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Multiple solutions and numerical analysis to the dynamic and stationary models coupling a delayed energy balance model involving latent heat and discontinuous albedo with a deep ocean

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We study a climatologically important interaction of two of the main components of the geophysical system by adding an energy balance model for the averaged atmospheric temperature as dynamic boundary condition to a diagnostic ocean model having an additional spatial dimension. In this work, we give deeper insight than previous papers in the literature, mainly with respect to the 1990 pioneering model by Watts and Morantine. We are taking into consideration the latent heat for the two phase ocean as well as a possible delayed term. Non-uniqueness for the initial boundary value problem, uniqueness under a non-degeneracy condition and the existence of multiple stationary solutions are proved here. These multiplicity results suggest that an S-shaped bifurcation diagram should be expected to occur in this class of models generalizing previous energy balance models. The numerical method applied to the model is based

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on a finite volume scheme with nonlinear weighted essentially non-oscillatory reconstruction and Runge–Kutta total variation diminishing for time integration.

1. Introduction

This paper presents new contributions on the mathematical study of a climate model coupling atmosphere and ocean under a simplified formulation. Our main goal is to exhibit the possible multiplicity of solutions owing to presence of an abruptly distributed coalbedo, such as it was formulated in terms of a discontinuous function by the climatologist M.I. Budyko (see [1]). Among the new effects considered with respect to previous mathematical treatments in the literature, we consider here a positive latent heat for the ocean and a general memory term for the top ocean surface temperature. Moreover, we present here the numerical approximation of solutions by means of finite volume methods. We shall also indicate many other references on the mathematical treatment of this class of problems, in a survey style, trying to be useful in the necessary dialogue between geophysical and mathematician experts.

Our model tries to understand the deterministic interactions between two of the main components of the climatic system. It is well known that in detailed mathematical models of the atmosphere, the ocean and ice sheets are available (see, for instance, the proceedings of several meetings devoted to this topic, as it was the case of the [2–5]). Nevertheless, investigating inherently transient phenomena with periods of 100–100 000 years is, of course, out of question for such sophisticated models. This is one reason why simpler models form useful tools in theoretical climatology. In addition, the mathematical treatment of such models is far to be obvious and requires the application of finer techniques of the mathematical and numerical analysis of nonlinear partial differential equations.

Our model takes in account, at least implicitly, the multiple spatial scales which arise in such complex coupling. Indeed, instead of considering the atmosphere temperature we shall work, as usual in the theory of energy balance model (EBM), with the averaged surface temperature on suitable spacial and time local scales. It is well known that in spite of its simplicity, this kind of averaged equations preserve a high sensitivity with respect to solar parameters. This is very useful for the study in very large timescales. Nevertheless, because the heat capacity of the ocean is so large, any departure from equilibrium in the ocean must have a fairly large effect on the thermodynamic state of the atmosphere. As for the ocean, although we can also simplify its modelling, we must maintain the fact that cold water in a few localized regions at high latitudes sinks is distributed throughout the deep ocean by currents and slowly rises towards the surface. So, following [6], we maintain the ocean depth scale for the deep ocean and identify the ocean-mixed layer with the averaged atmospheric surface. This type of models allows us to find some explanations for the Glacial–Holocene transition (see [6]). The inclusion of some stochastic internal and external variations imperfectly known, as it is the case of solar luminosity variations, volcanic aerosols and CO_2, has already studied for the associate surface EBM [7–10].

It is clear that more realistic ocean models can be also considered in order to investigate the interactions between time and space scales of both climate subsystems: for instance, the way in which averaging processes in media with different characteristic scales may produce the presence of memory terms in the averaged equations can be found in many texts [11]. This explains why different delayed terms, or more generally speaking non-local terms, may arise in the modelling of the EBMs owing to the own averaging method (see, e.g. the exposition made in [8] and the mathematical treatment made in [12,13]). But, sometimes, the presence of some memory terms can be argued from other modelling proposes (for instance, Bhattacharya et al. [14] study the effects of a delayed term to take into consideration the important role of the clouds on the albedo). Moreover, there are many reasons to consider the occurrence of a general delay term in some of the differential equations. For instance, a different justification can be obtained by regarding some others key phenomena, such as the El Niño/southern oscillation (ENSO) in the

tropical Pacific and its implications in the climate's interannual variability and in global warming. Although in this case the timescale must be shorter than in other palaeoclimate models, we recall that in many previous EBMs seasonal effects have been taken into account (for instance, the insolation function $S(t, x)$ is then taken as time-dependent) and so, some justification of the past glaciations were obtained in [15]. Here, we shall only include a delayed term in the deep ocean temperature equation following the approach initiated in the papers [16–19], among others, to simulate seasonally varying internal parameters.

As said before, even if the model under consideration responds to simplified modelling arguments, the presence of several nonlinear terms, some of them not always differentiable, makes that its mathematical treatment cannot be reduced to the mere direct application of the differential delayed equations (DDEs) theory [20,21]. In §2, we state the model under consideration and the main structural and constitutive assumptions. The study of solutions of the transient regime is presented in §3. Because there is no hope to obtain classical solutions of the system, we introduce the notion of weak solution we shall deal with. We prove the existence of such type of solutions under quite general assumptions on the data and, which is more unusual in the study of parabolic type systems, we prove that, in general, there is no uniqueness of solutions when the coalbedo is assumed to be discontinuous. Because we also prove that, in this case, there is a continuum of solutions for suitable initial data, it is not possible to apply the results of the classical bifurcation theory for transient systems. Instead of that, we prove the uniqueness of weak non-degenerate solutions (corresponding to the case in which the atmospheric surface temperature arrives not too flat near the boundary of the region where the coalbedo changes, i.e. on the surface where it becomes abruptly discontinuous). Let us recall that new aspects that have been taken into consideration in this type of coupled models are the ocean latent heat and the presence of a memory term.

Once we know the global existence of solutions on any arbitrary time horizon T, it can be proved (see [22,23] for a special case of the present system) that the assumptions made here on the data exclude any other elements in the omega limit set (when $t \to +\infty$) different from the solutions of the stationary system. Perhaps this is the moment to point out that in many other systems the memory may lead to different qualitative properties of solutions with respect to the same system but without memory. We shall not develop this approach here, but we refer the reader to a series of papers where this philosophy was carried out for different types of delayed systems (see [24–26]).

Coming back to the consideration of the associated stationary system, we show, in §4, the multiplicity of solutions in terms of the solar constant Q. Again, our result is not an automatic application of the general bifurcation theory but requires the construction of suitable families of super and subsolutions well adapted to our setting. An S-shaped bifurcation curve can be obtained in some special cases (see [27]).

The above-mentioned mathematical analysis of the model allows to start the study of the controllability of some models connected with the climate system and, in particular with EBMs and related problems (see [28,29] for the case of a single EBM equation and [30,31] for some related problems). Moreover, it is possible to obtain a mathematical meaning to the proposals already present in some late works by von Neumann (see [32,33]).

Finally, in §5, we present several numerical experiments on the coupled model by means of a finite volume approach with weighted essentially non-oscillatory (WENO)-7 spatial reconstruction and third-order TVD Runge–Kutta for time discretization (for the application of the finite-element method, see [34]). We compare the numerical solution of the model with and without the effect of the ocean latent heat, and we also present a numerical experiment carried out by considering the effects of the memory term. Although the data in such experiments could be made more realistic, we think that the main value of such a numerical approach is to show how it is possible to make accessible to the quantification some sophisticated mathematical analysis of complex nonlinear systems, involving, for instance discontinuous albedo data, for which the solutions satisfy the requirements only in a weak sense. As Jacques-Louis Lions (1928–2001) said:

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2. The mathematical model

EBMs were introduced, independently by Budyko [1] and Sellers [36] (some pioneering model is due to S. Arrhenius in 1896). Such type of climatological models has a diagnostic character and intended to understand the evolution of the global climate on a long timescale [8,37,38]. Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters. They have been used in the study of the *Milankovitch theory* of the ice-ages [15].

The EBMs study a distribution of surface atmospheric temperature, $u(t, x)$, which is expressed pointwise after some averaging process in space (the spatial variable x is in a small neighbourhood $B_\delta(x)$ in the Earth's surface) and in time (on a small interval $(t - \tilde{t}, t + \tilde{t})$)

$$u(t, x) = \frac{1}{2\tilde{t}|B_\delta(x)|} \int_{t-\tilde{t}}^{t+\tilde{t}} \int_{B_\delta(x)} T(a, s) \, \mathrm{d}a \, \mathrm{d}s.$$

The pointwise temperature $T(a, s)$ is obtained from the thermodynamics equation of the *atmosphere primitive equations* (see e.g. [39] for a mathematical study of those equations and [12,40] for the application of averaging processes in this context).

More simply, the energy balance model can be formulated by using the energy balance on the Earth's surface: internal energy flux variation $= R_a - R_e + D$, where R_a (respectively R_e) represents the absorbed solar (resp. the emitted terrestrial energy flux) and where D is the surface heat diffusion. By identifying the Earth's surface with a compact Riemannian manifold without boundary $\mathcal{M}$ (for instance, the sphere S^2 in $\mathbb{R}^3$), the distribution of temperature, $u(t, x)$, becomes a function of the spatial x and t time variables. The timescale is considered relatively long. The *absorbed energy* R_a depends on the planetary *coalbedo* β. The coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflection is greater than over oceans, therefore, the coalbedo is smaller. One observes that there is a sharp transition between zones of high and low coalbedo. In the energy balance climate models, a main change of the coalbedo occurs in a neighbourhood of a critical temperature for which ice become white, usually taken as $u = -10^\circ$C. The coalbedo can be modelled by different monotone increasing functions (discontinuous in the case of *Budyko model* and Lipschitz continuous for *Sellers model*). A more realistic albedo parametrization can be obtained by assuming that the coalbedo function β also depends on the spatial coordinates of each point of the Earth (specially on its latitude: see [38, §3.3]). Here, we mainly consider the Budyko model because it produces more clear answers when one studies the evolution of the icecaps.

With respect to the surface temperature diffusion, we send the reader to the modelling performed for instance in [8] for the case of a linear second-order differential operator. Nevertheless, a quasi-linear diffusion operator of the type $\mathrm{div}(k(x, u, \nabla u)\nabla u)$ was proposed in Stone [41] as a better eddy diffusive approximation to account for the effect of large-scale atmospheric circulation, where $k(x, u, \nabla u)$ is a nonlinear eddy diffusion coefficient, in particular, $k = b(x)|\nabla u|$. In our model, we shall follow Stone's approach to represent the eddy diffusive terms by setting $k(x, u, \nabla u) = k(x)|\nabla u|^{p-2}$, with $p \geq 2$ and $k(x) > \alpha > 0$.

With respect to the simplified model on the deep ocean, we shall follow the modelling derived in [6] but adding a positive latent heat, γ, which plays an important role in the formation of ice sheets. With respect to the memory term, we recall that such type of terms were proposed for the study of ENSO events. For instance, in [17], it is taken $\mathcal{G}(t, x, u, u(t - \tau)) = -u + u^3 + \alpha u(t - \tau)$, for some $\alpha, \tau > 0$. We could also include some memory terms inside the albedo and latent heat expressions (as in [42]), but the detailed mathematical treatment is much more technical. Note that because u will be a globally bounded function, without loss of generality, we can modify the previous example function outside a compact of $\mathbb{R}^2$ (concerning the values of u and $u(t - \tau)$) in order to obtain a globally Lipschitz function. Obviously, the case $\mathcal{G}(t, x, u, u(t - \tau)) = \mathcal{G}(t, x, u)$ represents the case without delayed effects, such as it was considered by many previous authors.

Note also, that if $\tau > 0$, then the initial condition for the unknown u needs to be given on the set $[-\tau, 0] \times \mathcal{M}$.

Summarizing, our model will represent the interior and surface temperature of a global ocean Ω, so that, the unknown are respectively given by $U : \Omega \times [0, T] \to \mathbb{R}$ and by $u : \mathcal{M} \times [-\tau, T] \to \mathbb{R}$, for an arbitrary $T > 0$. Here, we assume

(H1) Ω is a bounded and open set of $\mathbb{R}^3$ with maximum depth H and $\partial\Omega = \mathcal{M} \cup \mathcal{N}$. $\mathcal{M}$ and $\mathcal{N}$ are C^∞ two-dimensional compact connected orientated Riemannian manifold of $\mathbb{R}^3$ without boundary and $\mathrm{dist}(\mathcal{M}, \mathcal{N}) = H$.

Let (P_{3D}) be the problem

$$\frac{\partial \gamma(U)}{\partial t} - \mathrm{div}(\nabla U) + w \frac{\partial U}{\partial z} \ni 0 \quad \text{in } (0, T) \times \Omega,$$

$$\frac{\partial u}{\partial t} - \mathrm{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + \frac{\partial U}{\partial n} + F(x, \nabla_{\mathcal{M}} u) + \mathcal{G}(t, x, u, u(t - \tau))$$

$$\in \frac{1}{\rho c} Q S(t, x) \beta(u) + f(t, x) \quad \text{in } (0, T) \times \mathcal{M},$$

$$U|_{[0,T] \times \mathcal{M}} = u,$$

$$\hat{F}(x, \nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} = 0 \quad \text{in } (0, T) \times \mathcal{N},$$

$$U(0, x, z) = U_0(x, z) \quad \text{in } \Omega,$$

$$u(s, \cdot) = u_0(s, \cdot) \quad \text{on } [-\tau, 0] \times \mathcal{M}.$$

Here, $\nabla_{\mathcal{M}}$ and div are understood in the sense of the Riemannian metric on $\mathcal{M}$ [39,43]. The rest of structural conditions are the following:

(H2) β is a bounded maximal monotone graph, i.e. $|v| \leq M$ for all $v \in \beta(s)$, and all $s \in D(\beta) = \mathbb{R}$.

(H3) γ is the graph

$$\gamma(s) = \begin{cases} k_1 s & \text{if } s < 0, \\ [0, L] & \text{if } s = 0, \\ k_2 s + L & \text{if } s > 0, \end{cases}$$

with $k_1 > 0$, $k_2 > 0$ and $L > 0$.

(H4) $\mathcal{G} : (0, T) \times \mathcal{M} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{G}(t, x, \sigma, \eta)$ is a globally Lipschitz function with respect to σ (i.e. u) and η (i.e. $u(t - \tau)$), such that $\mathcal{G}(t, x, 0, 0) = 0$ and $|\mathcal{G}(t, x, \sigma, \eta)| \geq C(|\sigma|^r + |\eta|^r)$ for some $r > 0$. Moreover, $\tau \geq 0$.

(H5) $S : (0, T) \times \mathcal{M} \to \mathbb{R}$, $S_1 \geq S(t, x) \geq S_0 > 0$ a.e. $x \in \mathcal{M}$.

(H6) $f \in L^\infty((0, T) \times \Omega)$.

(H7) $F : \mathcal{M} \times T\mathcal{M} \to \mathbb{R}$ and $\hat{F} : \mathcal{N} \times T\mathcal{N} \to \mathbb{R}$ are linear on the tangent bundle spaces $T\mathcal{M}$ and $T\mathcal{N}$ with bounded coefficients.

(H8) $w \in C^1(\bar{\Omega})$.

Remark 2.1. We point out that, for the sake of simplicity, we have assumed here isotropic (and constant) diffusion matrices in both equations. The mathematical treatment of the case of non-constant definite positive diffusion matrices is quite similar and we drop the details.

Remark 2.2. The case in which the solar constant Q is assumed, in fact, as a periodic or almost periodic time function has been intensively studied in the literature (see, e.g. [8,44] and its many references).

We can reduce the dimension of the model by assuming that the surface $\mathcal{M}$ is a sphere simulating the Earth surface and that the temperature is constant over each parallel. So, the solution of the obtained model depends only on latitude, depth and time. For different

purposes, it is useful to particularize the above system to the simpler case of a one-dimensional EBM (the surface temperature is defined on $[0, T] \times [-1, 1]$: i.e. here $\mathcal{M} = \{(x, 0) : x \in [-1, 1]\} := \Gamma_0$) coupled with a two-dimensional deep ocean ($\Omega = [-1, 1] \times [0, -H]$, and so of boundaries $\mathcal{N} = \{(x, -H) : x \in [-1, 1]\} := \Gamma_H$, $\Gamma_{-1} := \{(-1, z) : z \in [-H, 0]\}$ and $\Gamma_1 := \{(1, z) : z \in [-H, 0]\}$). The resulting equations of the model (this time with non-isotropic diffusion coefficients and with $F(x, \nabla_{\mathcal{M}} u) := wx u_x$ and $\hat{F}(x, \nabla_{\mathcal{N}} U) := wx U_x$ and with a parameter $D > 0$ modelling the mixed layer depth) now with $x = \sin \varphi$, φ representing the latitude, and $z \in [-H, 0]$, are the following

$$\gamma(U)_t - \left(\frac{K_H}{R^2} (1 - x^2) U_x \right)_x - K_V U_{zz} + w U_z \ni 0 \quad \text{in } \Omega \times (0, T),$$

$$wx U_x + K_V U_z = 0 \quad \text{in } \Gamma_H \times (0, T),$$

$$D u_t - \frac{D K_{H_0}}{R^2} ((1 - x^2)^{p/2} |u_x|^{p-2} u_x)_x + K_V \frac{\partial U}{\partial n} + wx u_x + \mathcal{G}(t, x, u, u(t - \tau))$$

$$\in \frac{1}{\rho c} S(t, x) Q \beta(u) + f(t, x) \quad \text{in } \Gamma_0 \times (0, T),$$

$$U|_{[0,T] \times [-1,1]} = u,$$

$$(1 - x^2) U_x = 0 \quad \text{in } (\Gamma_{-1} \times (0, T)) \cup (\Gamma_1 \times (0, T)),$$

$$U(0, x, z) = U_0(x, z) \quad \text{in } \Omega,$$

$$u(s, x, 0) = u_0(s, x, 0) \quad \text{on } [-\tau, 0] \times \Gamma_0. \quad (P_{2D})$$

Remark 2.3. We note that we can introduce the change of variable $U = \alpha(V)$, with $\alpha := \gamma^{-1}$, and then the equation in the inner ocean can be written as

$$V_t - \left(\frac{K_H}{R^2} (1 - x^2) \alpha(V)_x \right)_x - K_V \alpha(V)_{zz} + w \alpha(V)_z = 0 \quad \text{in } \Omega \times (0, T),$$

where

$$\alpha(s) = \begin{cases} \dfrac{s}{k_1} & \text{if } s < 0, \\ 0 & \text{if } 0 < s < L, \\ \dfrac{1}{k_2}(s - L) & \text{if } s > L. \end{cases} \quad (2.1)$$

The terms γ and α (as well as β) are maximal monotone graphs (see [45]). The main difference between γ and α is that γ is always multivalued (once we assume $L > 0$) although, in the atmosphere temperature equation, the coalbedo β becomes a multivalued graph only when it is associated with a discontinuous coalbedo function, such as it was proposed in [1]. This is the reason why in the previous inner ocean equation and the surface EBM it appears the symbols $\in$ and $\ni$ instead of the usual equality symbol.

3. On the evolution problem

(a) Existence of solutions

We define the functional space $V := \{u \in L^2(\mathcal{M}) : \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})\}$, where $T\mathcal{M} = \cup_{p \in \mathcal{M}} T_p M$ is the tangent bundle space (see [46]). Owing to the presence of possible multivalued graphs (associated with discontinuous functions), and the possible choice $p \neq 2$, we cannot expect to solve the system in a classical sense but only in a weak way.

We say that the pair (U, u) with $U \in C([0, T] : L^2(\Omega))$, $u \in C([-\tau, T] : L^2(\mathcal{M}))$ is a bounded weak solution of (P_{3D}) if

    (i) $(U, u) \in L^\infty((0, T) \times \Omega) \times L^\infty((-\tau, T) \times \mathcal{M}) \cap L^2(0, T : H^1(\Omega)) \times L^p(-\tau, T : V)$,

(ii) there exist $Z \in L^\infty((0,T) \times \Omega)$ and $h \in L^\infty((-\tau, T) \times \mathcal{M})$ with $Z \in \gamma(U)$ a.e. $(t,x) \in (0,T) \times \mathcal{M}$, $h \in \beta(u)$ a.e. $(t,x) \in (-\tau, T) \times \mathcal{M}$ and such that

$$\int_\Omega Z(T,x)\phi(T,x)\,dA - \int_0^T \langle \phi_t(t,x), Z(t,x)\rangle_{H^1(\Omega)\times H^1(\Omega)'}\,dt + \int_0^T \int_\Omega \nabla U \nabla \phi\,dA\,dt$$
$$+ \int_0^T \int_\Omega w\frac{\partial U}{\partial z}\phi\,dA\,dt - \int_0^T \int_\mathcal{M} \frac{\partial U}{\partial n}\phi\,dS\,dt + \int_0^T \int_\mathcal{N} \hat{F}(x, \nabla_\mathcal{N} U)\phi\,dS\,dt$$
$$= \int_\Omega U_0(x)\phi(0,x)\,dA,$$

and

$$\int_\mathcal{M} u(T,x)\psi(T,x)\,dA - \int_0^T \langle \psi_t(t,x), u(t,x)\rangle_{V'\times V}\,dt + \int_0^T \int_\mathcal{M} |\nabla u|^{p-2}\nabla u \nabla \psi\,dS\,dt$$
$$+ \int_0^T \int_\mathcal{M} \mathcal{G}(t,x,u,u(t-\tau))\psi\,dS\,dt + \int_0^T \int_\mathcal{M} \frac{\partial U}{\partial n}\psi\,dS\,dt + \int_0^T \int_\mathcal{M} F(x, \nabla_\mathcal{M} u)\psi\,dS\,dt$$
$$= \int_0^T \int_\mathcal{M} QS(t,x)h(t,x)\psi\,dA\,dt + \int_0^T \int_\mathcal{M} f\psi\,dA\,dt + \int_\mathcal{M} u_0(0,x)\psi(0,x)\,dS$$

for every test function $(\phi, \psi) \in L^2(0,T:H^1(\Omega)) \times L^p(-\tau, T:W^{1,p}(\mathcal{M}))$ such that $(\phi_t, \psi_t) \in L^2(0,T;H^1(\Omega)') \times L^{p'}(0,T;V')$. Here, $\langle, \rangle_{V'\times V}$ denotes the duality product in $V' \times V$.

Theorem 3.1. *Let $U_0 \in L^\infty(\Omega)$ and $u_0 \in C((-\tau, 0]:L^\infty(\mathcal{M}))$. Then, there exists at least a bounded weak global solution of (P_{3D}).*

Proof. We write the inner ocean equation as

$$\frac{\partial V}{\partial t} - \mathrm{div}(\nabla \alpha(V)) + w\frac{\partial \alpha(V)}{\partial z} = 0 \quad \text{in } (0,T) \times \Omega,$$

with $U = \alpha(V)$ and $\alpha := \gamma^{-1}$, as mentioned in the remark 2.3 (note that now α is singlevalued and so we do not need the symbol $\in$). We approximate the maximal monotone graph α by some smooth increasing functions α_ϵ. Then, we obtain a family of new problems, that we shall denote by (P_ϵ). The main idea to solve (P_ϵ) is to apply theorem 5.3.1 of [47] related to abstract functional equations. We shall construct an operator $\mathcal{T}_\epsilon$ and to find a fixed point of it leading to a solution of (P_ϵ). This will consist of several intermediate steps.

Step 1. For every $h \in L^\infty((0,T) \times \mathcal{M})$, we consider the problem $(P_{h,\epsilon})$ by replacing the coalbedo term in (P_ϵ) by h. The proof of the existence of solution of $(P_{h,\epsilon})$ is inspired in [48,49]. We define the vectorial operator $\mathcal{A}_\epsilon$ by $\mathcal{A}_\epsilon(U,u) \longmapsto (A_\epsilon U, Bu)$ on the domain $D(\mathcal{A}_\epsilon) = \{(U,u) \in L^2(\Omega) \times L^2(\mathcal{M}) : A_\epsilon U \in L^2(\Omega), Bu \in L^2(\mathcal{M}), \alpha_\epsilon(U)_{|\mathcal{M}} = u\}$, where

$$A_\epsilon U = -\mathrm{div}(\nabla \alpha_\epsilon(U)) + w\frac{\partial \alpha_\epsilon(U)}{\partial z},$$
$$Bu = -\mathrm{div}(|\nabla_\mathcal{M} u|^{p-2}\nabla_\mathcal{M} u) + \frac{\partial \alpha_\epsilon(U)}{\partial n} + F(x, \nabla_\mathcal{M} u).$$

We also define the operator $G(t)u := \mathcal{G}(t,x,u,u(t-\tau))$. Then, the existence of solution of $(P_{h,\epsilon})$ is a consequence of the compactness of the semigroup associated with the operator $\mathcal{A}_\epsilon(U,u)$ (through theorem 5.3.1 of [47]) and the results of [48,49] leading, up very small variations, to the following properties of $\mathcal{A}_\epsilon$.

Lemma 3.2. *There exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, we have*

(i) $\mathcal{A}_\epsilon + \lambda I$ *is T-accretive in $L^1(\Omega) \times L^2(\mathcal{M})$.*
(ii) $R(\mathcal{A}_\epsilon + \lambda I) = L^1(\Omega) \times L^2(\mathcal{M})$.

Note that (i) allows us to prove a comparison principle for the system

$$\left.\begin{array}{l} \lambda U + A_\epsilon U = f \quad \text{in } L^1(\Omega), \\[4pt] \lambda u + Bu = g \quad \text{in } L^2(\mathcal{M}), \\[4pt] \alpha_\epsilon(U)_{|\mathcal{M}} = u \\[4pt] \hat{F}(x, \nabla_{\mathcal{N}} \alpha_\epsilon(U)) + \dfrac{\partial \alpha_\epsilon(U)}{\partial z} = 0 \quad \mathcal{N}. \end{array}\right\} \tag{3.1}$$

and

In fact, if $f_1 \le f_2$ and $g_1 \le g_2$, then the solutions of (3.1) with $f = f_1$, $g = g_1$ and of (3.1) with $f = f_2$, $g = g_2$ satisfy $U_1 \le U_2$ and $u_1 \le u_2$.

The small variation with respect to the proof given in [49] concerns the proof of (ii) in lemma 3.2. We note that the operator B can be expressed as $B_1 + B_2 + B_3$, where B_1 and B_2 are maximal monotone operators in $L^2(\mathcal{M})$,

$$B_1 u = -\text{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u)$$

and the pseudo-differential operator $B_2 u = \partial \alpha_\epsilon(U)/\partial n$, where U is the solution of the problem

$$\lambda U + A_\epsilon U = f \quad \text{in } L^2(\Omega)$$

$$\alpha_\epsilon(U)_{|\mathcal{M}} = u.$$

The operator B_3 is defined by $B_3 u = F(\nabla_{\mathcal{M}} u)$. This operator is not necessarily monotone but it is dominated (in the usual sense: see [45]) by the operators B_1 and B_2. Consequently, it is possible to apply the abstract results of perturbation of maximal monotone operators (see e.g. proposition 2.10 of [45]) and we arrive at the desired conclusion.

Step 2. We closely follow the proof of theorem 5.3.1 of [47] and the one given in theorem 3 of [43] for a related problem. We define the operator $\mathcal{T}_\epsilon : h \to g$, where $g \in \beta(u_h)$ and u_h is the solution of (P_h). It is easy to see that every fixed point of $\mathcal{T}_\epsilon$ is a solution of (P_ϵ). Moreover, $\mathcal{T}_\epsilon$ satisfies the hypotheses of Kakutani fixed point theorem [47], and so, if we denote $X = L^p(0, T : L^2(\mathcal{M}))$, then

(i) $K = \{h \in L^p((0, T), L^\infty(\Omega)) : ||h(t)|| \le C_0 \text{ a.e. } t \in (0, T)\}$ is a non-empty, convex and weakly compact set of X;

(ii) $\mathcal{T}_\epsilon : K \mapsto 2^X$ with non-empty, convex and closed values such that $\mathcal{T}_\epsilon(g) \subset K$, $\forall\, g \in K$ and

(iii) graph($\mathcal{T}_\epsilon$) is weakly×weakly sequentially closed.

Consequently, $\mathcal{T}_\epsilon$ has at least one fixed point in K. Finally, arguing as in the proof of theorem 5.3.1 of [47], we prove the existence of a weak solution of (P_ϵ).

Finally, we shall pass to the limit when $\epsilon \to 0$. To do that, we shall use several *a priori* estimates. First, owing to the assumptions on the initial data and the lemma 3.2, we know that there exists $M > 0$, independent of ϵ, such that

$$\max(\|U_\epsilon\|_{L^\infty((0,T) \times \Omega)}, \|u_\epsilon\|_{L^\infty((-\tau,T) \times \mathcal{M})}) \le M$$

and (by multiplying by U_ϵ and u_ϵ in the respective equations)

$$\max(\|U_\epsilon\|_{L^2(0,T:H^1(\Omega))}, \|u_\epsilon\|_{L^p(-\tau,T:V)}) \le M.$$

We also have that u_ϵ is a strong solution (see [43]) in the sense that

$$\left\| \frac{\partial u_\epsilon}{\partial t} \right\|_{L^2((-\tau,T) \times \mathcal{M})} \le M,$$

and that the family $\{U_\epsilon\}$ is equicontinuous (see proposition 6.3 of [50]). Then, there exists a subsequence of $\{U_\epsilon\}$ and $\{u_\epsilon\}$ (which we still label in the same way) such that $U_\epsilon \to \hat{U}$ weakly in $L^2(0, T : H^1(\Omega))$ and strongly in $C([0,T] : L^1(\Omega))$ (respectively $u_\epsilon \to u$ weakly in $L^p(-\tau, T : V)$ and strongly in $C([-\tau, T] : L^1(\Omega))$). Finally, by using that γ and β are maximal monotone graphs, and assumption (H$_4$) on $\mathcal{G}(t, x, \sigma, \eta)$, we can pass to the limit in all terms and we conclude that (U, u), where $U = \alpha(\hat{U})$, is a weak solution of the original problem (P_{3D}). ∎

Remark 3.3. Lemma 3.2 and similar arguments to those in lemma 3 of [43] allow us to prove the existence of maximal and minimal solutions.

(b) Non-uniqueness of solutions in the presence of a discontinuous coalbedo term

The presence of the multivalued coalbedo, β, (corresponding to a discontinuous function whose graph is completed as to generate a maximal monotone graph) allows us to prove that, for some special initial data, there exist more than one time-dependent solution. We assume here the following conditions.

(H_1^*) The coalbedo function is

$$
\beta(u) = \begin{cases} [m, M] & \text{if } u = -10, \\ m & \text{if } u < -10, \\ M & \text{if } u > -10, \text{ with } 0 < m < M. \end{cases} \tag{3.2}
$$

(H_2^*) $\mathcal{G}(t, x, u, u(t - \tau)) = Bu + C - \mu u(t - \tau)$ and $\gamma(u) = u$.
(H_3^*) B and C are positive constants verifying

$$
\frac{Qs_1 m}{\rho c} < -10B + C, \quad -10B + C + \mu \|u_0\|_{L^\infty(-\tau,0) \times L^\infty(-1,1)} < \frac{Qs_0 M}{\rho c}. \tag{3.3}
$$

(H_4^*) We also assume $w(x) \le 0$ for all $x \in (-1, 1)$.
(H_5^*) The initial data (U_0, u_0) satisfy

$$
U_0 \in C^\infty(\Omega), \quad u_0 \in C([-\tau, 0]) \times C^\infty(\Gamma_0),
$$
$$
u_0(s, x) = u_0(s, -x) = u_0(0, x), \quad x \in [-1, 1], \ s \in [-\tau, 0]
$$
$$
\frac{\partial u_0}{dx}(s, 0) = \frac{\partial^2 u_0}{\partial x^2}(s, 0) = 0, \quad u_0(s, 0) = -10,
$$
$$
\frac{\partial u_0}{dx}(s, x) < 0 \quad \text{if } x \in (0, 1), \quad \frac{\partial u_0}{dx} s, 1) = 0, \ s \in (-\tau, 0]
$$
$$
\frac{\partial U_0}{\partial z}(x, 0) > 0, \quad U_0(x, 0) = u_0(0, x), \quad \text{if } x \in (0, 1).
$$

Theorem 3.4. *Under the above conditions, problem (P_{2D}) has at least two bounded weak solutions.*

Proof. Step 1. First, we consider the problem (P_m)

$$
\frac{\partial U}{\partial t} - \frac{K_H}{R^2} \frac{\partial}{\partial x}\left((1 - x^2)\frac{\partial U}{\partial x}\right) - K_v \frac{\partial^2 U}{\partial z^2} + w \frac{\partial U}{\partial z} = 0 \quad (0, T) \times \Omega,
$$
$$
wx \frac{\partial U}{\partial x} + K_V \frac{\partial U}{\partial z} = 0 \quad (0, T) \times \Gamma_H
$$
$$
D\frac{\partial u}{\partial t} - \frac{DK_{H_0}}{R^2} \frac{\partial}{\partial x}\left((1 - x^2)^{p/2}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)
$$
$$
+ K_V \frac{\partial U}{\partial n} + wx \frac{\partial u}{\partial x} + Bu + C - \mu u(t - \tau, x) = \frac{1}{\rho c}QS(x)m \quad \text{on } (0, T) \times \Gamma_0
$$
$$
(1 - x^2)^{p/2}\left|\frac{\partial U}{\partial x}\right|^{p-2}\frac{\partial U}{\partial x} = 0 \quad \Gamma_1 \cup \Gamma_{-1}
$$
$$
U(0, x, z) = U_0(x, z) \quad \Omega,
$$
$$
U(0, x, 0) = u_0(x) \quad (-1, 1),
$$

We note that if $t < \tau$, then $u(t - \tau, x) = u_0(t - \tau)$. Denote (U^m, u^m) to the solution of (P_m). We note that if $u^m \le -10$, then (U^m, u^m) is also a solution of (P_{2D}) because $h(t, x) \equiv m \in \beta(u_m)$. Now, by

changing $U^* = -10 - U^m$ and $u^* = -10 - u^m$, we have that u^* verifies

$$Du_t^* - \frac{DK_{H_0}}{R^2}((1-x^2)|u_x^*|^{p-2}u_x^*)_x + Bu^*$$

$$= -\frac{QSm}{\rho c} - 10B + C - K_V \frac{\partial U^*}{\partial n} - wx\frac{\partial u^*}{\partial x} - \mu u_0(t - \tau, x).$$

From hypotheses (H_3^*) and (H_5^*), there exists $T_0 > 0$ s.t. if $t < T_0$ then the right-hand side term is positive. Consequently, $u^* = -10 - u^m$ is positive and $u^m < -10$. Note that $K_V(\partial U/\partial n) + wx(\partial U/\partial x) \leq 0$ in $(0, T_0) \times \Gamma_0$.

Step 2. Now, we prove that there exists a solution that takes values bigger than -10 in a subset of Γ_0 for $t < \tau$. To see the existence of this second solution, we shall construct a family of auxiliary functions U^λ (and the restrictions $U_{|\Gamma_0}^\lambda = u^\lambda$). We decompose $\Omega \times [0, \lambda] = \mathcal{Q}_1^\lambda \cup \mathcal{Q}_2^\lambda \cup \Sigma^\lambda$, where

$$\mathcal{Q}_1^\lambda = \left\{(x, z, t) \in \Omega \times [0, \lambda]: x^2 + z^2 > \frac{t^2}{\lambda^2}\right\},$$

$$\mathcal{Q}_2^\lambda = \left\{(x, z, t) \in \Omega \times [0, \lambda]: x^2 + z^2 < \frac{t^2}{\lambda^2}\right\},$$

$$\Sigma^\lambda = \left\{(x, z, t) \in \Omega \times [0, \lambda]: x^2 + z^2 = \frac{t^2}{\lambda^2}\right\}.$$

In the region $\mathcal{Q}_1^\lambda$. We consider (U^λ, u^λ) the solution of problem $(P_{\mathcal{Q}_1^\lambda})$ [51,52].

$$\frac{\partial U}{\partial t} - \frac{K_H}{R^2}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial U}{\partial x}\right) - K_v\frac{\partial^2 U}{\partial z^2} + w\frac{\partial U}{\partial z} = 0 \quad \mathcal{Q}_1^\lambda$$

$$wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = 0 \quad \mathcal{Q}_1^\lambda \cap (0, T) \times \Gamma_H$$

$$D\frac{\partial u}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{p/2}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)$$

$$+ K_V\frac{\partial U}{\partial n} + wx\frac{\partial u}{\partial x} + Bu + C - \mu u_0 = \frac{1}{\rho c}QS(x)m \quad \mathcal{Q}_1^\lambda \cap (0, T) \times \Gamma_0$$

$$(1-x^2)^{p/2}\left|\frac{\partial U}{\partial x}\right|^{p-2}\frac{\partial U}{\partial x} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_{-1}$$

$$U(0, x, z) = U_0(x, z), \quad U(0, x, 0) = u_0(x)$$

$$U^\lambda = -10 \quad \Sigma^\lambda$$

On the region $\mathcal{Q}_2^\lambda$, we define $U^\lambda = -10 - C^\lambda(t)(x^2 + z^2 - t^2/\lambda^2)$. Note that if $C^\lambda > 0$, then $U^\lambda > -10$ in $\mathcal{Q}_2^\lambda$. It is easy to see that (U^λ, u^λ) is a solution of problem (P_λ),

$$\frac{\partial U}{\partial t} - \frac{K_H}{R^2}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial U}{\partial x}\right) - K_v\frac{\partial^2 U}{\partial z^2} + w\frac{\partial U}{\partial z} = H^\lambda \quad \text{in } (0, T) \times \Omega,$$

$$wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = g^\lambda \quad \text{in } (0, T) \times \Gamma_H$$

$$D\frac{\partial u}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{p/2}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)$$

$$+ K_V\frac{\partial U}{\partial n} + wx\frac{\partial u}{\partial x} + Bu + C - \mu u_0 = h^\lambda \quad \text{in } (0, T) \times \Gamma_0$$

$$(1-x^2)^{p/2}\left|\frac{\partial U}{\partial x}\right|^{p-2}\frac{\partial U}{\partial x} = 0 \quad \text{in } (0, T) \times (\Gamma_1 \cup \Gamma_{-1})$$

$$U(0, x, z) = U_0(x, z) \text{ in } \Omega, \quad U(0, x, 0) = u_0(x) \text{ in } (-1, 1),$$

where, for $(t, x, z) \in Q_2^\lambda$, $H^\lambda = -(C^\lambda)'(t)(x^2 + z^2 - t^2/\lambda^2) - C^\lambda(t)[(-2t/\lambda^2) - (2K_H/R^2)(1 - 3x^2) - 2K_v + 2wz]$,

$$
h^\lambda = -D(C^\lambda)'(t)\left(x^2 - \frac{t^2}{\lambda^2}\right) - C^\lambda(t)\left[-\frac{2Dt}{\lambda^2} + 2wx^2 + B\left(x^2 - \frac{t^2}{\lambda^2}\right)\right.
$$

$$
- 2^{p-1}\frac{DK_{H_0}}{R^2}|C^\lambda(t)|^{p-2}(-p(1 - x^2)^{(p-2)/2}|x|^p
$$

$$
\left. + (p - 1)(1 - x^2)^{p/2}|x|^{p-2}\right] - 10B + C - \mu u_0),
$$

$$
g^\lambda = -2C^\lambda(t)(x^2 w - K_v H) \geq 0.
$$

Thus, there exist $\lambda > 0$ and $C^\lambda : [0, T_0] \to \mathbb{R}$ such that $h^\lambda \leq Q s_0 M/\rho c$. Then, (U^λ, u^λ) is a lower solution of problem (P_{2D}). Then, by upper and lower solution method, we deduce that there exists a solution (V, v) of (P_{2D}) satisfying $u^\lambda < v$. Consequently, $v > -10$ in some subset of positive measure. (V, v) is different from the solution of step 1. Finally, we obtain two different solutions of (P_{2D}) for an initial data satisfying (H_5^*). ∎

Remark 3.5. The above construction makes arise a parameter λ which is not uniquely determined. So, in fact, the proof shows the existence of a continuum of solutions, and not only two of them.

Remark 3.6. In the proof of the above result, the multivalued nature of β was a crucial element. As a matter of fact, if by the contrary we assume that β is a regular function, for instance a Lipschitz function then, by standard arguments we obtain the uniqueness of weak solutions.

(c) Uniqueness of non-degenerate solutions

Now, we wonder if it is possible to obtain uniqueness of time-dependent solutions for a model which may involve a multivalued coalbedo term but for some special initial data. The answer is positive but it will depend on a suitable property which must be satisfied by the weak solutions. By simplicity in the exposition, we shall assume here $\gamma(s) = s$ (the result remains true for the case of the graph γ corresponding to a positive latent heat but the details are too technical as to be presented here). We define a class of solutions called as *non-degenerate* on Γ_0. This notion was also useful in [43,53] where the EBM model without the deep ocean effect was studied.

Definition. Let $w \in L^\infty(\Gamma_0)$. We say that w satisfies the strong non-degeneracy property (resp. weak) if there exist $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $|\{x \in \Gamma_0 : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$ (resp. $|\{x \in \Gamma_0 : 0 < |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$).

Theorem 3.7. *(i) Assume that there exists a solution (U, u) of (P_{2D}) such that $u(t)$ verifies the strong non-degeneracy property for all $t \in [0, T]$ then (U, u) is the unique bounded weak solution of (P_{2D}). (ii) There exists at most one solution of (P_{2D}) verifying the weak non-degeneracy property.*

The idea of the proof is based on the fact that β generates a continuous operator from $L^\infty(\Gamma_0)$ to $L^q(\Gamma_0)$ $\forall q \in [1, \infty)$ when the domain of such operator is the set of functions verifying the strong non-degeneracy property. More precisely, we estimate the difference between two possible solutions $(U - V, u - v)$ by using the following

Lemma 3.8. *(i) Let $w, \hat{w} \in L^\infty(\Gamma_0)$. Assume w satisfies the strong non-degeneracy property. Then, for every $q \in [1, \infty)$, there exists $\tilde{C} > 0$ such that for every $z, \hat{z} \in L^\infty(\Gamma_0)$ verifying $z(x) \in \beta(w(x))$ and $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in \Gamma_0$, we have*

$$
\|z - \hat{z}\|_{L^q(\Gamma_0)} \leq (b_w - b_i)\min\{\tilde{C}\|w - \hat{w}\|_{L^\infty(\Gamma_0)}^{(p-1)/q}, \; |2|^{1/q}\}. \tag{3.4}
$$

(ii) If $w, \hat{w} \in L^\infty(\Gamma_0)$ satisfy the weak non-degeneracy property, then

$$
\int_{\Gamma_0} (z(x) - \hat{z}(x))(w(x) - \hat{w}(x)) \, dA \leq (b_w - b_i)C\|w - \hat{w}\|_{L^\infty(\Gamma_0)}^p. \tag{3.5}
$$

The idea of the proof (for the case of the simpler model (P_{2D})) of the uniqueness of solution follows closely theorem 5 of [43]. First, we argue on the time interval $[0, \tau]$ (it is enough to repeat the same arguments on subintervals of length τ to obtain the result on the whole interval $[-\tau, T]$ for any arbitrary $T > 0$). Assume there exist two solutions (U, u) and (V, v). By using Holder, Young and Friedrich inequalities and the lemma of non-degeneracy property (by introducing a suitable spatial rescaling $x \mapsto \lambda x$ to estimate some balance of the upper bounds), we obtain that

$$\frac{\partial}{\partial t} \|U - V\|_{L^2(\Omega)}^2 + \frac{\partial}{\partial t} \|u - v\|_{L^2(\Gamma_0)}^2 \leq K_1 \|U - V\|_{L^2(\Omega)}^2 + K_2 \|u - v\|_{L^2(\Gamma_0)}^2.$$

Finally, by Gronwall lemma, we conclude that $\|U - V\|_{L^2(\Omega)} = 0$ and $\|u - v\|_{L^2(\Gamma_0)} = 0$, which ends the proof.

Remark 3.9. The conclusion of theorem 3.7 also holds for the (P_{3D}), but its proof becomes more technical. It will be presented in a future work by the authors.

4. Multiplicity of steady states

The analysis of the stabilization, as $t \to +\infty$ of the solutions, cannot be carried out by means of any linearization principle owing to the presence of the possible multivalued graphs γ and β. An alternative method consists of characterizing the ω-limit set (once it is assumed that $f(t, .) \to f_\infty(.)$, when $t \to +\infty$, in some suitable sense). In that case, it can be shown that, given (U, u) bounded weak solution of (P_{3D}), any element of the ω-limit set of (U, u), defined by $\omega(U, u) = \{(U_\infty, u_\infty) \in (H^1(\Omega) \times V) \cap L^\infty(\mathcal{M}) \times L^\infty(\mathcal{M}): \exists t_n \to +\infty$ such that $(U(t_n, \cdot), u(t_n, \cdot)) \to (U_\infty, u_\infty)$ in $L^2(\Omega) \times L^2(\mathcal{M})\}$, is formed merely by solutions (U_∞, u_∞) of the associate stationary model, which we denote by (P_∞). The proof of this result follows the ideas of [22] (the details will appear in a future work). The associated stationary problem (P_∞) consists of the following set of equations

$$-\operatorname{div}(K\nabla U) + w\frac{\partial U}{\partial z} = 0 \quad \text{on } \Omega, \tag{4.1}$$

$$\hat{F}(x, \nabla_\mathcal{N} U) + \frac{\partial U}{\partial z} = 0 \quad \text{on } \mathcal{N}, \tag{4.2}$$

$$-\operatorname{div}_\mathcal{M}(|\nabla_\mathcal{M} u|^{p-2}\nabla_\mathcal{M} u) + K_V \frac{\partial U}{\partial n}$$
$$+ F(x, \nabla_\mathcal{M} u) + \hat{\mathcal{G}}(u) \in R_a(u) + f_\infty \quad \text{on } \mathcal{M}, \tag{4.3}$$

and
$$U_{|\mathcal{M}} = u, \tag{4.4}$$

where $\partial\Omega = \mathcal{N} \cup \mathcal{M}$ and with $\hat{\mathcal{G}}(x, u)$ given by the limit of $\mathcal{G}(t, x, u, u(t - \tau))$ when $t \to +\infty$. In this section, we shall assume the conditions

(H$_S$) $S: \Omega \to \mathbb{R}$, $S \in L^\infty(-1, 1)$, $S_1 \geq S(x) \geq S_0 > 0$ for some $S_1 > S_0$.
(H$_\mathcal{G}$) $\hat{\mathcal{G}}: \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\hat{\mathcal{G}}(0) = 0$ and $\lim_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$.
(H$_f$) $f_\infty \in L^\infty(\Omega)$ and there exist $C_f > 0$ such that $-\|f_\infty\|_\infty \leq f_\infty(x) \leq -C_f$ a.e. $x \in \Omega$.
(H$_\beta$) β is a bounded maximal monotone graph of $\mathbb{R}^2$, and there exist two real numbers $0 < m < M$ and $\epsilon > 0$ such that $\beta(r) = \{m\}$ for any $r \in (-\infty, -10 - \epsilon)$ and $\beta(r) = \{M\}$ for any $r \in (-10 + \epsilon, +\infty)$.
(H$_{C_f}$) $\hat{\mathcal{G}}(-10 - \epsilon) + C_f > 0$ and $(\hat{\mathcal{G}}(-10 + \epsilon) + \|f_\infty\|_\infty)/(\hat{\mathcal{G}}(-10 - \epsilon) + C_f) \leq S_0 M/S_1 m$.
(H$_w$) $w \in C^1(\bar{\Omega})$ (for the sake of simplicity).
(H$_K$) The constants $K_H, K_V, K_H, K_{H_0}, D, R, \rho, c$ and Q are positive.

One important technique that we shall use in the following result is the continuity of the solutions with respect to the coalbedo β. This allows us to approximate a discontinuous (i.e. multivalued graph) β by a smoother function. This also have some implications for the numerical treatment of the model.

Theorem 4.1. *Let* $(H_S), (H_{\mathcal{G}}), (H_f), (H_w), (H_K)$ *and* (H_β) *be satisfied. Then, for any* $Q > 0$*, there is a minimal solution* $(\underline{U}, \underline{u})$ *(resp. a maximal solution* $(\bar{U}, \bar{u})$*) of problem* (P_Q)*. Moreover, if* (H_{C_f}) *holds, then there exist* $Q_1 < Q_2 < Q_3 < Q_4$ *such that*

  (i) *if* $0 < Q < Q_1$*, then* (P_Q) *has a unique solution,*
  (ii) *if* $Q_2 < Q < Q_3$*, then* (P_Q) *has at least three solutions,*
  (iii) *if* $Q_4 < Q$*, then* (P_Q) *has a unique solution, where*

$$Q_1 = \frac{(\hat{\mathcal{G}}(-10 - \epsilon) + C_f)\rho c}{S_1 M} \qquad Q_2 = \frac{(\hat{\mathcal{G}}(-10 + \epsilon) + \|f_\infty\|_\infty)\rho c}{S_0 M}$$

$$Q_3 = \frac{(\hat{\mathcal{G}}(-10 - \epsilon) + C_f)\rho c}{S_1 m} \qquad Q_4 = \frac{(\hat{\mathcal{G}}(-10 + \epsilon) + \|f_\infty\|_\infty)\rho c}{S_0 m}.$$

Proof. This proof is the extension to (P_{3D}) of the results for (P_{2D}) given in [54] (see also [22]). Let us define the vectorial operator $\mathcal{A} : L^2(\Omega) \times L^2(\mathcal{M}) \to L^2(\Omega) \times L^2(\mathcal{M})$ by $\mathcal{A}(U, u) := (AU, Bu)$ with domain

$$D(\mathcal{A}) = \{(U, u) \in L^2(\Omega) \times L^2(\mathcal{M}) : AU \in L^2(\Omega), \; Bu \in L^2(\mathcal{M}), \; U_{|\mathcal{M}} = u\},$$

where $AU = -\text{div}(\nabla U) + w(\partial U/\partial z)$ and

$$Bu = -\text{div}(|\nabla_\mathcal{M} u|^{p-2} \nabla_\mathcal{M} u) + K_V \frac{\partial U}{\partial n} + F(x, \nabla_\mathcal{M} u) + \hat{\mathcal{G}}(x, u).$$

It is easy to find some constant functions $(\underline{V}, \underline{v})$ and $(\bar{U}, \bar{u})$ verifying

$$A\underline{V} = 0$$

$$B\underline{v} = \frac{1}{\rho c} QS_0 m - \|f_\infty\|_\infty \leq \frac{1}{\rho c} QS(x)\underline{\beta}(\underline{v}) + f_\infty,$$

$$A\bar{U} = 0$$

$$B\bar{u} = \frac{1}{\rho c} QS_1 M - C_f \geq \frac{1}{\rho c}(x)\bar{\beta}(\bar{u}) + f_\infty,$$

where $\underline{\beta}$ and $\bar{\beta}$ are some (eventually discontinuous) functions (i.e. single-valued sections of the graph β) such that $\underline{\beta}(s) \in \beta(s), \bar{\beta}(s) \in \beta(s)$ and $\underline{\beta}(u) \leq h \leq \bar{\beta}(u)$ for all $h \in \beta(u)$. Every solution (U, u) of (P_{3D}) verifies $\underline{V} \leq U \leq \bar{U}$ and $\underline{v} \leq u \leq \bar{u}$.

(i) If $Q < Q_1$ then $\underline{V} \leq \bar{U} \leq -10 - \epsilon$. So, every solution (U, u) of (P_Q) verifies $u < -10 - \epsilon$ and it is a solution of the problem

$$(P_Q^m) \quad \begin{cases} AU = 0 & \text{on } \Omega, \\ Bu = \frac{1}{\rho c} QS(x)m + f_\infty & \text{on } \mathcal{M}, \\ U = u & \text{on } \mathcal{M}, \\ \hat{F}(x, \nabla_\mathcal{N} U) + \frac{\partial U}{\partial z} = 0 & \text{on } \mathcal{N}, \end{cases}$$

which has a unique solution. To prove it, we assume there exist two solutions, (U_1, u_1) and (U_2, u_2) and we take the difference $U_1 - U_2$ as a test function in the weak formulation. The accretiveness of the operator allows us to conclude the uniqueness.

(ii) If $Q_4 < Q$ then $-10 + \epsilon \leq \underline{V} \leq \bar{U}$. So, every solution (U, u) verifies $-10 + \epsilon \leq u$ and $\beta(u) = M$. So, (U, u) is the unique solution of problem (P_Q^M) which is obtained by replacing m by M in problem (P_Q^m).

(iii) The proof of the multiplicity consists of three steps

Step 1. Construction of upper and lower solutions. If $Q_2 < Q < Q_3$, then,

$$\bar{U}_1 := \mathcal{G}^{-1}\left(\frac{1}{\rho c}QS_1M - C_f\right) \quad \text{is an upper solution of } (P_Q^M)$$

$$\underline{V}_1 := \mathcal{G}^{-1}\left(\frac{1}{\rho c}QS_0M - \|f_\infty\|_\infty\right) \quad \text{is a lower solution of } (P_Q^M)$$

$$\bar{U}_2 := \mathcal{G}^{-1}\left(\frac{1}{\rho c}QS_1m - C_f\right) \quad \text{is an upper solution of } (P_Q^m)$$

$$\underline{V}_2 := \mathcal{G}^{-1}\left(\frac{1}{\rho c}QS_0m - \|f_\infty\|_\infty\right) \quad \text{is a lower solution of } (P_Q^m).$$

Moreover, $\underline{V}_2 < \bar{U}_2 < -10 - \epsilon < -10 + \epsilon < \underline{V}_1 < \bar{U}_1$. Then, there exist two solutions (U_1, u_1) and (U_2, u_2) of (P_Q) such that u_1 and u_2 do not cross the level -10. To find the third solution, we shall apply a result of [55]. This is possible for the case where β is a Lipschitz function. In next step, we will approximate the graph β by some Lipschitz functions.

Step 2. Approximate problem. We define a new family of problems, $(P_{Q,\lambda})$ by replacing $\beta(u)$ by $\beta_\lambda(u)$ in (P_{3D}), where β_λ is the Lipschitz function $\beta_\lambda = (1/\lambda)(I - (I - \lambda\beta)^{-1})$, $\lambda > 0$ (the Yosida approximation of β). Because β verifies (H_β), we obtain that

$$\beta_\lambda \text{ is a bounded and non-decreasing function } \forall \lambda > 0,$$

$$\beta_\lambda(s) = \beta(s) \text{ for any } s \notin [-10 - \epsilon, -10 + \epsilon + \lambda M], \ \forall \lambda > 0,$$

$$\beta_\lambda(s) \to \beta(s) \text{ in the sense of maximal monotone graphs when } \lambda \to 0$$

(see [45]). In the case of β is a Lipschitz function, we take $\beta_\lambda = \beta$. Now, by applying the argument of step 1 to problem $(P_{Q,\lambda})$, there exit λ_0 such that $\underline{V}_2 < \bar{U}_2 < -10 - \epsilon < -10 + \epsilon + \lambda_0 M < \underline{V}_1 < \bar{U}_1$. Then, we have two families of solutions of $\{(P_{Q,\lambda})\}$ such that u_1^λ and u_2^λ do not cross the level -10. We have the third family of solutions by using the following lemma. We recall that X is a retract of E if there exists a continuous mapping $r: X \to E$ such that $r(x) = x$ for each $x \in X$.

Lemma 4.2 (Amann [55]). *Let X be a retract of some Banach space E and let $F: X \to X$ be a compact map. Suppose that X_1 and X_2 are disjoint retracts of X, and let Y_k, $k = 1, 2$ be open subset of X such that $Y_k \subset X_k$. Moreover, suppose that $F(X_k) \subset X_k$ and that F has no fixed points on $X_k - Y_k$, $k = 1, 2$. Then, F has at least three distinct fixed points x, x_1, x_2 with $x_k \in X_k$ and $x \in X - (X_1 \cup X_2)$.*

We see that the assumptions of this lemma are satisfied. Any solution u of the problem $(P_{Q,\lambda})$ is a fixed point of the equation $u = F(u)$ with $F: L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M})$ defined by

$$u = P_2\left(\mathcal{A}^{-1}\left(\frac{1}{\rho c}QS(\cdot)\beta_\lambda(u) + f_\infty(\cdot)\right)\right),$$

where P_2 is the projection over the second component. Let $E = L^\infty(\mathcal{M})$ which is an ordered Banach space with respect to the natural ordering whose positive cone is given by

$$L_+^\infty(\mathcal{M}) = \{v \in L^\infty(\mathcal{M}) : v(x) \geq 0 \text{ a.e. } x \in \mathcal{M}\},$$

having a non-empty interior. Let us define the intervals $X = [\underline{V}_2 - \delta, \ \bar{U}_1 + \delta]$, $X_1 = [\underline{V}_1 - \delta, \ \bar{U}_1 + \delta]$ and $X_2 = [\underline{V}_2 - \delta, \ \bar{U}_2 + \delta]$, where $\delta > \lambda_0 M$ is taken such that $\underline{V}_1 > -10 + \epsilon + \delta$, $\bar{U}_2 > -10 - \epsilon - \delta$. So, there exists an open set Y_k of $L^\infty(\mathcal{M})$ containing u_k^λ for $k = 1, 2$ such that $Y_k \subset X_k$. The sets X, X_1 and X_2 are retracts of $L^\infty(\mathcal{M})$ (resp. X), because they are non-empty closed convex subsets of $L^\infty(\mathcal{M})$ (resp. X). Moreover, $F(X) \subset X$ and $F(X_k) \subset X_k$. Finally, from the properties of β_λ and the compact embedding $W^{1,p}(\mathcal{M}) \subset L^\infty(\mathcal{M})$ for $p \geq 2$, we arrive at $F: X \to X$ is a compact map. So, by lemma 4.2, we conclude that F has at least three fixed points, or equivalently, $(P_{Q,\lambda})$ has at least three solutions: $u_1^\lambda \in X_1$, $u_2^\lambda \in X_2$ and $u_3^\lambda \in X - (X_1 \cup X_2)$.

Step 3. The proof ends with the convergence of a subsequence of $\{u_3^\lambda\}$ to u_3 such that (U_3, u_3) is a solution of $(P_{Q,\lambda})$. To obtain this limit, we need to use a result of maximal monotone graphs [56] which guarantees that the limit of $\beta_\lambda(u_\lambda)$ is in the graph $\beta(u_3)$. Finally, the convergence in $L^\infty(\mathcal{M})$ allows us to show that u_3 is different from u_1 and u_2. In particular, u_3 must cross the level -10. ∎

Table 1. Physical data used in the model.

parameter	value	units
K_H	0.049	$m^2 c^{-1}$
K_{H_0}	0.555×10^{-3}	$m^2 c^{-1}$
K_V	0.0125	$m^2 c^{-1}$
C, B	190, 2	Wm^{-2}, $Wm^{-2} K^{-1}$
c, ρ	3900,1004	$J(kg\,^{\circ}C)^{-1}$, $kg\,m^{-3}$
Q	340	Wm^{-2}
D	60	m

5. Numerical approximation

Here, we are concerned with computing a numerical solution for the problem (P_{2D}) with $p = 3$. The numerical approximation used is based upon the finite volume method with WENO reconstruction in space and third-order Runge–Kutta TVD for time integration. Details of WENO reconstruction can be found in many references, among them [57–60]. For each time step, a numerical solution of the EBM is computed and then used as a Dirichlet boundary condition for the deep ocean model. Other approximations are possible, for instance, we can mention the ADER–ENO scheme for nonlinear reaction–diffusion problems proposed in [61]. The numerical scheme follows the ideas put forward in [62]. Its application allows to obtain $\gamma_{i,j}^{n+1}$ for each control volume. Then, we use an iterative solver of nonlinear equations to compute the cell averages of the numerical solution for the deep ocean model $U_{i,j}^{n+1}$ from $\gamma_{i,j}^{n+1}$, solving the nonlinear equation

$$\gamma_{i,j}^{n+1} = \begin{cases} k_1 U_{i,j}^{n+1,iter} & \text{if } U_{i,j}^{n+1,iter-1} < 0 \\ \dfrac{L + k_2 \epsilon}{\epsilon} U_{i,j}^{n+1,iter} & \text{if } 0 \leq U_{i,j}^{n+1,iter-1} < \epsilon \ (iter = 1, 2, \ldots), \\ k_2 U_{i,j}^{n+1,iter} + L & \text{if } U_{i,j}^{n+1,iter-1} \geq \epsilon \end{cases}$$

for a given small ϵ. This iterative process ends up when $|U_{i,j}^{n+1,iter} - U_{i,j}^{n+1,iter-1}| < \delta$ for each control volume $V_{i,j}$ and with δ small enough. The iterative solver used consists of a combination of Newton's method and bisection method, in such a way that the method performing is the one that converges faster. Note that, following this idea, both methods can act at a particular time step. Finally, we assign the value $U_{i,j}^{n+1} = U_{i,j}^{n+1,iter}$. As for the cell averages of the delay term $u_i(t - \tau)$, an arithmetic mean of the values $u_i(t^k)$ and $u_i(t^{k+1})$ with $t - \tau \in [t^k, t^{k+1}]$ has been used.

The evolution of the temperature in the deep ocean is due to the combined effect of water sinking from the Earth poles with heating–cooling processes taking place in the interface atmosphere–ocean. In addition, water *upwelling* takes place at certain latitudes.

In the first numerical example, we compare the numerical solution of the model with and without the effect of the latent heat. The initial conditions considered are $U(0, x, z) = 18e^{-x^2-z^2} + 6e^{6z}(11e^{-x^2} - 10)$ for the ocean interior and $u(0, x) := U(0, x, 0) = 84e^{-x^2} - 60$. The data used in this example are depicted in table 1. In table 1, the unit c stands for century. The insolation function is taken as $S(x) = 1 - \frac{1}{2}P_2(x)$, where $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second Legendre polynomial in the interval $[-1, 1]$. The coalbedo $\beta(u)$ is given by (H_1^*), where $m = 0.4$ and $M = 0.69$. The numerical implementation of the coalbedo is performed considering that we are in the context of an explicit scheme therefore if, in the previous time step, at certain control volume, $u \leq -10$ then $\beta(u) = m$ otherwise, $\beta(u) = M$. As for the velocity, it depends only on x and in this work, it is defined as

$$\omega(x, z) = W(x) = \frac{10(x + 0.75)(x - 0.75)}{(0.1 + 10|x + 0.75|)(0.1 + 10|x - 0.75|)}. \tag{5.1}$$
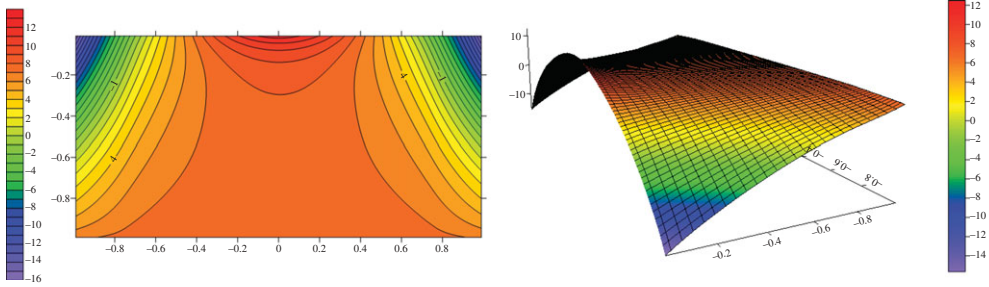
Figure 1. Temperature without latent heat ($\gamma(u) = u$) and $t = 5$. (Online version in colour.)
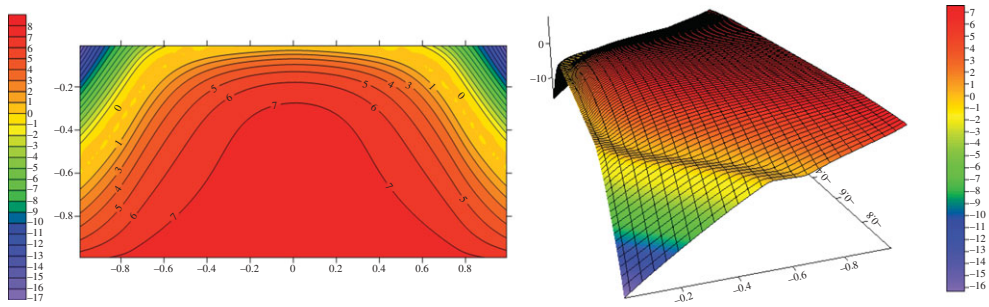
Figure 2. Temperature with latent heat and $t = 5$. (Online version in colour.)

This particular velocity is a way to represent sinking water near the poles and *upwelling* water in the vicinity of the Equator. The spatial discretization used is $\Delta x = 2/60$; $\Delta z = 1/60$, and the size of the time step is calculated in an iterative way according to the formula

$$(\Delta t)_n = \min \left(\alpha \Delta x^2 ((1 - x^2) K_H)^{-1}, \; \alpha \Delta z^2 (K_V)^{-1}, \; \alpha \Delta x^2 \left((1 - x^2) K_{H0} \left| \frac{d u_{n-1}}{dx} \right| \right)^{-1} \right), \qquad (5.2)$$

where $\alpha = 0.3$ for stability reasons.

Other values used here are $k_1 = k_2 = 1$, $\epsilon = 0.01$ and $L = 3$ (figures 1 and 2). The numerical experiment with latent heat shows more clearly (than the experiment without this term) the crucial role of the deep ocean: indeed, besides a suitable justification of the formation of sea-ice sheets (the level lines of the lower values of the sea temperature are now more separated, which corresponds to the presence of large regions without great abrupt temperature changes), most of the higher level lines of the sea temperature does not arrive to touch the sea surface (except, at most, some of them which do that around the Equator).

We can generate solutions of (P_{3D}) from the solutions of (P_{2D}) under suitable conditions. In figure 3, we can see the distribution of temperature on the Earth surface obtained by the numerical approximation of (P_{2D}) and rotated, thanks to the spherical coordinates. We observe that the surface temperature is lower in the case of presence of latent heat than when this effect is neglected. This is precisely what may be considered as an alarm about the gravity of the global change, because if a *realistic* deep ocean (that means with latent heat) is heated, the time to return to previous colder situations may be very large.

Another numerical experiment carried out considers the delay effect. The results can be seen in figure 4. The range of temperatures is narrower when considering this term than without its influence. Therefore, the delay term is like a memory one, which *remembers* the temperature of previous time steps and, therefore, tends to smooth the spatial evolution of the temperature.
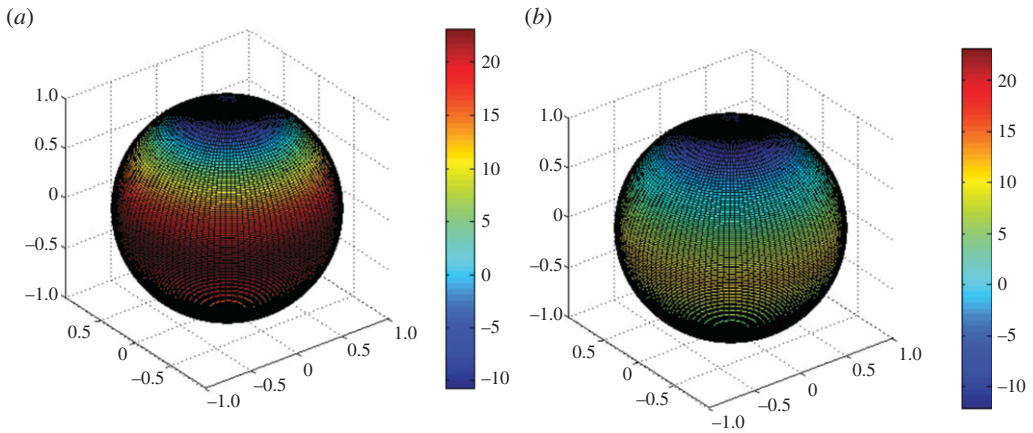
Figure 3. Mean surface temperature (*a*) without latent heat effect and (*b*) with latent heat effect at $t = 4$. (Online version in colour.)
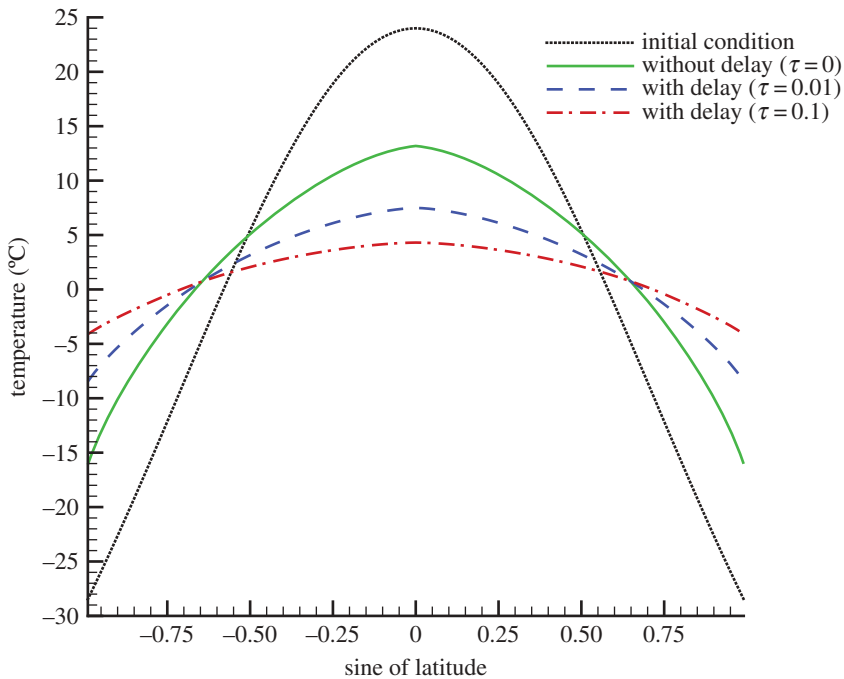
Figure 4. Delay effect in the upper boundary (without latent heat effect) at $t = 5$ for different values of the time of delay τ. (Online version in colour.)

In this example, we have taken $\mu = 0.5$, but no latent heat, for the sake of simplicity. Another interesting feature of the effect of the delay term is depicted in figure 5, where the temperature is plotted as a function of time for the particular latitude $38°$ S and for different values of the parameters μ and τ. The results show that, in both situations, a stationary state is reached. Nevertheless, when the time of delay τ is larger, the solution becomes more oscillating and takes a longer time in reaching the stationary state. This effect is more evident for larger values of μ. This conclusion is similar to that pointed out in [42]. In addition, figure 5 reveals that the consideration of the latent heat effect give rise to less oscillating solutions.
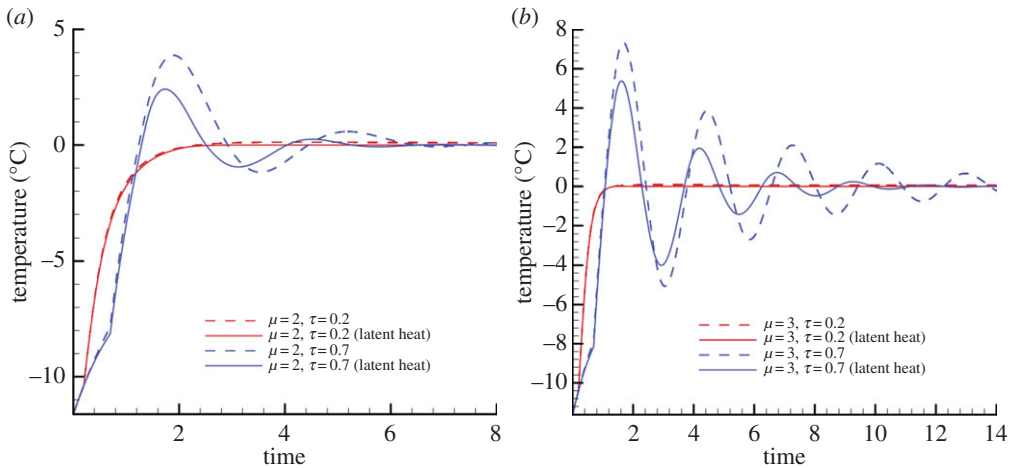
Figure 5. Temperature in the upper boundary as a function of time, with (full lines) and without (dashed lines) latent heat effect, for the latitude $38°$ S, being $\mu = 2$ (a) and $\mu = 3$ (b) and, in both cases, for two different values of the time delay τ. (Online version in colour.)

References

1. Budyko MI. 1969 The effects of solar radiation variations on the climate of the Earth. *Tellus* **21**, 611–619. (doi:10.1111/j.2153-3490.1969.tb00466.x)
2. Díaz JI, Lions JL (eds). 1994 *Environment, economics and their mathematical models*. Research Notes in Applied Mathematics, no. 35. Paris, France: Masson.
3. Díaz JI (ed.). 1996 *The mathematics of models in climatology and environment*. ASI NATO Global Change Series I, No. 48, Heidelberg, Germany: Springer.
4. Díaz JI (ed.). 2003 *Ocean circulation and pollution control*. *A mathematical and numerical inquiry*. Lecture Notes, EMS Volume. Berlin, Germany: Springer.
5. Brezis H, Díaz JI. 2003 Mathematics and environment. Special issue of *Rev. R. Acad. Cien. Serie A Matem* (RACSAM), 96, 3.
6. Watts RG, Morantine M. 1990 Rapid climatic change and the deep ocean, *Clim. Change* **16**, 83–97. (doi:10.1007/BF00137347)
7. North GR, Cahalan RF. 1982 Predictability in a solvable stochastic climate model. *J. Atmos. Sci.* **38**, 504–513. (doi:10.1175/1520-0469(1981)038<0504:PIASSC>2.0.CO;2)
8. Ghil M, Childress S. 1987 *Topics in geophysical fluid dynamics: atmospheric dynamics dynamo theory and climate dynamics*. Applied Mathematical Sciences. Berlin, Germany: Springer.
9. Díaz G, Díaz JI. 2002 On a stochastic parabolic PDE arising in climatology. *Rev. R. Acad. Cien. A Math.* **96**, 123–128.
10. Díaz JI, Langa JA, Valero J. 2009 On the asymptotic behaviour of solutions of a stochastic energy balance climate model. *Physica D* **238**, 880–887. (doi:10.1016/j.physd.2009.02.010)
11. Sanchez-Palencia E. 1980 *Non-homogeneous media and vibration theory*. Berlin, Germany: Springer.
12. Díaz JI, Hetzer G. 1998 *A functional quasilinear reaction-diffusion equation arising in climatology. Equations aux derivees partielles et applications. Articles dedies a J.-L. Lions*, pp. 461–480. Paris, France: Elsevier.
13. Díaz JI, Hetzer G, Tello L. 2006 An energy balance model climate model with hysteresis. *Nonlinear Anal.* **64**, 2053–2074. (doi:10.1016/j.na.2005.07.038)

14. Bhattacharya K, Ghil M, Vulis IL. 1982 Internal variability of an energy-balance model with delayed albedo effects. *J. Atmos. Sci.* **39**, 1747–1773. (doi:10.1175/1520-0469(1982)039<1747:IVOAEB>2.0.CO;2)

15. Mengel JG, Short DA, North GR. 1988 Seasonal snowline instability in an energy balance model. *Clim. Dyn.* **2**, 127–131. (doi:10.1007/BF01053470)

16. Cane MA, Zebiak SE. 1985 A theory for E1 Niño and the southern oscillation. *Science* **228**, 1084–1087. (doi:10.1126/science.228.4703.1085)

17. Suarez MJ, Schopf PS. 1988 A delayed action oscillator for ENSO. *J. Atmos. Sci.* **J5**, 3283–3287. (doi:10.1175/1520-0469(1988)045<3283:ADAOFE>2.0.CO;2)

18. Davey MK, Anderson DLT, Lawrence S. 1996 A simulation of variability of ENSO forecast skill. *J. Clim.* **9**, 240–246. (doi:10.1175/1520-0442(1996)009<0240:ASOVOE>2.0.CO;2)

19. Ghil M, Zaliapin I, Thompson S. 2008 A delay differential model of ENSO variability: parametric instability and the distribution of extremes. *Nonlin. Proc. Geophys.* **15**, 417–433. (doi:10.5194/npg-15-417-2008)

20. Hale JK. 1977 *Theory of functional differential equations*. New York, NY: Springer.

21. Wu J. 1996 *Theory and applications of partial functional differential equations*. New York, NY: Springer.

22. Díaz JI, Hernández J, Tello L. 1997 On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology. *J. Math. Anal. Appl.* **216**, 593–613. (doi:10.1006/jmaa.1997.5691)

23. Tello L. 2009 Estabilización de soluciones para un problema parabólico con condición de contorno dinámica y difusiva. Actas del XXI CEDYA/ VIII CMA. Universidad de Castilla La Mancha.

24. Casal AC, Díaz JI. 2006 On the complex Ginzburg–Landau equation with a delayed feedback. *Math. Models Methods Appl. Sci.* **16**, 11–17. (doi:10.1142/S0218202506001030)

25. Casal C, Díaz JI, Stich M. 2007 Control of turbulence in oscillatory reaction–diffusion systems through a combination of global and local feedback. *Phys. Rev. E* **76**, 0362091–0362099.

26. Casal AC, Díaz JI, Vegas JM. 2011 Finite extinction time property for a delayed linear problem on a manifold without boundary. *Discrete Cont. Dyn. Syst.* **2011** (Special issue), 265–271.

27. Arcoya D, Díaz JI, Tello L. 1998 S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology. *J. Diff. Equ.* **150**, 215–225. (doi:10.1006/jdeq.1998.3502)

28. Díaz JI. 1994 Approximate controlability for some simple climate models. In *Environment, economic and their mathematical models* (eds JI Díaz, JL Lions), pp. 29–43. Research Notes in Applied Mathematics, no. 35. Paris, France: Masson.

29. Díaz JI. 1994 Controllability and obstruction for some non linear parabolic problems in climatology. In *Modelado de Sistemas en Oceanografía, Climatología y Ciencias Medio Ambientales* (eds C Pares, A Valle), pp. 43–58. Málaga, Spain: Universidad de Málaga.

30. Díaz JI, Lions JL. 2003 On the approximate controllability of Stackelberg–Nash strategies. In *Ocean circulation and pollution control. A mathematical and numerical inquiry* (ed. JI Díaz). Lecture Notes, EMS Volume, Proceedings of the Diderot Videoconference Amsterdam-Madrid-Venice, Lecture-Notes, pp. 17–28. Berlin, Germany: Springer.

31. Dautray R, Díaz JI. 2006 Lo infinitamente pequeño, lo infinitamente grande y lo infinitamente complejo: el medio ambiente. *Rev. R. Acad. Cienc. Exact. Fis. Nat.* **100**, 13–24.

32. von Neumann J. 1955 Collected Works. Vol VI, Pergamon Press, 1963, pp. 504–519. (Reprinted original in *Fortune*, 1955).

33. Díaz JI. 2002 On the von Neumann problem and the approximate controllability of Stackelberg–Nash strategies for some environmental problems. *Rev. R. Acad. Cien. A Math.* **96**, 343–356.

34. Bermejo R, Carpio J, Díaz JI, Tello L. 2009 Mathematical and numerical analysis of a nonlinear diffusive climate energy balance model. *Math. Comput. Model.* **49**, 1180–1210. (doi:10.1016/j.mcm.2008.04.010)

35. Rasool I. 1993 *Système terre*. Paris, France: Flammarion.

36. Sellers WD. 1969 A global climatic model based on the energy balance of the earth–atmosphere system. *J. Appl. Meteorol.* **8**, 392–400. (doi:10.1175/1520-0450(1969)008<0392:AGCMBO>2.0.CO;2)

37. Crowley TL, North GR. 1991 *Paleoclimatology*. Oxford, UK: Oxford University Press.

38. Henderson-Sellers A, McGuffie K. 1987 *A climate modelling primer*. Chichester, UK: John Wiley & Sons.

39. Lions JL, Temam R, Wang S. 1992 New formulations of the primitive equations of atmosphere and applications, *Nonlinearity* **5**, 237–288. (doi:10.1088/0951-7715/5/2/001)

40. Kiehl JT. 1992 Atmospheric general circulation modeling. In *Climate system modeling* (ed. KE Trenberth), pp. 319–370. Cambridge, UK: Cambridge University Press.

41. Stone PH. 1972 A simplified radiative-dynamical model for the static stability of rotating atmospheres. *J. Atmos. Sci.* **29**, 405–418. (doi:10.1175/1520-0469(1972)029<0405:ASRDMF>2.0.CO;2)

42. Roques L, Chekroun MD, Cristofol M, Soubeyrand S, Ghil M. 2014 Parameter estimation for energy balance models with memory. *Proc. R. Soc. A* **470,** 20140349. (doi:10.1098/rspa.2014.0349)

43. Díaz JI, Tello L. 1999 A nonlinear parabolic problem on a Riemannian manifold without boundary arising in climatology. *Collectanea Math.* **50**, 19–51.

44. Badii M, Díaz JI. 2010 On the time periodic free boundary associated with some nonlinear parabolic equations. *Bound. Value Probl.* **2010**, 147301. (doi:10.1155/2010/147301)

45. Brezis H. 1973 *Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*. Amsterdam, The Netherlands: North Holland.

46. Aubin T. 1982 *Nonlinear analysis on manifolds. Monge–Ampere equations*. Berlin, Germany: Springer.

47. Vrabie II. 1995 *Compactness methods for nonlinear evolutions*. London, UK: Pitman Longman.

48. Díaz JI, Jimenez R. 1984 Aplicación a la teoria no lineal de semigrupos a un operador pseudodiferencial. In *Actas VII CEDYA*, pp. 137–142. Granada, Spain: University of Granada.

49. Bejenaru I, Díaz JI, Vrabie II. 2001 An abstract approximate controllability result and applications to elliptic and parabolic systems with dynamic boundary conditions. *Electron. J. Diff. Equ.* **50**, 1–19.

50. Di Benedetto E, Fasano A, Primicerio M. 1985 On a free boundary problem related to an irreversible process. *Control Cibernet.* **14**, 195–219.

51. Friedman A. 1964 *Partial differential equations of parabolic type*. Englewood, Cliffs, NJ: Prentice-Hall.

52. Díaz JI, Tello L. 2009 On the coupling between the deep ocean and an atmospheric balanced climate model. *Maths and Water. Monografías de la Real Academia de Ciencias Exactas, Físicas Químicas y Naturales de Zaragoza*. **31**, 67–76.

53. Díaz JI. 1993 Mathematical analysis of some diffusive energy balance climate models. In *Mathematics, climate and environment* (eds JI Díaz, JL Lions), pp. 28–56. Paris, France: Masson.

54. Díaz JI, Tello L. 2008 On a climate model with a dynamic nonlinear diffusive boundary condition. *Discrete Continuous Dyn. Syst. Ser. S*, **1**, 253–262. (doi:10.3934/dcdss.2008.1.253)

55. Amann H. 1976 Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18**, 620–709. (doi:10.1137/1018114)

56. Brezis H. 1971 Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In *Contributions to nonlinear functional analysis* (ed. E Zarantonello), pp. 101–156. New York, NY: Academic Press.

57. Balsara DS, Shu CW. 2000 Monotonicity preserving weighted essentially non-oscillatory schemes with increasingly high order of accuracy. *J. Comp. Phys.* **160**, 405–452. (doi:10.1006/jcph.2000.6443)

58. Titarev VA, Toro EF. 2004 Finite-volume WENO schemes for three-dimensional conservation laws. *J. Comp. Phys.* **201**, 238–260. (doi:10.1016/j.jcp.2004.05.015)

59. Titarev VA, Toro EF. 2005 ADER schemes for three-dimensional non-linear hyperbolic systems. *J. Comp. Phys.* **204**, 715–736. (doi:10.1016/j.jcp.2004.10.028)

60. Dumbser M, Enaux C, Toro EF. 2008 Finite volume schemes of very high order of accuracy for stiff hyperbolic balance laws. *J. Comp. Phys.* **7**, 3971–4001. (doi:10.1016/j.jcp.2007.12.005)

61. Toro EF, Hidalgo A. 2009 ADER finite volume schemes for nonlinear reaction–diffusion equations. *Appl. Numer. Math.* **59**, 73–100. (doi:10.1016/j.apnum.2007.12.001)

62. Hidalgo A, Tello L. 2011 A finite volume scheme for simulating the coupling between deep ocean and an atmospheric energy balance model. In *Modern mathematical tools and techniques in capturing complexity* (eds L Pardo, N Balakrishnan, M Angeles). Springer Series in Complexity, pp. 239–255. Berlin, Germany: Springer.