentation for all of the above claims is found in extensive tables in *SI Appendix*. For both cases cross-correlation among the variables $\overline{u}, u'_k, k=1,2,\ldots,5$ is negligible in the statistical equilibrium mean with values roughly less than the 5% level.

In the above paragraphs, we emphasized models with $K=5$ to mimic the many degrees of freedom in real anisotropic turbulence and their interaction with the mean flow. From a mathematical viewpoint, it is interesting to address the following: what is the lowest dimensional conceptual model with intermittency and satisfying all of the requirements in (A)–(D)? Versions of the conceptual model with $K=2$ already exhibit intermittency in u'_2 as well as all of the other features required in (A)–(D) for both positive and negative damping as shown in *SI Appendix*. However, the two mode models with $K=1$ always exhibit either sub-Gaussian or at most Gaussian behavior in u'_1 without intermittency as the noise level is varied in all of our numerical experiments.

Concluding Discussion

Conceptual dynamical models for anisotropic turbulence have been introduced here which, despite their simplicity, capture key features of vastly more complicated systems. The conceptual dynamical models introduced here in **4** involve a large-scale mean flow $\overline{u}$ and turbulent fluctuations, $u'_k, 1 \le k \le K$, on a variety of spatial scales and involve energy-conserving wave–mean-flow interactions as well as suitable degenerate stochastic forcing of the fluctuations u'_k. The models have a transparent mechanism where the mean flow $\overline{u}$ can destabilize the kth mode whenever

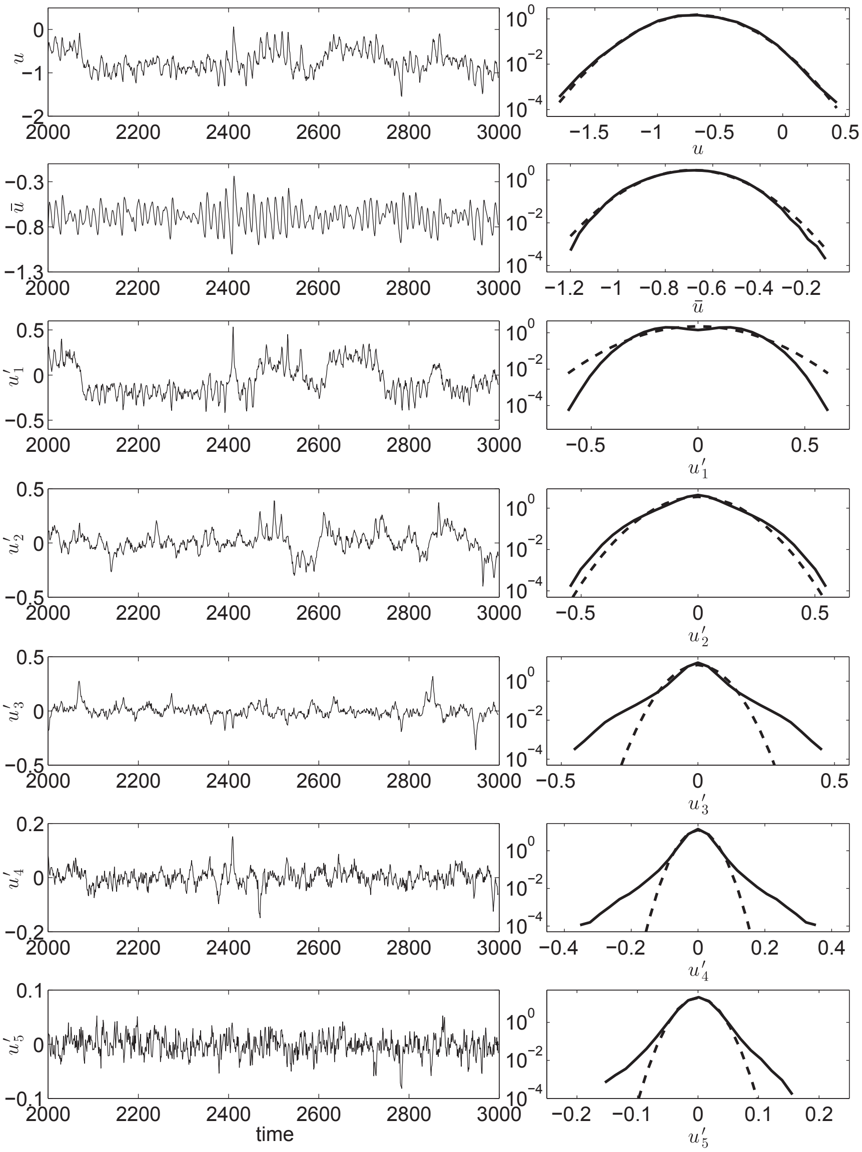

Fig. 2. Positive large-scale damping: time series (*Left*) and pdfs (*Right*) of the turbulent signal u, $\overline{u}$ and u'_k, $k = 1,2,\ldots,5$ with $\overline{F} = -0.080$. Note the logarithmic scale of pdfs in the y axis. Dashed lines are Gaussian distributions with the same mean and variance.

$d_k + \gamma\overline{u} < 0$; a phase plane analysis yields parameters and robust regimes of sufficiently strong large-scale external forcing $\overline{F}$, where the models have a statistical equilibrium mean $\langle u \rangle$ which is nearly neutrally stable in the sense that $d_1 + \gamma\langle\overline{u}\rangle \cong 0$ so that fluctuations in the mean $\overline{u}$ often introduce intermittent instability. Numerical experiments with a six-dimensional version of the model summarized here and in *SI Appendix* confirm that it captures key statistical features of vastly more complex anisotropic turbulent systems. These include chaotic statistical behavior of the mean flow $\overline{u}$ with a sub-Gaussian pdf for its fluctuations, whereas the turbulent fluctuations $u'_k, 1 \le k \le 5$ have decreasing energy and correlation times as k increases with nearly Gaussian pdfs for the large-scale fluctuations and fat-tailed non-Gaussian pdfs for the smaller-scale fluctuations; this last feature allows for intermittency of the small-scale fluctuations where turbulent modes with small variance can have relatively frequent large-amplitude extreme events which directly impact the mean flow $\overline{u}$. Remarkably, vastly more complex realistic turbulent systems often exhibit such marginal critical behavior on average (4). As mentioned above (**1** and **2**), we can regard $\overline{u}, u'_k$ for $1 \le k \le K$ as defining turbulent fluctuations at a grid point in a vastly more complex spatially extended system. There are straightforward generalizations of the conceptual model to allow for many large-scale grid points $\overline{u}_j, j = 1, 2, \ldots, J$ with associated turbulent fluctuations $u'_{j,k}, 1 \le k \le K$ satisfying a coupled system of equations on the large scales,

$$\begin{aligned} \frac{du'_{j,k}}{dt} &= -\left(d_k + \gamma\overline{u}_{j,k}\right)u'_{j,k} + \sigma_k \dot{W}_{j,k} \\ \frac{d\overline{u}_j}{dt} &= L\overline{u}_j + \gamma\sum_{k=1}^{K}\left(u'_{j,k}\right)^2 - \overline{d}\,\overline{u}_j - \overline{\alpha}\,\overline{u}^3 + \overline{F}_j, \end{aligned} \qquad [13]$$

where L can be a linear or nonlinear operator coupling the $\overline{u}_j$. The conceptual models in **13** are nonlinear generalizations with transparent physical mechanisms of those introduced to study stochastic superparameterization in anisotropic turbulence (6, 10). Besides their role as qualitative analog models of vastly more complicated anisotropic turbulence, the conceptual dynamical models introduced here are potentially useful as a simplified test bed for algorithms and strategies for prediction, uncertainty quantification (11), and data assimilation (8) in vastly more complex anisotropic turbulent systems. It also would be interesting to derive the limiting statistical behavior of the conceptual models as the number of fluctuating components k becomes large.

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