Proc. Natl. Acad. Sci. USA
Vol. 76, No. 12, pp. 6030–6034, December 1979
Applied Mathematical Sciences

# Dynamics of technological evolution: Random walk model for the research enterprise

(dimensionless constants/scaling)

ELLIOTT W. MONTROLL* AND KURT E. SHULER†

*Institute for Fundamental Studies, Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627; and †Department of Chemistry, University of California, San Diego, La Jolla, California 92093

*Contributed by Elliott W. Montroll, September 27, 1979*

**ABSTRACT** Technological evolution is a consequence of a sequence of replacements. The development of a new technology generally follows from model testing of the basic ideas on a small scale. Traditional technologies such as aerodynamics and naval architecture involved feasibility experiments on systems characterized by only one or two dimensionless constants. Technologies of the "future" such as magnetically confined fusion depend upon many coupled dimensionless constants. Research and development is modeled and analyzed in terms of random walks in appropriate dimensionless constant space.

The word "*re*search" connotes an activity that does not yield its sought goal upon one's first attempt to achieve it. Hence that activity is endowed with a random component. Its proper description must characterize the stochastic process that generates the random component. The aim of this paper is to model that process by a random walk (or flight) in an appropriate space. This is a fourth report based on considerations of the nature of technological evolution (1–3). It has been emphasized (2) that the application of technology to the industrial arts and to our life style has evolved through a sequence of replacements.

The continuing improvement of any technology eventually becomes limited by some physical principal (3) so that a new technology overtakes the old by becoming more cost effective and permitting a broader range of operating characteristics (greater speed, broader bandwidth, etc.). The speed of a traditional displacement ship is limited by the dissipation of available power into bow wave formation at the expense of increased thrust in the high-speed regime. The memory capacity of the old vacuum tube computers was limited by statistics of tube lifetimes.

It was also observed that feasibility studies for successful quickly developed old technologies such as aerodynamics and nuclear fission reactors were expedited through model tests of systems characterized by only one or two dimensionless constants. Those technologies that always seem to be a technology of the future, such as magnetically confined fusion, require many coupled dimensionless constants for their characterization.

We (*i*) review the nature of model testing and the importance of exhibiting results in terms of dimensionless constants to provide scaling laws; (*ii*) emphasize the tyranny of many dimensionless constants in the investigation of processes involving many highly correlated variables: and (*iii*) model the development of a technology as the consequence of a random walk in the space of the dimensionless constants appropriate to that technology.

The theories upon which new technologies are based often involve sets of nonlinear partial differential equations subject to complicated boundary conditions. With today's state of the art they generally cannot replace experimental feasibility studies. Equations serve as a guide to experiment rather than as a touchstone to conclusions. We show through hydrodynamic examples how basic nonlinear equations aid in design of experiments and in appreciation of the complexity of a technology.

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## USE OF DIMENSIONLESS CONSTANTS IN DESIGN OF MODELING EXPERIMENTS

Students of Stokes and Rayleigh generally admire their employment of dimensional analysis to deduce simple explanations of complex phenomenon. Stokes derived his law for the drag force on a sphere pulled with a velocity $v$ through a fluid, and Rayleigh introduced his explanation of the blue of the sky by dimensional analysis. It was a natural next step from dimensionless constants to engineering model testing. Without modeling, the design of airplanes, ships, harbors and even certain electric devices would be impossible.

We review the theoretical basis and the practice of modeling strategy in terms of the Navier–Stokes equation for the flow field of an incompressible fluid. Let $v \equiv v(r,t)$ be the velocity of a fluid element at $r$ at time $t$ of an incompressible fluid of density $\rho$ and with kinematic viscosity $\nu$. The Navier–Stokes equation is

$$\partial v/\partial t + v \cdot \nabla v = -\nabla(p/\rho) + \nu\nabla^2 v + F/\rho, \qquad [1]$$

$p \equiv p(r,t)$ being the pressure and $F$ the extreme force on the fluid element. The equation of continuity for an incompressible fluid is $\nabla \cdot v = 0$. We specialize the force to be gravitational with $F/\rho = g$.

Scaling theory (4) is based on the transformation of Eq. 1 to an equivalent equation for dimensionless quantities. The velocity, pressure, and even $g$ are all variables with dimensions—their values depend on the units chosen. To obtain a dimensionless equivalent of Eq. 1, local velocities and pressures are measured as multiples of some important basic dimensions of the object responsible for the flow pattern. Consider the flow of water around a moving ship. Let

$V$ = average velocity of body being investigated,
$L$ = an important length (say the length of the ship), and
$P$ = average pressure in absence of body.

Then we can define a set of dimensionless quantities $u'$, $p'$, $x'$, etc. by

$$v = Vv' \qquad x = Lx' \qquad p = Pp'. \qquad [2]$$

If we are concerned with steady flows, $\partial v/\partial t = 0$ in Eq. 1, and

$$\nabla' = i\partial/\partial x' + \ldots = L\nabla \qquad [3]$$

transforms Eq. 1 to

$$v' \cdot \nabla v' = -P\nabla' p' + (1/R)\nabla'^2 v' + (1/F), \text{ with} \quad [4]$$

$$R = \text{Reynolds number} = VL/\nu, \quad [5a]$$

$$P = \text{pressure number} = P/\rho V^2, \text{ and} \quad [5b]$$

$$F = \text{Froude number} = V^2/Lg. \quad [5c]$$

Sommerfeld called the combination $VL/\nu$ the Reynolds number to honor Osborne Reynolds' pioneering studies on the onset of turbulence in flow of fluids through pipes. $F = V^2/Lg$ is named after William Froude, a junior naval architect under Isambard Brunel and Scott–Russell (of recently revived soliton fame) in the design of the *Great Eastern*. That great, underpowered, unprofitable iron ship (1858), from which the first successful Atlantic cable was laid, was a wonder of its time. Unfortunately, since its design required a giant leap from the state of the art, it was plagued by numerous engineering and management faults (5) (including poor cost estimation, a common curse on giant leaps). Froude's experiences with the *Great Eastern* motivated him to consider the possibility of estimating power requirements for ships from model tests.

In certain flow regimes two out of the three terms on the right-hand side of Eq. 4 are negligible relative to the third. Suppose only the $1/F$ term need be retained. Then the flow field and engineering design parameters depending upon the flow field would be a function of only $F$; small scale model experiments could yield design data for full-scale engineering.

For a 1000-foot ship operating at 40 feet/sec, with pressures being measured in units of atmospheric pressure (using the kinematic coefficient of viscosity of water at 15°C, $\nu = 1.23 \times 10^{-5}$ foot$^2$/sec), $1/F = 20$, $P = 0.69$, and $R^{-1} = 10^{-9}$. Hence, only the $1/F$ term need be retained on the right-hand side of Eq. 4. Then, ship modeling can, to a first approximation, be based on Froude modeling; that is, modeling with a dimensionless constant that depends on $g$.

A 10-foot ship model moving 4 feet/sec has the Froude number of a 1000-foot real ship at 40 feet/sec. Hence, by plotting the ratio of pounds of resistance per ton of displacement (a dimensionless quantity) of a 10-foot model towed at 4 feet/sec in a towing tank as a function of $1/F$, one can determine the power required to overcome the resistance expected by the full-scale ship.

In aerodynamics the first term on the right-hand side of Eq. 4 is most important. Consider an airplane with a wing of width 10 feet designed to operate at a speed of 800 feet/sec (about 545 miles/hr). Measuring the pressure in atmospheres (and using the kinematic coefficient of viscosity of air at 15°C, $\nu = 1.59 \times 10^{-4}$ foot$^2$/sec), $1/F = 5 \times 10^{-4}$, $P = 1.45$, and $R^{-1} = 2 \times 10^{-8}$.

If, as suggested by these numbers, we need retain only the first term on the right-hand side of Eq. 4, the resulting equation is the Bernoulli equation of a nonviscous fluid $\nabla[\frac{1}{2}v^2 + p(\rho)] = 0$. Since the pressure difference between the bottom and top of the wing section of an airplane, as developed by circulation of air around the wing, determines the "lift" of the wing, it is not surprising that the pressure term is most important in our regime of interest. A wind tunnel (4) is the traditional device for measuring the lift and drag (and their ratio) on a model airplane in a flow stream. Since the length $L$ does not enter into the pressure number, the lift-to-drag ratio would be the same on a small airplane model as on a full-sized object of the same shape.

## TYRANNY OF MANY DIMENSIONLESS CONSTANTS

We have noted that the design of airplanes and ships, and research on flow of fluids through pipes, are expedited by model experiments on systems characterized by a single dimensionless constant.

It is remarkable that within 10 years of the Wright brothers' first motor-powered flight, Igor Sikorsky (1914) built a successful four-engined giant, his *Ilya Mourometz*, capable of remaining airborne for 6½ hr, carrying six passengers. Twenty-one years after Kitty Hawk, the Imperial Airways flew Handley–Page airliners on routes from Cairo to distant parts of Africa and India. The 1928 Handley–Page 42s seated 38 passengers. The interval from Enrico Fermi's Stagg Field experiments on sustained fission (1942) to the first commercial nuclear power plant was about 15 years. On May 31, 1935, Robert Goddard fired a rocket vertically to an altitude of more than 1 mile (1.6 km). Twenty-six years later a Soviet cosmonaut encircled the earth in a rocket-launched artificial satellite.

Such successes can easily hypnotize one to believe that with a little money and ingenuity any scientific goal is achievable. Unfortunately, this is not always true and sometimes, even if it is true for a particular goal, the time scale may not be fully appreciated. Consider Project Sherwood (1951), the initiation of the U.S. program on magnetically confined fusion. Since 1/6500 of the hydrogen in sea water is the deuterium isotope, it was believed by optimists that with the success of the program our energy problems would be solved. Unfortunately, 28 years and hundreds of millions of dollars later, energy by magnetically confined fusion still remains a technology of the future. What has happened? Why has this branch of physics failed to live up to expectations?

We contend that the magnetically confined fusion program has fallen victim to the tyranny of many dimensionless constants. Old great engineering successes involved processes that could, to a first approximation, be characterized by a small number of dimensionless constants. Hence only a small number of model experiments sufficed to establish feasibility and to estimate the cost and difficulties to be surmounted. Even the space program was compartmentalized into numerous independent subprojects, each being analyzed in terms of a small number of dimensionless constants. The combined results of many modeling experiments then formed a basis for full-scale engineering designs.

A complication of magnetically confined fusion seems to be that at least eight hydrodynamic, electromagnetic, and nuclear dimensionless constants are intimately connected in the process of transforming a low-density low-temperature plasma to a higher-density very-high-temperature plasma. Since, as we shall now indicate, the cost involved in, or the time required for, the understanding of the nature of a process characterized by $N$ interacting dimensionless constants can be expected to grow exponentially with $N$, we should not be surprised with the slow progress in the field of magnetically confined fusion.

Let $N$ be the number of dimensionless constants required to characterize a process. Then an experimental program must sample $n_1 \times n_2 \times \ldots \times n_N$ points in the $N$-dimensional space of characterization. The cost of the program $P$ should be proportional to the number of sampling tests; i.e.,

$$P = kn_1 \times n_2 \times \ldots \times n_N = k' \exp N\left\{\frac{1}{N}\sum_{j=1}^{N} \log n_j\right\}. \quad [6]$$

Hence, if $\lambda$ is the average value of the logarithm of the number of observations for each dimensionless constant, $P = k \exp N\lambda$ as was suggested.

The genius of individual inventors sometimes allows them to cut costs and time by going directly to the correct regime of the dimensionless constant of interest without conducting model tests over a broad range. A probabilistic argument similar to that given above indicates (1) that the probability of an individual's being identified as a genius by going "directly to the point" in the development of a technology that involves $N$ connected dimensionless constants decreases exponentially with $N$.

Consider now the search problem of starting at a point in the space of the dimensionless constants relevant for the development of a new process and proceeding to the location of the operative regime of the system of interest. For the construction of an abstract search model we divide the space of dimensionless constants into cells, each cell representing an experiment or observation. After each observation one moves to a neighboring cell and makes a new observation. This is continued until a cell is reached that yields the observation that indicates that a system will be operative with the set of dimensionless constants appropriate to that cell. In our first primative model we assume that no special forces exist to give a special direction to the search. A next model would include the effectiveness of clues.

To develop some intuition about our search process we list some theorems concerning random walks on lattices and indicate their significance for our enterprise. Each cell corresponds to a lattice point in the following random walk results.

## RANDOM WALKS ON SPACE LATTICES

Consider a simple hypercubic $d$-dimensional space lattice with $N^d$ lattice points, with a typical lattice point as $(s_1, s_2, \ldots s_d)$. Each $s_j$ ranges through $1, 2, \ldots, N$. We choose periodic boundary conditions

$$(s_1 + n_1N, s_2 + n_2N, \ldots, s_d + n_dN) \equiv (s_1, s_2, \ldots, s_d), \quad [7]$$

each $n_j$ being an integer positive, negative, or zero. With $d = 1$ our lattice would form a circle, with $d = 2$ a torus, etc. We define random walks on these lattices to be unbiased and postulate that a walker at any given lattice point may step with equal likelihood to any nearest neighbor point with probability $p = (2d)^{-1}$. Single steps to more distant points are prohibited. All theorems quoted here are based on this model.

An ergodic theorem exists for these walks (6). The mean number of steps required for a walker to return to his starting point is $\langle n(0)\rangle = N^d$. The number to reach a point displaced by a vector $s \neq 0$ from the origin is

$$\langle n(s)\rangle = s(N - s) \text{ for } d = 1 \text{ and } s = 1, 2, \ldots, N - 1 \quad [8a]$$

$$\langle n(s)\rangle \sim N^2(2/\pi)\log|s| \text{ for } d = 2 \quad [8b]$$

$$\langle n(s)\rangle \sim N^3\left\{P_3(0,1) - \frac{3}{2\pi|s|} + O(|s|^{-2})\right\} \text{ for } d = 3 \quad [8c]$$

with

$$P_3(0,1) = 1.51638\ldots \quad [9]$$

An interpretation of $P(0,1)$ will be given later. The $d = 2$ and $d = 3$ results are asymptotic for $|s|$ large but yet small compared with $N$. Generally for all $d$, $\langle n(s)\rangle = O(N^d)$. $N^d$ is precisely the total number of lattice points.

The above results corroborate the observation that an expert is one who has made all the possible mistakes in his field. In his search for the truth he has visited all lattice points in his relevant space.

Eqs. **8a–9** represent situations with information about the starting and end points of a walk. In the process of technological discovery, the destination is sometimes unknown. Then we must search for a target of unknown position. Our lattice being homogeneous and unbiased, the number of steps required to locate our target is equivalent to that of reaching a known target from an undefined starting point. The appropriate results for this: Let a lattice $L_d$ of $N^d$ lattice points be composed of one special trapping point and $N^d - 1$ ordinary points and suppose that a walker starts with equal likelihood at any regular point. How many steps $\langle n\rangle$ on the average are required for the walker to reach the trapping point for the first time? For large $N$ one finds (7)

$$\langle n\rangle = \begin{cases} N(N+1)/6 & d = 1 \quad [10a] \\ 2\pi^{-1}N^2\log N + 0.195056\,N^2 & \\ \quad - 0.1170 + 0(N^{-2}) & d = 2 \quad [10b] \\ P_3(0,1)N^3 + 0(N^2) & d = 3 \quad [10c] \end{cases}$$

For $d \geq 3$, $\langle n\rangle = 0(N^d)$. These results are also discussed in ref. 8.

Clearly, as $N \rightarrow \infty$ the mean number of steps required to reach any new point (or to return to the starting point) becomes infinite. A deeper insight is achieved by examining theorems concerning walks on infinite lattices. A walker who has taken many steps in a random walk of long duration but yet a number of steps small compared with the total number of lattice points has not yet "felt" the finiteness of the lattice and might as well have been walking on an infinite lattice. The delicate role of dimensionality becomes important in walks on infinite lattices.

Polya (9) noticed that on infinite $1d$ and $2d$ lattices a random walker executing our subject type of walk is certain to eventually return to his starting point. On lattices of dimension $d \geq 3$ an escape probability exists. On a $d = 3$ simple cubic lattice the probability of eventual return to the starting point is (10) [with $P_3(0,1)$ defined by Eq. **9**],

$$F_3(0,1) = 1 - [P_3(0,1)]^{-1} = 0.340537330\ldots \quad [11a]$$

with escape probability being 0.659462670. $P_3(0,1)$ is then the reciprocal of the escape probability. For $d \gg 3$ the probability of return is (10)

$$F_d(0,1) \sim \frac{1}{2d}\left\{1 + \frac{2}{2d} + \frac{7}{(2d)^2} + \frac{35}{(2d)^3} + \frac{215}{(2d)^4} + \ldots\right\}$$

$$\sim \frac{1}{2d}\left\{\frac{1 + \frac{1}{2d} + \frac{2}{(2d)^2} + \frac{11}{(2d)^3} + \frac{69}{(2d)^4}}{1 - \frac{1}{2d} - \frac{3}{(2d)^2} - \frac{11}{(2d)^3} - \frac{68}{(2d)^4}}\right\}. \quad [11b]$$

When $d = 3$ this Padé approximant type formula yields $F_3(0,1) = 0.3419$, deviating from **11a** by less than 0.5%. Return probabilities, $F_d$, are given in Table 1.

When $d = 3$, the probability of eventually reaching a preassigned point displaced by a vector $s$ from the starting point decreases as $|s| \rightarrow \infty$ as

$$F_3(s,1) \sim 0.31950\ldots |s|^{-1} \text{ with } s^2 = s_1^2 + s_2^3 + s_3^2. \quad [12]$$

For several moderate $s$ vectors we exhibit return probabilities in Table 2. Similar results (6) can be obtained for $d > 3$.

Table 1. Return probabilities, $F_d$, to starting point as a function of dimension, $d$

| | d | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $F_d$ | 0.341 | 0.189 | 0.134 | 0.104 | 0.086 | 0.073 | 0.064 | 0.056 |

Table 2. Probability, $F(1,s)$, that a walker who starts from the origin ever reaches a point $s = (s_1,s_2,s_3)$ on a simple cubic lattice

| $(s_1,s_2,s_3)$ | $F(1,s)$ | $(s_1,s_2,s_3)$ | $F(1,s)$ |
|---|---|---|---|
| 001 | 0.3405 | 023 | 0.0873 |
| 002 | 0.1697 | 111 | 0.1724 |
| 003 | 0.1090 | 112 | 0.1265 |
| 011 | 0.2184 | 113 | 0.0951 |
| 012 | 0.1422 | 122 | 0.1035 |
| 013 | 0.1010 | 123 | 0.0837 |
| 022 | 0.1110 | 223 | 0.0896 |

In this symmetrical walk $F(1;s_1,s_2,s_3) = F(1;s_2,s_1,s_3)$, etc.

It is evident from Tables 1 and 2 that for $d \geq 3$, a considerable likelihood exists for a walker starting from the origin never to reach a preassigned point only a few lattice spacings away (or even returning to the origin itself). Let us restrict ourselves to the class of successful walks connecting the origin with a prespecified point $s$.

It has been shown (11) on a five-or-more-dimensional hypercubic lattice that the average number of steps required to successfully complete the path from 0 to $s$ is

$$\langle n(s)\rangle = [d/(d-4)]|s|^2 \rightarrow |s|^2 \text{ as } d \rightarrow \infty. \quad [13]$$

The $|s|^2$ behavior is characteristic of $d - 1$ walks as is indicated by letting $N = \alpha s$ in Eq. **8a**. Then $\langle n(s)\rangle = (\alpha - 1)|s|^2$, which is equivalent to Eq. **13** if $\alpha = 2(d-2)/(d-4)$. When $d$ is very large $\alpha \rightarrow 2$. Hence the mean number of steps required by a walker who certainly reaches $s$ (starting from 0) is about the same as that required to reach a point halfway around a $1d$ ring of $2s$ lattice points. For the lower limiting value of $d = 5$, $\alpha = 6$. An alternative interpretation of Eq. **13** is (11) that the number of steps required to go from 0 to 5 for a walker certain to arrive at $s$ is the same as the number required in a unidirectional walk to the right along a line terminated on the left by a reflecting barrier.

Then, if a walker who starts from 0 on a $d \geq 5$ dimensional lattice ventures but slightly from the direct line path connecting 0 with $s$ his chance of becoming lost and never reaching $s$ becomes close to certainty. Similar remarks (11) with certain caveats are also valid for $d = 3$ and 4.

A vignette from life in harsh climes such as that of Greenland in the winter might convince the reader that qualitative features of the above observation are evident even when $d$ is as small as 2. When small villages such as Ivigtut were first settled in Greenland, houses were built several hundred or more feet apart. During blizzards accompanied by strong wind gusts, visibility was reduced to a few feet. Villagers innocently venturing from home on such occasions sometimes became lost, executing a $2d$ random walk without finding another house or even their own. They were sometimes found frozen to death within 50 feet of a dwelling or store. After several of these tragedies it became customary to connect buildings by guide ropes in the autumn to prepare for winter storms. As villages grew, more sophisticated networks evolved analogous to the early telephone networks, with central "stations" playing a role. In the spirit of our random walk theorem, in the more primitive period, a walker on stormy days had a good chance of becoming lost by venturing slightly from the direct path to his destination. In an analogous situation, projects may die from lack of funds if they stray too far from the direct path to success.

It has been said that a common difference between successful and unsuccessful scientists is that successful ones abandon foolish ideas quickly and get on to something new. Less successful ones continue to nurse and hang on to them too long without getting anywhere. This remark is in the spirit of our random walk theorem. Go directly to the target along a linear path, allowing for some fluctuations! Otherwise, it is doubtful that you will ever get there. Unfortunately this advice is difficult to follow, because the target location is frequently unknown.

## DISCUSSION

The results of the last section make one wonder how any complex system ever becomes understood and how any complicated device can be developed. Today two classes of organizations, the invisible college and the industrial (and government) research laboratories, nurture technological innovation. A typical invisible college is a collection of scientists, generally scattered, who work on similar problems. They attend conferences together, read each others reports, visit and generally stimulate each other in their investigations of the topics that fascinate them. A region in some space of dimensionless constants is their realm of inquiry.

Exploration of a region occasionally exposes a "bright spot" of special significance. Then many researchers concentrate upon the neighborhood of that spot. The sphere of understanding around it expands until some natural boundary in the dimensionless constant space impedes progress, or perhaps until someone discovers a new bright spot that distracts their attention. Sometimes new devices or deeper understanding of nature are consequences of the expansion of expertise.

A successful industrial laboratory operates similarly, except that most of its researchers are housed together to provide for easier personal cooperation and management. The management may fine tune its program to a higher degree than that attempted by the foundation and government sponsorers of the invisible colleges. The laboratory directorate attempts to choose the dimensionless constant space of inquiry so that existent bright spots yield a large probability of contributing to the improvement of old products and leading to the development of new products profitable to the supporting firm. A management that insists upon the direct achievement of a preconceived complex goal may suffer from the tyranny of many dimensionless constants and miss opportunities imbedded in regions near bright spots. One that allows the staff to indefinitely admire the bright spots may never develop a product. It is the subtle interplay between action and permissiveness that is a characteristic of progressive management.

In device and production technology cost is of course one of the first dependent variables to be considered. Technological improvement is generally associated with the discovery of a path, in the neighborhood of some bright spot in an appropriate space, along which either cost decreases or quality increases. Technological revolutions are often the consequence of finding new bright spots and exploiting them. In maximizing some essential attribute of a device or process one must be cautious of great peaks in the neighborhood of an instability such that a slight displacement from optimum becomes disastrous. Broader peaks are preferred by the more prudent.

It is sometimes possible in the early phases of the development of a technology to avoid the tyranny of many dimensionless constants through the design of a cascade with each of its stages depending upon only a small number of dimensionless constants. The stages may then be perfected independently. For example, when Kammerlingh Onnes raced James Dewar to be the first to liquify helium he designed the best nitrogen liquifier, the best hydrogen liquifier, and the best final stage equipment that he could so that the cascade performed perfectly on its first trial. As a technology advances, cascading may become less necessary. Today's integrated circuits are preferred over networks of individual components.

Social dynamics and environmental processes might also depend on a large number of dimensionless constants. The understanding of these processes is not exempt from the tyranny of many dimensionless constants, nor is an attempt to make government regulations concerning them exempt. It sometimes seems even more difficult to make a regulation work in the manner expected than to make a device perform as required.

This research was partly supported by the Advanced Research Projects Agency of the Department of Defense under contracts with the University of Michigan and the La Jolla Institute, by the National Science Foundation under Grant CHE 78-21460, and by a grant from Charles and Renée Taubman.

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