Published in final edited form as:
Am Math Mon. 2014 March 1; 121(3): 258–259. doi:10.4169/amer.math.monthly.121.03.258.

Hadamard's Determinant Inequality

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Abstract

This note is devoted to a short, but elementary, proof of Hadamard's determinant inequality.

Let $\boldsymbol{X}$ be an $n \times n$ matrix with complex entries and columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Hadamard's determinant inequality [2] reads

$$|\det \boldsymbol{X}| \le \prod_{i=1}^{n} ||\boldsymbol{x}_i||_2.$$

Horn and Johnson [1] supply at least a half dozen proofs for this classical result. Hadamard's inequality is obviously true if any $\boldsymbol{x}_i = \boldsymbol{0}$. If we assume otherwise and divide both sides by the right-hand side, then Hadamard's inequality reduces to the inequality $|\det \boldsymbol{X}| \le 1$ subject to the Euclidean length constraints $\|\boldsymbol{x}_i\|_2 = 1$. Equality is attained by taking $\boldsymbol{X}$ to be the identity matrix $\boldsymbol{I}$ or any unitary matrix. Recall that a unitary matrix $\boldsymbol{X}$ has orthonormal columns and consequently satisfies

$$|\det \boldsymbol{X}|^2 = \det \boldsymbol{X}^* \det \boldsymbol{X} = \det(\boldsymbol{X}^* \boldsymbol{X}) = \det \boldsymbol{I} = 1.$$

In view of Weierstrass' theorem, the necessarily positive maximum of $|\det \boldsymbol{X}|$ is attained on the compact set defined by the constraints. Suppose $\boldsymbol{X}$ has columns of unit length and yields the maximum value, but two columns of $\boldsymbol{X}$, say the first two, are not orthogonal. Let us demonstrate that $|\det \boldsymbol{X}|$ can be increased by replacing the first column by the linear combination $\boldsymbol{y}_1 = a\boldsymbol{x}_1 + b\boldsymbol{x}_2$ for carefully chosen complex scalars a and b. The determinant of the new matrix $\boldsymbol{Y}$ satisfies

$$\det \boldsymbol{Y} = \det(a\boldsymbol{x}_1 + b\boldsymbol{x}_2, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_n) = a \det \boldsymbol{X}.$$

We force $\boldsymbol{y}_1$ to be a unit vector by imposing the constraint

$$|a|^2 + |b|^2 + 2\mathrm{Re}(\overline{a} b s) = 1, \qquad s = \boldsymbol{x}_1^* \boldsymbol{x}_2.$$

The Cauchy-Schwarz inequality implies $|s| \leq 1$. Furthermore, $s \neq 0$ by assumption, and $|s| \neq 1$, for otherwise $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are collinear and $\det \boldsymbol{X}$ vanishes. The reader can check that we may choose

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-|s|^2}} \\ -\frac{\bar{s}}{\sqrt{1-|s|^2}} \end{pmatrix}.$$

Since $a > 1$, the identity $\det \boldsymbol{Y} = a \det \boldsymbol{X}$ shows that $|\det \boldsymbol{Y}| > |\det \boldsymbol{X}|$. This contradiction proves that $\boldsymbol{X}$ has orthonormal columns, so it is unitary and $|\det \boldsymbol{X}| = 1$.

References

1. Horn, RA.; Johnson, CR. Matrix Analysis. 2. Cambridge University Press; Cambridge: 2013.
2. Hadamard J. Résolution d'une question relative aux déterminants. Bulletin des Sciences Math. 1893; 2:240–246.