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Mathematics

Stability of Coulomb systems in a magnetic field

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ABSTRACT I study N electrons and M protons in a magnetic field. It is shown that the total energy per particle is bounded below by a constant independent of M and N, provided the fine structure constant is small. Here, the total energy includes the energy of the magnetic field.

A proof of stability of matter in the presence of a magnetic field is announced here. The problem arises partly because intense magnetic fields occur in the interior of stars. More importantly, however, stability of matter with magnetic fields implies stability of nonrelativistic quantum electrodynamics (QED) with ultraviolet cutoffs. It was J. Fröhlich who discovered this implication and posed the problem of stability of matter with magnetic fields. Nonrelativistic QED is a natural setting for the study of the interaction of matter and radiation under ordinary conditions. The main result here, *Theorem 2* below, was first conjectured in a series of papers by Fröhlich, Lieb, Loss, and Yau (1–3). These papers proved various special cases, which are not reproduced here. (See also Avron, Herbst, and Simon, ref. 4.)

Stability of matter is an estimate for the energy of a quantum-mechanical system composed of N electrons and M nuclei. The nuclei lie at fixed locations $y_1, y_2, \ldots, y_M \in \mathbf{R}^3$. For simplicity, we take the nuclei to be protons. When there is no magnetic field, we may set up the problem as follows. The state of the system is an antisymmetric wave function $\psi(x_1 \cdots x_N) \in L^2(\mathbf{R}^{3N})$ with norm 1. (For simplicity, we take ψ scalar-valued when there is no magnetic field.) The energy is given by $\langle H_{NM}\psi, \psi\rangle$, where (in suitable units)

$$H_{NM} = \sum_{k=1}^{N}(-\Delta_{x_k}) + V_{\text{Coulomb}}, \qquad [1]$$

with

$$V_{\text{Coulomb}} = \sum_{1 \le j < k \le N} |x_j - x_k|^{-1} + \sum_{1 \le j < k \le M} |y_j - y_k|^{-1} - \sum_{j=1}^{N}\sum_{k=1}^{M}|x_j - y_k|^{-1}. \qquad [2]$$

The following result is well known.

THEOREM 1. [*Stability of Matter; see Dyson and Lenard* (5, 6) *and Lieb and Thirring* (7)]. $\langle H_{NM}\psi, \psi\rangle \ge -C(N + M)$ *for a universal constant* C.

The value of the constant C is important in understanding how atoms and molecules form from a large collection of electrons and nuclei (see ref. 8).

When there is a magnetic field present, the Hamiltonian Eq. 1 has to be modified as follows. First of all, because the electron has spin ½, the wave function $\psi(x_1 \cdots x_N)$ takes its values not in $\mathbf{C}$, but rather in $\mathbf{C}^{(2^N)}$, regarded as the tensor product $\mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ of N copies of $\mathbf{C}^2$.

Second, we introduce a vector potential $A = (A_\mu(x))_{\mu=1,2,3}$ for the magnetic field. Without loss of generality, we may work in the Coulomb gauge—i.e., we may assume that $\operatorname{div} A = 0$.

Next, we bring in the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The σ_μ act on $\mathbf{C}^2$. For $j = 1, 2, \ldots, N$ and $\mu = 1, 2, 3$ we set $\sigma_{j\mu} = I \otimes \cdots \otimes I \otimes \sigma_\mu \otimes I \otimes \cdots \otimes I$, where σ_μ is the jth factor and I is the 2×2 identity matrix. Thus, $\sigma_{j\mu}$ acts on $\mathbf{C}^{(2^N)}$.

The Pauli matrices and the vector potential give rise to a Dirac operator $\sigma_j \cdot (i\nabla_{x_j} - A(x_j)) = \Sigma_{\mu=1,2,3}\sigma_{j\mu}(i\partial/\partial x_{j\mu} - A_\mu(x_j))$ acting on $L^2(\mathbf{R}^{3N}, \mathbf{C}^{(2^N)})$. Here, $(x_{j\mu})_{\mu=1,2,3}$ are the coordinates of $x_j \in \mathbf{R}^3$. The Hamiltonian that replaces Eq. 1 for electrons and protons in a magnetic field is

$$H_{NM}^{A} = \sum_{1 \le j \le N} [\sigma_j \cdot (i\nabla_{x_j} - A(x_j))]^2 + V_{\text{Coulomb}}, \qquad [3]$$

and the energy of the particles is $\langle H_{NM}^{A}\psi, \psi\rangle$.

In contrast to *Theorem 1*, the energy of the particles need not be bounded below, even when $N = M = 1$ (see ref. 1). To rescue stability, one therefore has to bring in the energy of the magnetic field, which, in our present units and in the Coulomb gauge, is given by

$$\text{field energy} = \Gamma \int_{\mathbf{R}^3} |\nabla A|^2 dx. \qquad [4]$$

Here, Γ is a dimensionless constant; in fact, $\Gamma = \alpha^{-2}/8\pi$, where α is the fine structure constant. Since $\alpha \simeq 1/137$, we have $\Gamma \simeq 750$. In view of Eqs. 3 and 4, the total energy of the system is $\langle H_{NM}^{A}\psi, \psi\rangle + \Gamma \int_{\mathbf{R}^3}|\nabla A|^2 dx$. The main result on stability of matter in magnetic fields is as follows.

THEOREM 2. *There exist universal constants* C, $\Gamma > 0$ *such that* $\langle H_{NM}^{A}\psi, \psi\rangle + \Gamma \int_{\mathbf{R}^3} |\nabla A|^2\, dx \ge -C(N + M)$. *In particular,* Γ *and* C *are independent of* N, M, *and* A.

It would be very interesting to prove that one can take $\Gamma \simeq 750$ in *Theorem 2*. The proof of *Theorem 2* also gives the following slight refinement.

THEOREM 3. *Let* $\rho(x) = \min_{1 \le k \le M}|x - y_k|$ *be the distance from* x *to the nearest nucleus. Then*

$$\langle H_{NM}^{A}\psi, \psi\rangle + \Gamma \int_{\mathbf{R}^3} e^{-\rho(x)}|\nabla A(x)|^2 dx \ge -C(N + M)$$

for universal constants Γ *and* C.

Thus, we need only use the energy of the magnetic field near the nuclei. It is from *Theorem 3* that Fröhlich's argument derives the stability of nonrelativistic QED with ultraviolet cutoffs.

Finally, note that the proof of *Theorems 2* and *3* generalizes from protons to nuclei of higher atomic number. As conjectured in refs. 1 and 2, one finds stability provided Γ exceeds $\max\{\Gamma_0, \Gamma_1 Z\}$, where Γ_0 and Γ_1 are universal constants and Z is the maximum of the atomic numbers of the nuclei. If Γ is not large enough, then examples in ref. 3 show that stability fails.

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Abbreviation: QED, quantum electrodynamics.

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