Developmental and Individual Differences in Understanding of Fractions

Robert S. Siegler and **Aryn A. Pyke**
Carnegie Mellon University

Abstract

We examined developmental and individual differences in 6th and 8th graders' fraction arithmetic and overall mathematics achievement and related them to differences in understanding of fraction magnitudes, whole number division, executive functioning, and metacognitive judgments within a cross sectional design. Results indicated that the difference between low achieving and higher achieving children's fraction arithmetic knowledge, already substantial in 6th grade, was much greater in 8th grade. The fraction arithmetic knowledge of low achieving children was similar in the two grades, whereas higher achieving children showed much greater knowledge in 8th than 6th grade, despite both groups having been in the same classrooms, using the same textbooks, and having the same teachers and classmates. Individual differences in both fraction arithmetic and mathematics achievement test scores were predicted by differences in fraction magnitude knowledge and whole number division, even after the contributions of reading achievement and executive functioning were statistically controlled. Instructional implications of the findings are discussed.

Keywords

fractions; numerical magnitude representations; arithmetic; mathematics difficulties; mathematics achievement

Research on numerical development has focused on the development of whole numbers. This knowledge is obviously important, but numerical development also involves understanding of other types of numbers, such as fractions. Fractions are crucial not only because they express values that cannot be expressed with whole numbers but also because they provide most children's first opportunity to learn that many properties of whole numbers are not properties of numbers in general. All whole numbers can be represented by a single numeral, have unique successors, never decrease with multiplication, never increase with division, and so on. None of these properties, however, is true of fractions or of numbers in general. Instead, the only property that all real numbers have in common is that they have magnitudes that can be located and ordered on number lines.

Correspondence concerning this article should be addressed to Robert S. Siegler or Aryn A. Pyke, Psychology Department, Carnegie Mellon University, Pittsburgh, PA 15213. rs7k@andrew.cmu.edu or apyke@andrew.cmu.edu.

Current Understanding of the Development of Fraction Knowledge

Many children have great difficulty acquiring fraction knowledge. Although children receive substantial fraction instruction beginning in 3rd or 4th grade (National Council of Teachers of Mathematics (NCTM), 2006), a recent National Assessment of Educational Progress revealed that 50% of 8th graders could not correctly order three fractions (2/7, 1/12, and 5/9) from least to greatest. The difficulty continues in high school and college; for example, on another NAEP item, fewer than 30% of 11th graders translated .029 into the correct fraction (Kloosterman, 2010). The pattern is longitudinally stable; children who have early difficulties with fractions tend to have later difficulties as well (Hecht & Vagi, 2010; Mazzocco & Devlin, 2008).

Failure to master fractions has large consequences. It impedes acquisition of more advanced mathematics and precludes participation in many occupations (McCloskey, 2007; National Mathematics Advisory Panel (NMAP), 2008). A nationally representative sample of 1,000 U.S. Algebra 1 teachers rated fractions as one of the two largest weaknesses in students' preparation for their course (National Opinion Research Center (NORC), 2007). Consistent with these informed opinions, analyses of two large, longitudinal data sets, one from the U.S. and one from the UK, indicated that fifth graders' knowledge of fractions uniquely predicted those students' knowledge of algebra and overall mathematics achievement in high school 5–6 years later, even after statistically controlling for other types of mathematical knowledge, verbal and non-verbal IQ, reading comprehension, working memory, and family income and education (Siegler et al., 2012).

Understanding fractions requires not only knowledge of procedures for solving fraction arithmetic problems but considerable conceptual knowledge as well. Conceptual knowledge of fractions involves knowing what fractions are: that they are numbers that stretch from negative to positive infinity; that between any two fractions are an infinite number of other fractions; that the numerator-denominator relation, rather than either number alone, determines fraction magnitudes; that fraction magnitudes increase with numerator size and decrease with denominator size; that fractions can be represented as points on number lines; and so on.

Understanding magnitudes appears to be a particularly important aspect of conceptual understanding of fractions. Several measures of fraction magnitude representations, including number line estimation, magnitude comparison, and ordering of multiple fractions, correlate highly with knowledge of fraction arithmetic and overall mathematics achievement from 5th to 8th grade (Bailey, Hoard, Nugent, & Geary, 2012; Mazzocco & Devlin, 2008; Siegler, Thompson, & Schneider, 2011). This relation between fraction magnitude knowledge and overall math achievement remains strongly present even when fraction arithmetic competence is statistically controlled.

Acquiring conceptual and procedural knowledge of fractions poses several serious challenges. One challenge in acquiring conceptual knowledge is that children's massive prior experience with integers leads to a whole number bias, in which properties of positive integers are assumed to extend to fractions (Ni & Zhou, 2005). For example, even high

school students often claim that there are no numbers between fractions such as 5/7 and 6/7 (as there are no integers between 5 and 6), and that multiplication cannot yield products smaller than both multiplicands (Vamvakoussi & Vosniadou, 2004).

Acquiring procedural knowledge of fraction arithmetic poses different difficulties. Components of different fraction arithmetic procedures are complexly related. Addition and subtraction of fractions require common denominators, but multiplication and division of fractions do not. Fraction multiplication problems can be solved by applying the arithmetic operation independently to the numerators and to the denominators, but fraction addition and subtraction problems cannot be solved in this way. In fraction division, but in no other operation, answers can be obtained through the invert and multiply procedure.

This analysis suggests that conceptual and procedural knowledge of fractions should be related, because without conceptual understanding, the procedures are highly confusable and difficult to remember. Consistent with this reasoning, conceptual and procedural knowledge of fractions correlate substantially over a wide age range (Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Siegler, et al., 2011).

Goals of the Present Study

Describing Fraction Development among Lower and Higher Achieving Children

The present study was designed to provide a more nuanced description of developmental and individual differences in fraction arithmetic knowledge than currently available. To pursue this goal, we investigated differences between 6th and 8th graders in conceptual and procedural knowledge of fractions among low achieving (LA) and higher achieving (HA) children. We examined three tasks to assess conceptual knowledge of fractions (magnitude comparison and 0–1 and 0–5 number line estimation), five aspects of fraction arithmetic (accuracy, strategy use, strategy variability, confidence, and calculation errors), and two types of processes that seemed likely to underlie fraction proficiency (whole number division and executive functioning).

This design allowed comparison of HA and LA children on all of the measures, as well as comparing the types of errors made by the two groups, which have been found to vary in previous studies (Mazzocco & Devlin, 2008). Of special interest was whether increases in fraction knowledge between 6th and 8th grade would differ for HA and LA children.

One plausible position was that low achieving children generally benefit less from instruction than typically achieving children (Fuchs et al., 2005); they might be expected to make less progress in this domain too. Consistent with this argument, Hecht and Vagi (2010) and Mazzocco and Devlin (2008) found poorer fraction learning among LA than HA children between 4th and 5th grade, and 7th and 8th grade, respectively. The LA and HA children in our sample were drawn from the same classrooms; thus, they were exposed to the same teachers, classmates, and textbooks and to very similar instruction. However, this does not imply that their learning was similar.

Another logic yielded the opposite prediction. Having learned less from initial instruction in fractions, LA children would have more to learn later. Such patterns of change have been

observed at times with whole number procedures (Chong & Siegel, 2008). Instruction during middle school might help LA children catch up to classmates whose fraction understanding was already relatively good.

A third possibility was that patterns for conceptual and procedural knowledge would differ in ways partially consistent with both of the other hypotheses. Conceptual knowledge of fractions is usually taught before fraction arithmetic (National Council of Teachers of Mathematics (NCTM), 2006). If HA children leave elementary school with greater conceptual understanding of fractions, they might make greater strides in learning fraction arithmetic during middle school. In contrast, LA children might make as great or greater progress in conceptual understanding during middle school, due to their having more to learn in that area, but their arithmetic learning might be poorer, due to their not initially possessing adequate conceptual understanding to understand the arithmetic procedures. The present design allowed us to determine which of these scenarios most accurately described changes in fraction knowledge during middle school

Identifying Sources of Individual Differences in Fraction Knowledge

A second goal of the study was to examine the contributions of four potential sources of individual differences in fraction knowledge: weak executive functioning, inadequate understanding of whole number division, limited knowledge of fraction magnitudes, and inaccurate metacognitive assessments of one's fraction arithmetic knowledge.

Contributions of executive functioning—Executive functioning, which involves control of working memory and attention, was one likely contributor to individual differences in fraction arithmetic and mathematics achievement. Inhibition and updating, two processes identified as central to executive functioning by Miyake et al. (2000), have been found to be related to whole number arithmetic; moreover, children with LA tend to exhibit executive functioning deficits relative to typically achieving peers (Bull & Scerif, 2001; Mazzocco & Kover, 2007; Passolunghi & Pazzaglia, 2004, 2005; Swanson & Beebe-Frankenberger, 2004).

Although most research relating executive functioning to math knowledge has been done with whole numbers, executive functioning might be at least as strongly related to fraction knowledge. Conceptual knowledge of fractions requires inhibiting the tendency to view fractions as two independent whole numbers and instead focusing on the relation between numerator and denominator. Procedural knowledge of fractions requires frequent updating of working memory to keep track of completed and yet to-be-completed sub-goals and procedures (mentally add 2/3 + 4/5 to get a sense of the demands of fractions arithmetic on updating).

Contributions of fraction magnitude knowledge—Logically, fraction magnitude knowledge could be unrelated to fraction arithmetic. Children could learn such procedures through rote memorization, without reference to the numerical magnitudes that the fractions represent. However, consistent relations have emerged, probably for several reasons. Fraction magnitude knowledge allows children to check the plausibility of their answers and to reject procedures that generate implausible answers (Hiebert & LeFevre, 1986).

Magnitude knowledge also might inhibit use of flawed arithmetic procedures that treat numerators and denominators as independent whole numbers, rather than as parts of a single magnitude that can be combined with another magnitude. A third reason involves motivation. Without knowledge of fraction magnitudes, fraction arithmetic, pre-algebra equation solving, and other aspects of middle school mathematics become arbitrary procedures that generate meaningless outcomes. When material is meaningless, people have little motivation to learn it (Reyna, Chapman, Dougherty, & Confrey, 2011). For example, without understanding what 1/3 means, there is no particular reason why on 1/3X = Y, X must be larger than Y. As this example illustrates, fraction knowledge is necessary for subsequent mathematics, including pre-algebra and algebra, to make sense. Thus, fraction magnitude knowledge was expected to be related to overall math achievement as well as to fraction arithmetic, even after other aspects of intellectual functioning and mathematical knowledge were statistically controlled.

Contributions of whole number division—From one perspective, the relation of whole number division to fractions is obvious. Any fraction can be viewed as a division problem (N/M); in this sense, fractions *are* division. However, this logical relation does not imply that the two are closely related psychologically. Indeed, knowledge of whole number division and of fractions in 5th grade independently predicted the children's mathematics achievement in 10th grade; neither predictive relation explained the other (Siegler, et al., 2012).

Whole number division might be important for understanding fractions for several reasons. Alone among whole number operations, division often yields fractional answers (e.g., 14 ÷ 3 = 4 2/3). Experience with whole number division also could provide knowledge of the magnitude of some fractions (e.g., 8/2) and a rough sense of the magnitudes of others (e.g., 10/3 is between 3 and 4). In contrast, whole number addition, subtraction, and multiplication never yield fraction answers and thus do not present opportunities to learn about them.

Contributions of metacognition—A fourth potential source of individual differences involves assessments of one's own knowledge. Inadequate metacognitive knowledge might produce either overconfidence in arithmetic answers across the board or lack of differences in confidence in correct and incorrect answers. If children are confident that incorrect procedures and the answers they yield are correct, they might not check their work or consider whether the answers are plausible (Mazzocco, 2007).

Previous findings indicate that children who are more accurate on a given math task tend to be more confident in the correctness of their answers, and that typically achieving children are more confident in their answers than children with math difficulties (Desoete, Roeyers, & Buysse, 2001; Garrett, Mazzocco, & Baker, 2006; Luncangeli & Cornoldi, 1997). However, the children in the current study were considerably older than those in most previous studies (middle school vs. early elementary school). At these older ages, both high and low achieving children might accurately evaluate the correctness of their answers.

Thus, the main purposes of the present study were to describe the development of fraction knowledge during middle school among low achieving and higher achieving children and to

test four potential sources of individual differences in fraction arithmetic and overall math achievement: knowledge of fraction magnitudes, knowledge of whole number division, executive functioning, and metacognitive awareness of one's own fraction competence.

Method

Participants

Participants were 60 6th graders ($M\ CA$ = 12.04 years, SD = 0.61; 50% girls; 68% Caucasian, 25% African American, 2% Asian, 5% Biracial) and 60 8th graders ($M\ CA$ = 14.08 years, SD = 0.49; 53% girls; 67% Caucasian, 30% African American, 2% Asian, 1% Biracial) recruited from three low-income public school districts near Pittsburgh, PA. All students who returned consent forms were included in the study. Testing was done by two female research associates and the second author (one Asian, two Caucasians).

The math section of the previous year's Pennsylvania System of School Assessments (PSSA), the standardized achievement test used in Pennsylvania, provided a measure of overall mathematics knowledge. Children whose scores were in the bottom 35% were classified as low achieving (LA,) and those in the top 65% were classified as high achieving (HA). This threshold has been used to good effect in previous studies (Hanich, Jordan, Kaplan, & Dick, 2001; Jordan, Hanich, & Kaplan, 2003). PSSA results for 8 children were unavailable because they had transferred from another state or because their parents did not release their scores. Among children whose scores were known, 33% (15 6th graders, and 22 8th graders) were LA. Test scores of 6th and 8th graders were comparable: for HA students, means = 66th and 67th percentiles, p =.916; for LA students, means = 18th and 17th percentiles, p =.894.

Tasks

Number line estimation—On the 0–1 task, children were presented number lines on a computer screen, each with 0 at the left end, 1 at the right, and the fraction to be estimated above the line. Children responded by moving the cursor to the position on the line that they thought corresponded to the number being estimated and clicking the track pad. Then, a new number line with a different fraction appeared, and the process repeated. To acquaint children with the track pad, the experimenter asked them to first locate the practice fraction 1/4.

After the practice trial, children estimated the positions of the 10 fractions that provided the experimental data: from smallest to largest, 1/19, 2/13, 1/5, 1/3, 3/7, 7/12, 5/8, 3/4, 7/8, and 13/14. Two of these fractions were drawn from each fifth of the number line. Here as on all experimental tasks, presentation order of items was random, and no feedback was provided.

The 0–5 number line task was identical, except that the right endpoint was labeled "5," the practice fraction was 7/2, and the 10 fractions to be estimated were 1/5, 7/8, 11/7, 9/5, 13/6, 7/3, 13/4, 10/3, 9/2, and 19/4.

Magnitude comparison—Children were asked to compare 3/5 to a series of other fractions shown one at a time in random order on a computer screen: 2/7, 1/3, 5/11, 4/7, 2/3,

3/4, 7/9, and 5/6. The task was to press the "a" key if the fraction was smaller than 3/5 and to press the "l" key if the fraction was larger than 3/5.

Fraction arithmetic—Children were presented 16 problems, 4 for each of the 4 arithmetic operations, one at a time on a computer screen. For each operation, the four problems combined the operand 3/5 with 1/5, 1/4, 2/3, and 4/5. Thus, half of the problems for each operation had operands with equal denominators (e.g., 3/5 + 1/5; 4/5 + 3/5) and half had operands with unequal denominators (e.g., 3/5 + 1/4; 3/5 + 2/3). For subtraction and division problems, the larger operand was always listed first. On each item, children received a sheet of paper that stated the problem at the top and afforded ample space for paper-and-pencil calculations. After generating an answer, children were asked to rate their confidence in it on a 5-point scale (1=not confident at all, 5=extremely confident) and then to describe how they solved the problem. These explanations were audio recorded for later coding.

Whole number division—Children were presented three whole number division problems: $56 \div 8$, $306 \div 9$, and $91 \div 4$. Pencil and paper for calculations were provided. For the two problems with whole number quotients, children received 1 point for being correct; for the problem whose quotient included a fraction, children received 1 point for a completely correct answer (e.g., 22.75 or 22 ¾), 0.75 point for an answer with a correct remainder (e.g., 22, R 3), and 0.5 point for answers where only the whole number was correct (e.g., 22 or 22.3).

Standardized math and reading achievement tests—We obtained children's scores from the PSSA mathematics and reading sections given toward the end of 5th and 7th grades, about half a year before the study began. The PSSA math section examined a range of skills, including knowledge of whole number and fraction arithmetic; probability and statistics; interpretation of tables, graphs, and figures; pre-algebra; geometry; and series extrapolation. The PSSA reading section examined vocabulary, general comprehension, and inferential reasoning skills. For example, children read about a baker who argued that a man enjoying smells from his shop should have to pay for the pleasure; the children then were asked if the baker lacks: A. talent, B. popularity, C. success, or D. generosity.

Executive function—Children were presented two EF tasks adapted from Miyake, et al. (2000), one measuring updating of working memory and one measuring inhibition.

Working memory updating—Letters were presented one at a time on a computer screen for 2750 ms each. The task was to recall, in order, the three most recently presented letters when "???" appeared on the screen. For example, after seeing "Q, D, X, R, M, ???", the child was to respond "XRM." Children were encouraged to rehearse aloud the three most recent letters as each new letter appeared. This task required updating one's memory by adding the most recently presented letter and dropping the 4th most recently presented. After three practice items, children performed nine trials on which five, seven, or nine letters appeared before the recall request. The nine trials included three of each length, and were ordered randomly. The dependent measure was percent recall of the 27 letters from the nine trials.

Inhibition—A fixation cross was presented at the center of the computer screen for a period varying from 1500 to 3500 ms. A black square (0.4 degrees wide) then appeared for 200 ms on a randomly determined side of the screen. Next, a target letter appeared, either in the square's previous location (pro-saccade trials) or on the opposite side of the screen (anti-saccade trials); the task was to identify the letter. Because the letter was present for only 150 ms before being masked by a gray square, initially looking toward the wrong side generally ruled out correct letter identification.

Participants first completed 26 practice trials: 18 pro-saccade trials followed by 8 anti-saccade trials. Then they completed the 36 anti-saccade trials that provided the experimental data. These anti-saccade trials were the trials of interest, because they required participants to inhibit the inclination to first look toward the cue that appeared in their periphery (the black square) and instead to attend to the opposite side of the screen. The dependent measure was the percent of anti-saccade target trials on which children correctly identified the letter.

Design and Procedure

Children were tested individually in a quiet room in their school and completed all tasks on a laptop computer during a single 45-minute session. Both the items within each mathematics task and the tasks themselves were ordered randomly. The two executive function tasks (updating and inhibition) were presented in random order following the math tasks.

Results

We describe, in order, findings on development of fraction magnitude representations (number line and magnitude comparison tasks), fraction arithmetic, whole number division, executive functioning, and predictors of individual differences in fraction arithmetic and overall math achievement. Pairwise comparisons using the Bonferroni correction were used to interpret statistical interactions. Mauchly's test indicated that the assumption of sphericity was violated in a few cases; when that occurred, Greenhouse-Geisser corrections to the degrees of freedom were used. Performance of the eight children whose PSSA scores were unavailable was excluded from all analyses that included LA/HA status as a variable.

Fraction Magnitude Tasks

Number line estimation—Accuracy of number line estimation was indexed by percent absolute error (PAE), defined as:

PAE = (|Child's Answer – Correct Answer|) / Numerical Range * 100%

For example, if a child was asked to locate 9/2 on a 0–5 number line, and the child marked the location corresponding to 7/2, the PAE would be 20% ((ú3.5 – 4.5ú) / 5 * 100%). PAE is an index of error; the higher the PAE, the *less* accurate the estimate.

A 2 (grade: 6th or 8th) X 2 (achievement status: HA or LA) X 2 (number line: 0–1 or 0–5) ANOVA revealed that 8th graders' estimates were more accurate than those of 6th graders (PAEs = 17% and 22%), $F(1, 108) = 7.31$, $p = .008$, $\eta_p^2 = .06$; HA students' estimates were

more accurate than those of LA students (PAEs = 13% and 26%), $F(1,108) = 63.62$, $p < .001$, $\eta_p^2 = .37$; and estimates were more accurate on 0–1 than 0–5 number lines (PAEs = 16% and 23%), $F(1,108) = 23.17$, $p < .001$, $\eta_p^2 = .18$. No interactions were significant.

Magnitude comparison—A grade X achievement status ANOVA indicated that comparison accuracy increased marginally from 6th to 8th grade (69% vs. 75% correct), $F(1,108) = 3.17$, $p = .078$, $\eta_p^2 = .03$, and was higher among HA than LA students (81% vs. 63%), $F(1,108) = 24.05$, $p < .001$, $\eta_p^2 = .18$. In both grades, the greater the distance of the comparison fraction from 3/5, the greater the number of correct comparisons: $r(6) = .73$, $p = .039$ for 6th graders, and $r(6) = .66$, $p = .076$ for 8th graders.

Fraction Arithmetic

To obtain a nuanced understanding of fraction arithmetic development, we examined accuracy, strategy use, strategic variability, calculation errors, and confidence ratings.

Accuracy—A 2 (grade) X 2 (achievement status) X 4 (arithmetic operation) X 2 (denominator: equal or unequal) ANOVA on number of correct answers yielded main effects for all four variables: grade (41% correct for 6th graders vs. 57% for 8th graders), $F(1,108) = 15.57$, $p < .001$, $\eta_p^2 = .13$; achievement status (36% correct for LA students vs. 61% for HA students), $F(1,108) = 37.97$, $p < .001$, $\eta_p^2 = .26$; arithmetic operation (60% correct for addition, 68% for subtraction, 48% for multiplication and 20% for division), $F(1.93, 208.3) = 45.82$, $p < .001$, $\eta_p^2 = .30$; and denominator equality (55% correct for problems with equal denominators vs. 43% for those with unequal ones), $F(1,108) = 48.87$, $p < .001$, $\eta_p^2 = .31$. Interactions were present for grade X achievement status, $F(1,108) = 4.24$, $p = .042$, $\eta_p^2 = .04$; achievement status X arithmetic operation, $F(1.93, 208.3) = 8.89$, $p < .001$, $\eta_p^2 = .08$; achievement status X denominator equality, $F(1,108) = 34.34$, $p < .001$, $\eta_p^2 = .24$; denominator equality X arithmetic operation, $F(2.13, 230.0) = 39.15$, $p < .001$, $\eta_p^2 = .27$; and among arithmetic operation, denominator equality, and achievement status, $F(2.13, 230.0) = 5.85$, $p = .003$, $\eta_p^2 = .05$.

The three-way interaction and several of the two-way interactions reflected differences between HA and LA children in the effects of denominator equality on the four arithmetic operations (Table 1). Among HA children, accuracy was higher on equal than on unequal denominator problems for addition (87% correct on equal vs. 74% on unequal denominator problems, $p = .013$), subtraction (91% vs. 77%, $p = .002$), and division problems (36% vs. 27%, $p = .002$). In contrast, HA children's multiplication accuracy was *lower* on equal than on unequal denominator problems (34% vs. 61%, $p < .001$). This difference reflected children often extending equal denominators in the operands to the answer (e.g., $1/5 \times 3/5 = 3/5$), an approach that was correct for addition and subtraction but incorrect for multiplication.

LA children exhibited effects of denominator equality that were similar but much larger than those of their HA peers for both addition (63% correct on equal denominators problems vs. 12%, on unequal denominator ones, $p < .001$) and subtraction (75% vs. 28%, $p < .001$). For multiplication, the effect of denominator equality was somewhat smaller among children

with LA than among their HA peers, but in the same direction (40% on equal vs. 55% on unequal, $p = .050$). For division, the accuracy of children with LA was low regardless of whether denominators were equal or unequal (12% and 6% correct, $p = .114$).

The forest and the trees—It is easy to get lost in the details of these data and to miss the most striking finding: the differing pattern of change during middle school in fraction arithmetic knowledge of HA and LA children. As shown in Table 2, increases in percent correct between 6th and 8th grade were greater for HA than LA children on all eight types of fractions arithmetic problems. HA children's accuracy was significantly higher among 8th than 6th graders on 6 of the 8 types of problems, compared to 0 of 8 for LA children. The LA children were already less accurate as 6th graders, but the gap between their and the HA children's accuracy widened considerably between 6th and 8th grade.

Strategies—Children's fraction arithmetic strategies fell into four categories. *Correct strategies* (54% of trials) were generally the standard fraction arithmetic algorithms. *Independent whole numbers strategies* (14%) involved performing the arithmetic operation on numerators and denominators separately, as if they were unrelated whole numbers (e.g., 3/5 + 1/4 = 4/9). *Wrong fraction operation strategies* (26%) involved importing components of procedures that are correct for another fraction arithmetic operation but inappropriate for the requested operation (e.g., leaving a common denominator unchanged on a multiplication problem, as in 3/5 × 4/5 = 12/5). Finally, the *None/Other* classification (7%) included trials on which children used idiosyncratic strategies (e.g., inverting the wrong fraction on a division problem) or refused to try any approach.

The analysis of frequency of correct strategies yielded highly similar results to the analysis of frequency of correct answers, despite correct strategies yielding incorrect answers on 9% of trials, usually because of whole number arithmetic errors (e.g., 5 × 3 = 20). Therefore, we only report analyses of frequencies of the two common incorrect strategies.

Independent whole numbers strategy—Treating numerators and denominators as independent whole numbers leads to incorrect answers for addition and subtraction, and to unnecessarily complex answers for division. We excluded multiplication from analyses of this strategy, due to it and the correct strategy being identical.

A grade X achievement status X arithmetic operation X denominator equality ANOVA indicated that the independent whole numbers strategy was more frequent among 6th than 8th graders (25% vs. 16%), $F(1, 108) = 3.09$, $p = .082$, $\eta_p^2 = .03$; among LA than HA children (32% vs. 10%), $F(1, 108) = 17.66$, $p < .001$, $\eta_p^2 = .14$; on addition than on subtraction or division problems (26%, 20%, and 15%, respectively), $F(1.79, 193.4) = 10.37$, $p < .001$, $\eta_p^2 = .09$; and on problems with unequal than equal denominators (25% vs. 16%), $F(1, 108) = 16.58$, $p < .001$, $\eta_p^2 = .13$.

These effects were qualified by interactions between arithmetic operation and denominator equality, $F(1.74, 188.1) = 11.48$, $p < .001$, $\eta_p^2 = .10$; between arithmetic operation and achievement status, $F(1.79, 193.4) = 9.49$, $p < .001$, $\eta_p^2 = .08$; and among arithmetic operation, denominator equality, and achievement status, $F(1.74, 188.1) = 4.04$, $p = .024$,

$\eta_p{}^2 = .04$. The three-way interaction and one two-way interaction reflected differing strategy use by HA and LA children. Among HA children, use of the independent whole numbers strategy was comparable for equal and unequal denominator problems for addition (10% and 12%, $p =.643$), subtraction (6% and 10%, $p =.209$), and division (12% and 8%, $p =.306$). Among LA students, the same was true for division (20% and 21%, $p = .735$), but these children used the independent whole numbers strategy twice as often on problems with unequal as equal denominators for addition (55% vs. 28%, $p < .001$) and subtraction (45% vs. 21%, $p < .001$). This suggests that LA children sometimes used the whole number strategy because they did not know how to generate a common denominator, rather than because they thought it was correct.

Wrong fraction operation strategies—Children often applied components of procedures that were correct for another fraction arithmetic operation but incorrect for the operation specified in the problem. A grade X achievement status X arithmetic operation X denominator equality ANOVA indicated that such wrong fraction operation strategies were more common among 6th than 8th graders (30% vs. 22% of problems), $F(1, 108) = 5.31$, $p = .023$, $\eta_p{}^2 = .05$, and much more common on division and multiplication than on addition and subtraction problems (55%, 46%, 1%, and 1%, respectively), $F(1.77, 190.0) = 130.98$, $p <.001$, $\eta_p{}^2 = .55$. The disproportionate frequency of these errors on multiplication and division problems seems attributable to children transferring components of earlier learned fraction arithmetic operations (addition and subtraction) to later learned ones.

Interactions were present for achievement status and grade, $F(1, 108) = 5.55$, $p = .020$, $\eta_p{}^2 = .05$, achievement status and arithmetic operation, $F(1.77, 191.0) = 3.30$, $p = .045$, $\eta_p{}^2 = .03$, and achievement status, grade, and arithmetic operation, $F(1.77, 191.0) = 5.12$, $p = .009$, $\eta_p{}^2 = .05$. The three-way interaction reflected different patterns of differences between 6th and 8th grade for HA and LA children. Among HA children, wrong fraction operation strategies were more common in 6th than 8th grade on multiplication and division problems (65% to 34% and 65% to 29%, p's < .001). Similar decreases on addition and subtraction were impossible, because wrong fractions operation strategies were already uncommon in 6th grade (0% and 3%). In contrast, among LA children, use of the strategy was comparable for 6th and 8th graders for multiplication (47% and 40%, $p =.596$) and division (57% and 68%, $p =.385$). As with HA children, wrong operation errors were already rare in 6th grade (3% and 1%), so no decrease was possible. Thus, another way that the arithmetic performance of HA children improved between 6th and 8th grade, and that of LA children did not, was in reduction of use of wrong fraction operation errors for multiplication and division.

Variability of fraction arithmetic strategies—Highly consistent strategy use might have been expected within the pairs of virtually identical problems (e.g., multiplication problems with common denominators, such as 3/5 * 1/5 and 3/5 * 4/5). However, most participants used different strategies on at least one pair of highly similar problems. This variability was present among a greater percentage of LA than HA children, 79% vs. 51%, $F(1,108) = 7.94$, $p =.006$, $\eta_p{}^2 = .07$. LA children also generated a greater number of pairs of

trials that showed such strategic variability (21% for LA and 11% for HA children), $F(1,108) = 13.10$, $p < .001$, $\eta_p^2 = .11$.

This variability often did not reflect children lacking knowledge of the correct strategy and therefore vacillating between different incorrect strategies. Children used one correct and one incorrect strategy on 65% of the inconsistent pairs of trials (67% for HA children, 64% for LA children, *n. s.*).

Calculation errors during fraction arithmetic strategies—Incorrect answers sometimes resulted from whole number fact errors, such as writing "$5 \times 3 = 20$." A grade X achievement status ANOVA indicated that 6th graders made more such errors than 8th graders (25% vs. 16%), $F(1,108) = 9.76$, $p = .002$, $\eta_p^2 = .08$, and that LA children made more of them than did HA children (28% vs. 13% of trials), $F(1,108) = 28.60$, $p < .001$, $\eta_p^2 = .21$.

Confidence in fraction arithmetic answers—A grade X achievement status X arithmetic operation X denominator equality ANOVA indicated that confidence in the correctness of answers was greater among HA than LA children (means of 4.1 vs. 3.4 on the 1–5 rating scale), $F(1, 108) = 20.17$, $p < .001$, $\eta_p^2 = .16$; on addition than subtraction, subtraction than multiplication, and multiplication than division problems (means = 4.2, 3.9, 3.7 and 3.1, respectively), $F(2.23, 240.7) = 75.56$, $p < .001$, $\eta_p^2 = .41$; and on problems with equal than unequal denominators (3.8 vs. 3.6), $F(1, 108) = 19.63$, $p < .001$, $\eta_p^2 = .15$.

Denominator equality and arithmetic operation interacted, $F(3, 324) = 20.20$, $p < .001$, $\eta_p^2 = .16$. Confidence was higher on equal than unequal denominator problems for addition (4.4 vs. 4.1, $p < .001$), subtraction (4.1 vs. 3.7, $p < .001$), and division (3.3 vs. 2.9, $p < .001$), but lower on equal than unequal denominator multiplication problems (3.6 vs. 3.8, $p = .002$). This interaction for confidence ratings paralleled the one for accuracy; just as children erred more often when multiplication problems had equal denominators, they were also less confident in their answers on such problems. A three-way interaction of denominator equality, arithmetic operation, and grade was also present, $F(3, 324) = 4.19$, $p=.006$, $\eta_p^2 = .04$, though the reasons for it were unclear.

Among the 92% of children who generated at least one correct and one incorrect answer on the 16 arithmetic problems, a within-subjects analysis revealed that children were more confident in their correct than incorrect answers (mean confidence rating = 4.2 vs. 3.5), $F(1,109) = 126.61$, $p < .001$, $\eta_p^2 = .54$. Mean confidence in correct versus incorrect answers was higher among most children (79%-87%) in all four grade X achievement status groups.

Whole Number Division

A grade X achievement status ANOVA revealed that HA students were much more accurate than students with LA at whole number division (89% vs. 53% correct), $F(1,108) = 43.26$, $p < .001$, $\eta_p^2 = .29$. No other effects were significant ($Fs \quad 1$).

Executive Functioning

A grade X achievement status ANOVA indicated higher memory updating scores for HA than LA children (78% vs. 69% correct), $F(1,108) = 8.54$, $p = .004$, $\eta_p^2 = .07$. A grade X

achievement status interaction, $F(1,108) = 5.26$, $p =.024$, $\eta_p^2 =.05$, reflected higher updating scores among 8th than 6th grade HA children (82% vs. 74%, $p =.042$), but not among LA peers (65% for 8th graders, 73% for 6th graders, $p =.176$).

A grade X achievement status ANOVA indicated that HA children were more successful than LA ones in inhibiting inappropriate responses (40% vs. 34% correct), $F(1,108) = 4.07$, $p =.046$, $\eta_p^2 =.04$. No effect for age or interaction was present.

Individual Differences

Predictors of fraction arithmetic performance. I—In both grades, almost all measures were related to fraction arithmetic accuracy (Table 3). To examine predictors of fraction arithmetic accuracy in a more structured way, we constructed a hierarchical regression model. We first entered the child's PSSA reading score, to control for general intellectual ability. Next, we entered measures of performance on the experimental tasks in the order in which the competencies seem likely to be acquired during the developmental/ educational process: 1) executive functions; 2) whole number division; and 3) fraction magnitude measures. The two EF measures were entered in a single step of the hierarchical regression, as were the three measures of fraction magnitude knowledge.

Reading scores accounted for 14% of the variance in 6th graders' fraction arithmetic scores, $F(1,54) = 8.53$, $p =.005$, and 33% of the variance in 8th graders' scores, $F(1,54) = 26.98$, $p<.001$. The two executive function measures explained an additional 2% of variance for 6th graders (*n.s.*), and 1% for 8th graders (*n.s.*). Whole number division accounted for an additional 8% of variance for 6th graders, $F(1,51) =5.43$, $p =.024$, and 6% for 8th graders, $F(1,51) = 5.31$, $p =.025$. The fraction magnitude tasks contributed an additional 12% of variance for 6th graders, $F(3,48) = 3.04$, $p =.038$, and 8% for 8th graders, $F(3,48) = 2.37$, $p =.082$. In all, these predictors accounted for 36% of variance in 6th graders' fraction arithmetic scores, $F(7,48) = 3.78$, $p =.002$, and 48% of variance in the 8th graders' scores, $F(7,48) = 6.36$, $p<.001$.

This ordering of variables might be viewed as underestimating the contribution of executive functioning, due to it sharing variance with reading achievement. To evaluate this hypothesis, we re-ordered the predictors so that executive functioning was entered before reading achievement (the order of the other variables, and their contribution, remained constant). In this analysis, EF accounted for 4% of variance in 6th graders' fraction arithmetic accuracy, $F(2,53) = 1.10$, $p = .342$ (*n.s.*) but 17% of the variance in 8th graders' accuracy, $F(2,53) = 5.28$, $p = .008$. Reading achievement accounted for an additional 11% of variance in 6th graders' fraction arithmetic accuracy, $F(1,52) = 6.83$, $p = .012$, and 18% of variance in 8th graders' performance, $F(1,52) = 13.91$, $p <.001$.

Predictors of mathematics achievement—Bivariate correlations indicated that among both 6th and 8th graders, all three fraction magnitude tasks, whole number division, calculation errors and at least one EF measure were related to overall math achievement test scores. The relations tended to be stronger for the older children (Table 3). The relations of PSSA scores to fraction performance were not due to fraction questions being major parts of the test. Analysis of test guidelines suggested that only 10%-15% of PSSA mathematics test

items are dedicated to fractions for the grades in question http://www.portal.state.pa.us/portal/server.pt/community/assessment_anchors/7440.

To examine predictors of individual children's PSSA math scores more systematically, we conducted a hierarchical regression analysis that paralleled the one for fraction arithmetic. Because instruction in fraction arithmetic generally comes after instruction in fraction magnitudes, we entered fraction arithmetic after fraction magnitudes.

PSSA reading scores accounted for 36% of the variance in 6th graders' PSSA math scores, $F(1,54) = 30.40$, $p < .001$, and 56% of the variance in 8th graders' scores, $F(1,54) = 67.44$, $p<.001$. The two executive function measures explained 8% of variance beyond that explained by the reading scores for 6th graders, $F(2,52) = 3.93$, $p = .026$, and 2% for 8th graders, (*n.s.*). Whole number division accounted for an additional 9% of variance for both 6th graders, $F(1,51) = 9.38$, $p = .004$, and 8th graders, $F(1,51) = 14.13$, $p < .001$. Fraction magnitude tasks contributed an additional 10% of variance for 6th graders, $F(3,48) = 4.10$, $p = .011$, and 9% for 8th graders, $F(3,48) = 5.50$, $p < .002$. The fraction arithmetic measure added a non-significant 1% to the variance explained by the other measures for both 6th and 8th graders. Together, these predictors accounted for 64% of variance in 6th graders' PSSA math scores, $F(8,47) = 10.33$, $p < .001$, and 76% of variance in 8th graders' scores, $F(8,47) = 18.83$, $p < .001$.

When entered first, EF accounted for 12% of variance in 6th graders' math achievement scores, $F(2,53) = 3.59$, $p = .035$, and 28% in 8th graders' scores, $F(2,53) = 10.05$, $p<.001$. Reading achievement explained 33% additional variance in 6th graders' math achievement scores, $F(1,52) = 30.40$, $p<.001$, and 30% in 8th graders' scores, $F(1, 52) = 36.10$, $p<.001$.

Fraction magnitudes, fraction arithmetic, and mathematics achievement—
Fazio, Bailey, Thompson, and Siegler (under review) found that children who based number line estimates predominantly on fraction magnitudes did better on other mathematics tasks and on a math achievement test than children whose number line estimation was based predominantly on numerators or denominators alone. To examine whether this relation was again present, we followed Fazio et al.'s procedure of computing gamma correlations between each child's 0–1 number line estimates and the order of the fractions if ranked by numerator, denominator, and fraction magnitude. The predictor corresponding to the largest of these three correlations was interpreted as the child's predominant approach, unless all relations were weak (|gamma|<.25), which resulted in a strategy classification of "unknown". Two-thirds of children (69%) relied predominantly on fraction magnitudes, 17% on numerators, and 12% on denominators.

As in Fazio et al., children who relied mainly on fraction magnitudes on the 0–1 number line task generated much better performance on all measures of mathematical knowledge than children who did not. This was true on 0–5 number line estimation among 6th graders (PAE = 18 vs. 30), $t(58) = -3.73$, $p<.001$, $d = -1.03$, and 8th graders (PAE = 17 vs. 25), $t(58) = -2.75$, $p = .008$, $d = -.82$; on magnitude comparison among 6th graders (82% vs. 60% correct), $t(58) = 4.64$, $p < .001$, $d = 1.28$, and 8th graders (83% vs. 55%), $t(58) = 5.48$, $p<.001$, $d = 1.63$; on fraction arithmetic among 6th graders (50% vs. 39% correct), $t(58) = 1.88$,

$p = .066$, $d = .52$, and 8th graders (64% vs. 40%), $t(58) = 3.13$, $p = .003$, $d = .93$; on whole number division among 6th graders (83% vs. 62% correct), $t(30.81) = 2.55$, $p = .016$, $d = .78$, and 8th graders (87% vs. 39%), $t(58) = 5.83$, $p < .001$, $d = 1.73$; and on overall math achievement test scores among 6th graders (PSSA = 1597 vs. 1320), $t(54) = 4.39$, $p < .001$, $d = 1.26$, and 8th graders (1547 vs. 1213), $t(54) = 5.91$, $p < .001$, $d = 1.86$. Thus, the differentiation based on 0–1 number line estimation was strong not only on other measures of numerical magnitude knowledge but also on fraction arithmetic, whole number division, and mathematics achievement.

Focusing on the numerator alone or the denominator alone seemed likely to be related to LA/HA status, and it was. Reliance on fraction magnitudes was much higher among HA than LA children in both 6th grade (81% versus 27%) and in 8th grade (94% versus 45%).

Discussion

Development of Fraction Knowledge in LA and HA Children

The most striking finding from the study was that differences between the fraction arithmetic knowledge of HA and LA children widen substantially between 6th and 8th grade. HA children's fraction arithmetic accuracy was much higher in 8th than 6th grade, but LA children's accuracy was similarly low in both grades.

At least three factors contributed to this pattern. One was differentiating between properties of whole numbers and fractions. Among LA children, whole number strategies were equally common on fraction arithmetic problems in 6th and 8th grade; in contrast, among HA children, whole number strategies were fairly common in 6th grade but almost nonexistent in 8th grade. A second likely contributor to the differential improvement involved establishing common denominators. On operations that require equal denominators (addition and subtraction), LA 6th and 8th graders showed equally poor performance, but HA 8th graders were considerably more successful than HA 6th graders. A third likely contributor involved variability of strategies. Both 6th and 8th grade LA children and 6th grade HA children fairly often used a correct and an incorrect procedure to solve pairs of virtually identical problems, but HA 8th graders consistently used correct procedures.

Sources of Individual Differences in Fraction Arithmetic and Overall Math Achievement

Fraction magnitude knowledge—As found previously (Siegler, et al., 2011), individual differences in fraction magnitude knowledge were closely related to individual differences in fraction arithmetic and mathematics achievement. The present results showed that this relation remains present even after controlling for reading achievement and executive functioning.

The relation to math achievement test scores seems likely to be due in part to the importance of fractions for solving pre-algebra and algebra problems. Consistent with this interpretation, in both Siegler et al. (2011) and the present study, the relations of fraction magnitude knowledge to mathematics achievement test scores were stronger in 8th grade, when children would have taken or be taking pre-algebra and algebra courses, than in 6th grade. The fact

that in both studies the relation was also present, albeit weaker, in 6th grade suggests that fraction magnitude knowledge also facilitates learning of less advanced mathematics.

Whole number division—Whole number division accounted for substantial variance in 6th and 8th graders' fraction arithmetic and overall math achievement test scores, even after controlling for reading achievement and executive function. This predictive relation probably had several sources: Whole number division and fractions are logically equivalent (N/M); whole number division is assessed on middle school mathematics achievement tests; and it is essential to solving many pre-algebra, algebra, and statistics problems that also appear on the tests. Consistent with this analysis, HA and LA children's whole number division accuracy differed greatly (89% versus 53% correct.)

In contrast to this large difference between LA and HA children's division accuracy, no difference was present between 6th and 8th graders in either group. A likely reason is that whole number division is not a focus of middle school math instruction. The lack of improvement during middle school helps explain why 5th graders' whole number division uniquely predicts 10th graders' mathematics achievement (Siegler, et al., 2012). If children do not learn whole number division when it is taught, they may be unlikely to learn it later.

Executive functioning—Our findings regarding relations of executive functioning to fraction arithmetic and overall mathematics achievement were complex. Eighth graders' working memory updating and inhibitory skills correlated with their fraction arithmetic accuracy and math achievement. These relations made sense; success at mixed series of problems, like the present ones, requires inhibiting inappropriate procedures, such as maintaining equal denominators in the product during multiplication, and it also requires continuously updating working memory contents, such as which steps in a computation have been completed and which remain. In contrast, 6th graders' fraction arithmetic accuracy was not related to either executive function, and their mathematics achievement was related to inhibitory skill but not working memory updating. Adding to the complexity, when reading achievement was statistically controlled, the relation between executive functioning and fraction arithmetic disappeared at both grade levels, and the relation of executive functioning to overall math achievement was present for 6th but not 8th graders. These results were less consistent than those obtained in some previous studies (e.g., Bailey, et al., 2012) but similar to those obtained in others (e.g., Mazzocco & Kover, 2007). The reasons for this inconsistent pattern, both within the present study and in the literature in general, remain to be established.

Metacognitive judgments—Previous studies have not tested whether children who use inappropriate arithmetic procedures believe those procedures are correct or whether they suspect that the procedures are incorrect but use them anyway because they cannot recall a correct strategy. The present study provided evidence for both interpretations.

Consistent with the view that children often believe that incorrect procedures are correct, HA children were wrong on 26% of answers in which they indicated maximal confidence, and LA children were wrong on 46%. Such overconfidence may have negative

consequences for learning. Children who are confident that their incorrect answers are correct might not check them or attend to subsequent instruction.

On the other hand, the present results also reflected considerable metacognitive knowledge. In both grades, children were more confident in their correct than in their incorrect answers, and their mean confidence on each problem correlated positively with their overall accuracy on the problem. The overall pattern suggests that both overconfidence in incorrect strategies and uncertainty about the correct strategy contribute to use of incorrect approaches.

Several limitations of this study should be noted. The study was cross-sectional, so the stability of individual differences could not be examined. The sample was relatively small and included few Hispanic or Asian students. Results regarding unique contributions of executive functioning did not yield clear conclusions.

Instructional Implications

A general instructional implication of these findings is that skills that are prerequisite for subsequent mathematics need to be taught much more effectively than they are at present. Illustratively, the recently adopted U. S. Common Core State Standards prescribe that whole number division be taught and learned in 4th grade, but the present results demonstrate that many 8th graders have not learned it. The strong relations of whole number division to both fraction arithmetic and math achievement test scores found in this study, as well as the inherently foundational role of division for later mathematics, show the necessity of improving teaching and learning in this area.

Another instructional implication of the present findings involves a target for early intervention that might avoid many subsequent difficulties that children have with fractions. Roughly one-third of children in the present study, and a similar percentage in Fazio et al. (under review), based their fraction magnitude estimates primarily on the numerator alone or the denominator alone. This tendency was associated with many negative outcomes, including poor performance on fraction arithmetic, whole number division, and overall mathematics achievement test scores. Relying on the numerator or denominator alone also overlapped with LA classification and with the tendency to treat numerators and denominators in fraction arithmetic problems as independent whole numbers.

Viewing fraction magnitudes as independent whole numbers is understandable, because fractions with larger numerators do tend to be larger, as do fractions with smaller denominators. Moreover, the approach is appropriate for fraction multiplication and yields correct answers to some other classes of problems that are often presented in textbooks, for example comparing fraction magnitudes with equal denominators (e.g., 1/5 versus 3/5) or equal numerators (e.g., 3/5 versus 3/7). However, it is not the right way to view fractions, and does not yield consistently correct performance across unrestricted sets of fraction comparison or arithmetic problems. Placing greater instructional emphasis on the need to view each fraction as an integrated magnitude that expresses the relation between its numerator and its denominator might avoid subsequent difficulties not only in fraction arithmetic but in learning of mathematics more generally.

Acknowledgments

References

Bailey DH, Hoard MK, Nugent L, Geary DC. Competence with fractions predicts gains in mathematics achievement. Journal of Experimental Child Psychology. 2012

Bull R, Scerif G. Executive functioning as a predictor of children's mathematics ability: Inhibition, switching, and working memory. Developmental Neuropsychology. 2001; 19(3):273–293. [PubMed: 11758669]

Chong SL, Siegel LS. Stability of computational deficits in math learning disability from second through fifth grades. Developmental Neuropsychology. 2008; 33(3):300–317. [PubMed: 18473201]

Desoete A, Roeyers H, Buysse A. Metacognition and mathematical problem solving in Grade 3. Journal of Learning Disabilities. 2001; 34(5):435–449. [PubMed: 15503592]

Fazio LK, Bailey DH, Thompson CA, Siegler RS. Relations of different types of numerical magnitude representations to each other and to mathematics achievement. (under review).

Fuchs LS, Compton DL, Fuchs D, Paulsen K, Bryant JD, Hamlett CL. Responsiveness to intervention: Preventing and identifying mathematics disability. Teaching Exceptional Children. 2005; 37(4):60–63.

Garrett AJ, Mazzocco MMM, Baker L. Development of the metacognitive skills of prediction and evaluation in children with or without math disability. Learning Disabilities Research & Practice. 2006; 21(2):77–88. [PubMed: 20084181]

Hanich LB, Jordan NC, Kaplan D, Dick J. Performance across different areas of mathematical cognition in children with learning difficulties. Journal of Educational Psychology. 2001; 93(3):615–626.

Hecht SA. Toward an information-processing account of individual differences in fraction skills. Journal of Educational Psychology. 1998; 90:545–559.

Hecht SA, Close L, Santisi M. Sources of individual differences in fraction skills. Journal of Experimental Child Psychology. 2003; 86:277–302. [PubMed: 14623213]

Hecht SA, Vagi KJ. Sources of group and individual differences in emerging fraction skills. Journal of Educational Psychology. 2010; 102(4):843–858. [PubMed: 21170171]

Hiebert, J.; LeFevre, P. Conceptual and procedural knowledge in mathematics: An introductory analysis. In: Hiebert, J., editor. Conceptual and procedural knowledge: The case of mathematics. Hillsdale, NJ: Erlbaum; 1986. p. 1-27.

Jordan NC, Hanich LB, Kaplan D. A longitudinal study of mathematical competencies in children with specific mathematics difficulties versus children with comorbid mathematics and reading difficulties. Child Development. 2003; 74(3):834–850. [PubMed: 12795393]

Kloosterman P. Mathematics skills of 17-year-olds in the United States: 1978 to 2004. Journal for Research in Mathematics Education. 2010; 41(1):20–51.

Luncangeli D, Cornoldi C. Mathematics and metacognition: What is the nature of the relationship? Mathematical Cognition. 1997; 3(2):121–139.

Mazzocco, MMM. Defining and differentiating mathematical learning disabilities and difficulties. In: Berch &, DB.; Mazzocco, MMM., editors. Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities. Baltimore, MD: Paul H Brookes Publishing; 2007.

Mazzocco MMM, Devlin KT. Parts and 'holes': Gaps in rational number sense among children with vs. without mathematical learning disabilities. Developmental Science. 2008; 11(5):681–691. [PubMed: 18801123]

Mazzocco MMM, Kover ST. A longitudinal assessment of executive function skills and their association with math performance. Child Neuropsychology. 2007; 13(1):18–45. [PubMed: 17364562]

McCloskey, M. Quantitative literacy and developmental dyscalculias. In: Berch, DB.; Mazzocco, MMM., editors. Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities. Baltimore, MD: Paul H Brookes Publishing; 2007. p. 415-429.

Miyake A, Friedman NP, Emerson MJ, Witzki AH, Howerter A. The unity and diversity of executive functions and their contributions to complex 'frontal lobe' tasks: A latent variable analysis. Cognitive Psychology. 2000; 41(1):49–100. [PubMed: 10945922]

National Council of Teachers of Mathematics (NCTM). Curriculum focal points for prekindergarten through grade 8 mathematics. Washington, DC: National Council of Teachers of Mathematics; 2006. Pdf available at http://www.nctm.org/focalpoints/down-loads.asp

National Mathematics Advisory Panel (NMAP). Foundations for success: The final report of the National Mathematics Advisory Panel. Washington, DC: U.S. Department of Education; 2008.

National Opinion Research Center (NORC). Final report on the National Survey of Algebra Teachers for the National Math Panel. NORC at the University of Chicago; 2007.

Ni YJ, Zhou YD. Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. Educational Psychologist. 2005; 40(1):27–52.

Passolunghi MC, Pazzaglia F. Individual differences in memory updating in relation to arithmetic problem solving. Learning and Individual Differences. 2004; 14(4):219–230.

Passolunghi MC, Pazzaglia F. A comparison of updating processes in children good or poor in arithmetic word problem-solving. Learning and Individual Differences. 2005; 15(4):257–269.

Reyna, VF.; Chapman, SB.; Dougherty, M.; Confrey, J. The adolescent brain: Learning, reasoning, and decision making. Washington, DC: American Psychological Association; 2011.

Siegler RS, Duncan GJ, Davis-Kean PE, Duckworth K, Claessens A, Engel M, Chen M. Early predictors of high school mathematics achievement. Psychological Science. 2012; 23(7):691–697. [PubMed: 22700332]

Siegler RS, Thompson CA, Schneider M. An integrated theory of whole number and fractions development. Cognitive Development. 2011; 62:273–296.

Swanson HL, Beebe-Frankenberger M. The relationship between working memory and mathematical problem solving in children at risk and not at risk for serious math difficulties. Journal of Educational Psychology. 2004; 96(3):471–491.

Vamvakoussi X, Vosniadou S. Understanding the structure of the set of rational numbers: a conceptual change approach. Learning and Instruction. 2004; 14(5):453–467.

Table 1

Percent Correct on Fraction Arithmetic by Problem Type for Low Achieving (LA) and Higher Achieving (HA) Students

Operation	Denominators	Achievement Status	
		LA	HA
Addition	Equal	63	87
	Unequal	12	74
Subtraction	Equal	75	91
	Unequal	28	77
Multiplication	Equal	40	34
	Unequal	55	61
Division	Equal	12	36
	Unequal	6	27

Table 2

Gains in Fraction Arithmetic Accuracy from 6th to 8th Grade Among Low Achieving (LA) and Higher Achieving (HA) Children

Operation	Denominators	% Accuracy Gain (8th minus 6th)	
		LA	HA
Addition	Equal	6	12
	Unequal	4	22[*]
Subtraction	Equal	10	13
	Unequal	3	20[*]
Multiplication	Equal	6	20[*]
	Unequal	9	31[*]
Division	Equal	11	36[*]
	Unequal	11	40[*]

Note: Gains = 8th grade – 6th grade percent correct.

[*] indicates significance at $p < .05$

Table 3

Correlations among Experimental Tasks and Math Achievement Tests for Grade 6 (above diagonal) and *Grade 8 (below diagonal)*

| | Math Achievement (PSSA) | Reading Achievement (PSSA) | Fraction Magnitude Comparison | Number line 0–1 (PAE) | Number line 0–5 (PAE) | Fraction Arithmetic | | | Whole Number Division | Executive Function | |
						Correct Strategy Use	Answer Accuracy	Basic Calculation Errors		Update (Letter Memory)	Inhibit (Anti-Saccade)
Math Achievement (PSSA)	1	.600**	.456**	-.477**	-.499**	.164	.234	-.470**	.466**	.010	.345**
Reading Achievement (PSSA)	.745**•	1	.460**	-.315*	-.361*	.351**	.369**	-.405**	.299*	.206	.149
Fraction Magnitude Comparison	.567**	.427**	1	-.493**	-.318*	.524**	.559**	-.519**	.418**	.088	.123
Number line 0–1 (PAE)	-.763**•	-.651**•	-.568**	1	.527**•	-.266*	-.295*	.271*	-.341**	-.295*	-.109
Number line 0–5 (PAE)	-.628**	-.507**	-.476**	.457**	1	-.325*	-.343**	.368**	-.484**	-.357**	-.190
Fraction Arithmetic Correct Strategy	.621**•	.579**	.455**	-.483**	-.507**	1	.960**	-.586**	.393**	.256*	.137
Fraction Arithmetic Answer Accuracy	.615**•	.577**	.405**	-.433**	-.540**	.917**	1	-.626**	.422**	.243	.076
Fraction Arithmetic Calculation Errors	-.540**	-.561**	-.383**	.470**	.353**	-.493**	-.635**	1	-.348**	-.136	-.230
Whole Number Division	.703*•	.603**•	.421**	-.708**•	-.488**	.656**•	.628**	-.562**	1	.287*	.231
EF: Update (Letter Memory)	.507**•	.569**	.306*	-.550**	-.457**	.363**	.446**	-.300*	.392**	1	.140
EF: Inhibit (Anti-Saccade)	.306*	.277*	.228	-.381**	-.354**	.248	.265*	-.332**	.387**	.364**	1

Notes: N=60 per grade (N=56 per grade for PSSA achievement correlations)

*
p<.05;

**
p<.01.