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Critical thresholds in flocking hydrodynamics with non-local alignment

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We study the large-time behaviour of Eulerian systems augmented with non-local alignment. Such systems arise as hydrodynamic descriptions of agent-based models for self-organized dynamics, e.g. Cucker & Smale (2007 *IEEE Trans. Autom. Control* **52**, 852–862. (doi:10.1109/TAC.2007.895842)) and Motsch & Tadmor (2011 *J. Stat. Phys.* **144**, 923–947. (doi:10.1007/s10955-011-0285-9)) models. We prove that, in analogy with the agent-based models, the presence of non-local alignment enforces *strong* solutions to self-organize into a macroscopic flock. This then raises the question of existence of such strong solutions. We address this question in one- and two-dimensional set-ups, proving global regularity for *subcritical* initial data. Indeed, we show that there exist *critical thresholds* in the phase space of the initial configuration which dictate the global regularity versus a finite-time blow-up. In particular, we explore the regularity of non-local alignment in the presence of vacuum.

1. Introduction

The main system we are concerned with is the 'pressure-less' compressible Euler equations with non-local alignment

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad x \in \mathbb{R}^n, \ t \geq 0 \tag{1.1a}$$

and

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \int_{\mathbb{R}^n} \frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\Phi(\mathbf{x},t)} (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{1.1b}$$

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subjected to compactly supported initial density $\rho(\mathbf{x},0)\equiv\rho_0(\mathbf{x})$ and uniformly bounded initial velocity $\mathbf{u}(\mathbf{x},0)\equiv\mathbf{u}_0(\mathbf{x})$,

$$\rho_0\in L^1_+(\mathbb{R}^n),\quad \mathbf{u}_0\in W^{1,\infty}(\mathbb{R}^n). \tag{1.1c}$$

Such systems arise as a macroscopic description of the agent-based models (e.g. [1]), in which every agent adjusts its velocity to that of its neighbours through the process of *alignment*, dictated by the *interaction kernel* $a(\cdot,\cdot)$,

$$\dot{\mathbf{x}}_i=\mathbf{v}_i,\quad \dot{\mathbf{v}}_i=\sum_{j=1}^N a(\mathbf{x}_i,\mathbf{x}_j)(\mathbf{v}_j-\mathbf{v}_i).$$

The expected long-time behaviour of these agents is to self-organize into finitely many *clusters*, and in particular, depending on the properties of the interaction kernel, $a(\cdot,\cdot)$, to flock into one such cluster; consult the recent reviews [2,3]. The goal of this paper is to study the *flocking* phenomenon of the corresponding hydrodynamic description (1.1), or flocking hydrodynamics for short.

We shall discuss two prototype models: the celebrated model of Cucker–Smale (CS) [4,5] which employs a symmetric interaction kernel quantified in terms of an influence function $\phi=\phi(r)$,

$$a(\mathbf{x},\mathbf{y})=\phi(|\mathbf{x}-\mathbf{y}|),$$

and a related model of Motsch & Tadmor (MT) introduced in [6], which provides a better description of far-from-equilibrium flocking dynamics, using a non-symmetric (and time-dependent) interaction of the form,

$$a(\mathbf{x},\mathbf{y})=\frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\int_{\mathbb{R}^n}\phi(|\mathbf{x}-\mathbf{z}|)\rho(\mathbf{z},t)\,\mathrm{d}\mathbf{z}}.$$

Thus, the scaling function $\Phi(\mathbf{x},t)$ in (1.1*b*) takes one of two forms

$$\Phi(\mathbf{x},t)\begin{cases}\equiv 1 & \text{CS model}\\ =\displaystyle\int_{\mathbb{R}^n}\phi(|\mathbf{x}-\mathbf{z})|)\rho(\mathbf{z},t)\,\mathrm{d}\mathbf{z} & \text{MT model,}\end{cases} \tag{1.1d}$$

corresponding to the CS and MT models. Here, $\phi=\phi(r)$ is the *influence function* which throughout the paper is assumed (i) non-increasing, reflecting the intuition that alignment becomes weaker as the distance increases (but consult [3]); (ii) Lipschitz continuous; and (iii) non-local in the sense of having a divergent tail

$$\int^\infty\phi(r)\,\mathrm{d}r=\infty. \tag{H}$$

Without loss of generality, we assume $\phi(0)=1$.

To put our discussion into context, consider the hydrodynamic model augmented by the symmetric CS alignment $a(\mathbf{x},\mathbf{y})=\phi(|\mathbf{x}-\mathbf{y}|)$,

$$\rho_t+\operatorname{div}(\rho\mathbf{u})=0,\quad \mathbf{x}\in\operatorname{supp}(\rho(\cdot,t))\subset\mathbb{R}^n,\ \ t\ge 0 \tag{1.2a}$$

and

$$(\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla P=\int_{\mathbb{R}^n}\phi(\mathbf{x}-\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\rho(\mathbf{x})\,\mathrm{d}\mathbf{y}, \tag{1.2b}$$

with an entropic pressure law, $P=P(\rho)$, and subjected to prescribed initial conditions $(\rho_0,\mathbf{u}_0)$ at $t=0$. Observe that strong solutions of (1.1), which are defined over the whole space $\mathbb{R}^n$, offer themselves as pressure-less solutions of (1.2) inside the non-vacuum region, $\mathbf{x}\in\operatorname{supp}(\rho(\cdot,t))$, without worrying about the propagation of the free boundary $\rho(\cdot,t)=0$. In particular, the interpretation of the hydrodynamic model as in our main system (1.1) extends the global existence result to initial density which is supported over disconnected blobs.

There are three different regimes of interest for (1.2), depending on the behaviour of ϕ near the origin and at infinity, which emphasize local dissipation, fractional dissipation and non-local alignment. Here is a brief overview.

1. *Local dissipation.* Assume ϕ is bounded and decays sufficiently fast at infinity, such that $\int_0^\infty \phi(r) r^{n+1}\,\mathrm{d}r$ is finite (in particular, including compactly supported ϕs). We process the hyperbolic scaling, $(\mathbf{x}, t) \mapsto (\mathbf{x}/\varepsilon, t/\varepsilon)$ and $\phi \mapsto \phi_\varepsilon := (1/\varepsilon^{n+2})\phi(|\mathbf{x}|/\varepsilon)$, $\rho \mapsto \rho/\varepsilon$, and let ε go to zero; one arrives at the following system (expressed in the usual conservative form in terms ω_n, the surface area of the unit sphere in $\mathbb{R}^n$):

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{1.3a}$$

and

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = C \operatorname{div}(\mu(\rho) \nabla \mathbf{u}), \quad C := \frac{\omega_n}{2n} \int_0^\infty \phi(r) r^{n+1}\,\mathrm{d}r, \tag{1.3b}$$

with pressure $P(\rho) := \int^\rho s p'(s)\,\mathrm{d}s$ and viscosity coefficient $\mu(\rho) = \rho^2$. System (1.3*a*) and (1.3*b*) belongs to the class of compressible Navier–Stokes equations with degenerate viscosity $\mu = \rho^\theta$, $\theta > 0$ which vanishes at the vacuum. The study of such equations is mostly limited to one dimension. For existence and uniqueness of a weak solution with 'moderate degeneracy', $\theta < 1/2$, we refer to [7–9]. Mellet & Vasseur [10] proved that the degenerate viscosity is nevertheless strong enough to enforce global existence and uniqueness of the strong solution away from vacuum assuming $\rho_0 > 0$.

2. *Fractional dissipation.* Consider an influence function with a sufficiently strong singularity at the origin, $\phi(r) \sim r^{-n+2\alpha}$, associated with fractional dissipation of order 2α. The corresponding incompressible set-up (setting, formally, $\rho \equiv 1$ in (1.2)) reads

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = -(-\Delta)^\alpha \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0.$$

The L^2-energy bound implies that global smooth solutions exist for $\alpha > \frac{1}{2} + n/4$ [11,12]. Additional pointwise bounds, available in the one-dimensional case, imply that the Burgers equation with fractional dissipation, $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -(-\Delta)^\alpha \mathbf{u}$, admits global solutions for $\alpha > 1/2$; the critical case, $\alpha = 1/2$, was the subject of extensive recent studies [13–15].

3. *Non-local alignment.* In this paper, we focus our attention on the remaining case where ϕ is bounded at the origin and decays sufficiently slow at infinity. As a prototypical example, we may consider $\phi(r) = (1 + r)^{-\alpha}$ with $\alpha < 1$: the non-local alignment (owing to the divergent tail of $\int^\infty \phi(r)\,\mathrm{d}r$) enforces an unconditional flocking of (1.1*a*) and (1.1*b*) for both the CS and MT models (1.1*d*); consult [1,3,4,16].

Our first main result, stated in theorem 2.2, shows, in analogy with the agent-based models, that Lipschitz solutions of (1.1) driven by non-local alignment are self-organized into a macroscopic flock. It is therefore natural to ask: when does the system (1.1) preserve the Lipschitz regularity of $\mathbf{u}(\cdot, t)$ for all time? This question of the uniform global bound $\|\nabla_{\mathbf{x}} \mathbf{u}(\mathbf{x}, t)\|_{L^\infty} < \infty$ occupies the rest of the paper. We begin with the one-dimensional case: our second main result, summarized in corollary 2.9, shows that there exists a large set of so-called *subcritical initial configurations*, under which $\|u_x(x, t)\|_{L^\infty}$ remains bounded for all time. On the other hand, there exists a set of super-critical initial configurations which lead to the loss of regularity at finite time, $\|u_x(\cdot, t)\|_{L^\infty} \to \infty$ as $t \uparrow T_c$. The so-called *critical threshold* phenomenon for Eulerian dynamics was first systematically studied in [17] in Euler–Poisson equations, followed by a series of related studies [18–21]. In particular, Liu & Tadmor [22] studied the critical threshold phenomenon of the one-dimensional Burgers equation with non-local convolution source term,

$$u_t + u u_x = \int_{-\infty}^{\infty} \phi(|x - y|)(u(y) - u(x))\,\mathrm{d}y,$$

corresponding to the one-dimensional CS alignment system (1.1*b*) with $\rho \equiv 1$. Here, we extend this result, proving the critical threshold phenomenon for the non-vacuum density-dependent model (theorem 4.3). Thus, global regularity and hence flocking follow for subcritical data, which in turn enable us to significantly improve the critical threshold derived in [22].

We remark in passing that the same phenomenon of flocking hydrodynamics for subcritical initial data occurs in the presence of a pressure term, $\nabla p(\rho)$, added to the left-hand side of (1.1*b*) [21]; this issue is left for future work.

Next, we extend the global regularity and hence flocking result of CS hydrodynamics to two-space dimensions (theorem 2.10). These global regularity results are complemented by the unique feature of the non-local alignment, which prevents finite-time blow-up dynamics within regions of vacuum (theorem 2.7). Finally, the flocking of one- and two-dimensional MT hydrodynamics is summarized in theorem 2.12.

The paper is organized as follows. We state the main results in §2. In §3, we prove that the global strong, i.e. C^1, solution implies flocking. In §4, we prove critical thresholds for CS system (1.1), in one and two spatial dimensions. The fast alignment enhances the dynamics and leads to improved critical thresholds; this is carried out in §5.

In §6, we discuss the question of regularity inside the vacuum region where $\rho(\cdot, t) \equiv 0$. It is here that the non-local alignment in (1.1*b*) plays a key role in bounding $\nabla \mathbf{u}$ for subcritical initial configurations. This is in sharp contrast to local systems such as (1.3), where the dynamics of its pressure-less momentum (1.3*b*) inside the vacuum region is reduced to the inviscid Burgers equation, $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0$, with generic formation of shock discontinuities. Here, we show how non-local alignment prevents the formation of such shock discontinuities.

We conclude in §7, with an extension of the above results on hydrodynamic flocking to the MT system.

2. Statement of main results

(a) Flocking of strong solutions

We begin with the notion of flocking [6].

Definition 2.1 (flock). We say that a solution $(\rho, \mathbf{u})$ of (1.1) converges to a flock if the following hold:

(i) Spatial diameter is uniformly bounded, i.e. there exists $D < +\infty$ such that

$$S(t) := \sup\{|\mathbf{x} - \mathbf{y}|,\ \mathbf{x}, \mathbf{y} \in \operatorname{supp}(\rho(t))\} \le D, \quad \forall t > 0.$$

(ii) Velocity diameter decays to 0 for large time, i.e.

$$\lim_{t\to\infty} V(t) = 0, \quad V(t) := \sup\{|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{y}, t)|,\ \mathbf{x}, \mathbf{y} \in \operatorname{supp}(\rho(t))\}.$$

If $V(t)$ decays exponentially fast in time, we say the flock has a *fast alignment* property.

Theorem 2.1 shows the flocking property for *strong* (Lipschitz) solutions of the non-local alignment system (1.1). In fact, the following fast alignment holds for strong solutions of both CS and MT models. This is quantified in terms of the following interaction bound, which will be used throughout the paper:

$$\int_{\mathbf{y}} a(\mathbf{x}, \mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} \begin{cases} \le m & \text{CS model} \\ \equiv 1 & \text{MT model.} \end{cases}$$

Recalling that $\phi(\cdot) \le 1$, one may use the CS interaction bound $m := \int_{\mathbb{R}^n} \rho_0(\mathbf{y})\, d\mathbf{y}$, whereas $m = 1$ for the MT model.

Theorem 2.2 (flock with fast alignment). *Let $(\rho, \boldsymbol{u})$ be a global strong solution of system (1.1) subjected to a compactly supported density $\rho_0 = \rho(\cdot, 0)$ and bounded velocity $\boldsymbol{u}_0 = \boldsymbol{u}(\cdot, 0) \in L^\infty$. Assume*

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an influence function, ϕ, is global in the sense that

$$m \int_{S_0}^{\infty} \phi(r)\,\mathrm{d}r > V_0. \tag{2.1}$$

Then, $(\rho, \mathbf{u})$ converges to a flock with fast alignment; specifically, there exists a finite number D, such that

$$\sup_{t\geq 0} S(t) \leq D, \quad V(t) \leq V_0\,\mathrm{e}^{-m\phi(D)t}.$$

Remark 2.3. With the slow decay assumption (H) on ϕ, condition (2.1) automatically holds with finite $S_0 = S(0)$ and $V_0 = V(0)$. The constant D depends on ϕ, S_0 and V_0. An explicit expression of D is given in (3.2).

It is well known that strong solutions persist as long as $\|\nabla \mathbf{u}(\cdot, t)\|_{L^\infty}$ remains bounded. Motivated by theorem 2.2, we study below the set of initial configurations which guarantee the uniform boundedness of $\nabla \mathbf{u}$ globally in time, which in turn implies the emergence of a flock.

(b) Critical thresholds in the one-dimensional Cucker–Smale model

We study the uniform boundedness of u_x for the one-dimensional CS alignment system

$$\rho_t + (\rho u)_x = 0 \tag{2.2a}$$

and

$$u_t + u u_x = \int_{\mathbb{R}} \phi(|x-y|)(u(y) - u(x))\rho(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \ t \geq 0, \tag{2.2b}$$

subjected to initial conditions (1.1*c*), with a non-local interaction (H). In §§4 and 5, we prove that if the initial velocity has a bounded diameter V_0, and if its slope is not too negative relative to V_0, then $\|u_x\|_{L^\infty(\mathrm{supp}(\rho))}$ remains uniformly bounded for all time.[1]

Theorem 2.4 (one-dimensional critical thresholds for non-vacuum). *Consider the initial value problem of* (2.2). *There exist threshold functions $\sigma_+ > \sigma_-$ (depending on ϕ), such that the following hold.*

— *If the initial condition satisfies*

$$d_0 := \inf_{x\in\mathrm{supp}(\rho_0)} u_{0x}(x) > \sigma_+(V_0), \quad V_0 := \sup_{x,y\in\mathrm{supp}(\rho_0)} |u_0(x) - u_0(y)|, \tag{2.3}$$

then $u_x(x, t)$ remains bounded for all $(x, t) \in supp(\rho)$.

— *If the initial condition satisfies $d_0 < \sigma_-(V_0)$, then there exists a finite-time blow-up $t = T_c > 0$ such that $\inf_{x\in supp(\rho(\cdot,t))} u_x(x, t) \to -\infty$ as $t \to T_c-$.*

Remark 2.5. Detailed expressions of threshold functions σ_+ and σ_- are given in §5*b*. Figure 1 illustrates the two thresholds. To ensure boundedness of u_x, there are two requirements for the initial configurations:

(i) the initial slope of velocity u_{0x} is not too negative and
(ii) the initial diameter of velocity V_0 is not too large.

Note that, owing to symmetry, the steady state of the CS system (2.2) is given by the average value, $u = \bar{u}_0$, and the upper threshold condition (2.3) tells us that if the initial configuration is not far away from that equilibrium, then a strong solution exists and non-local alignment enforces its flocking towards steady state.

Remark 2.6. The darker areas in figure 1 represent the thresholds result stated in theorem 4.3. It is an extension of the result in [22] for the case where $\rho \equiv 1$. Taking advantage of the fast alignment property, we are able to improve the result to the lighter area.

[1]Observe that if $\phi_0 \equiv 0$ then (2.2*b*) is reduced to the inviscid Burgers equation with generic finite-time blow-up unless $d_0 > 0$. Thus, the addition of non-local alignment has a regularization effect, by increasing the initial threshold for global regularity.

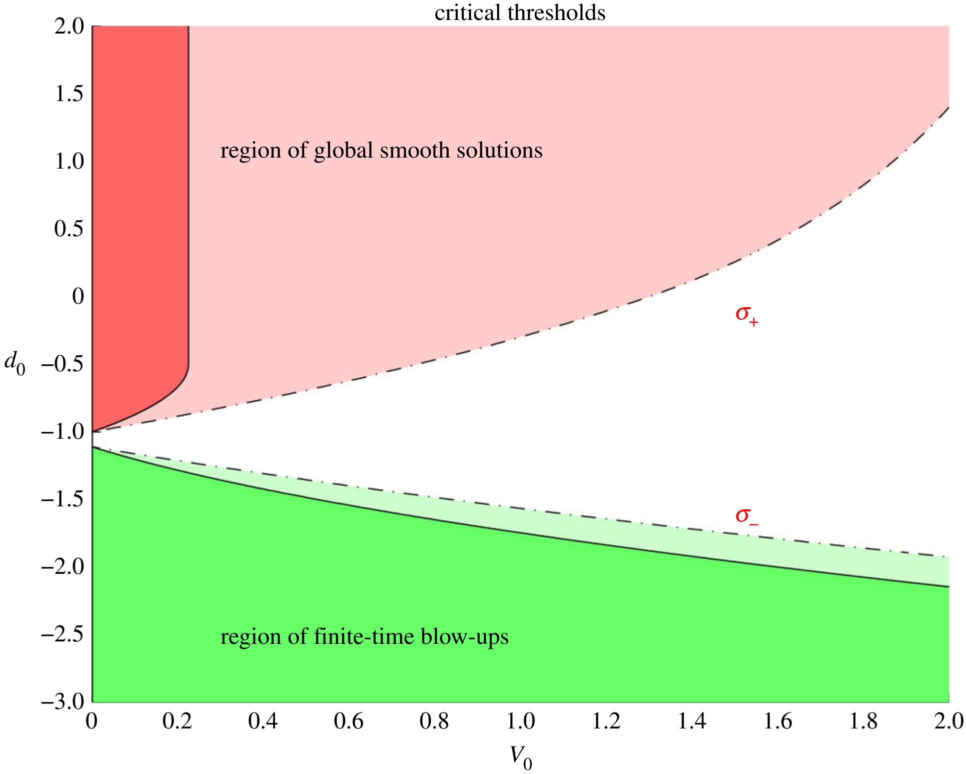

Figure 1. Illustration of the critical thresholds in one dimension. (Online version in colour.)

Theorem 2.4 is restricted to the non-vacuum portion of the solution. For local systems, e.g. (1.3), the dynamics inside the vacuum acts the same way as the inviscid Burgers equation, with generic formation of shock discontinuities. It is here that we take advantage of the non-local character of the alignment model (1.1): theorem 2.7 identifies an upper threshold which ensures that u_x remains bounded even outside the support of ρ. Thus, the non-local interaction helps with smoothing the equation and enables us to find thresholds in the vacuum region.

Theorem 2.7 (one-dimensional upper threshold for the vacuum region). *Consider the initial value problem of* (2.2). *Let V_0^λ denote the diameter of the initial velocity between a point in the non-vacuum region and a point at most λ away from that region, $V_0^\lambda := \sup\{|u_0(x) - u_0(y)| : \operatorname{dist}(x, \operatorname{supp}(\rho_0)) \le \lambda, y \in \operatorname{supp}(\rho_0)\}$.*

If the initial configuration satisfies

$$V_0^\lambda \le \frac{m\phi^2(\lambda + D)}{4|\phi'(\lambda)| + 2|\phi'(\lambda + D)|} \quad \text{for all } \lambda \ge 0 \tag{2.4a}$$

and

$$u_{0x}(x) \ge -\frac{m}{2}\phi(\operatorname{dist}(x, \operatorname{supp}(\rho_0)) + D), \tag{2.4b}$$

then $u_x(x,t)$ remains bounded for all $(x,t) \notin \operatorname{supp}(\rho)$.

Remark 2.8. Condition (2.4) has the same flavour as (2.3) for the non-vacuum area: the diameter of the initial velocity is not too large, and the slope of the initial velocity is not too negative. For (2.4*a*), when λ approaches zero, the condition is equivalent to the non-vacuum case. On the other hand, when λ approaches infinity, if $\phi(r) \sim r^{-\alpha}$, the right-hand side is proportional to $r^{1-\alpha}$. Thanks to the slow decay assumption on ϕ, i.e. $\alpha < 1$, (2.4*a*) provides no restrictions on V_0^∞. Note that, if $\alpha > 1$, the condition requires $V_0^\infty = 0$, which cannot be achieved unless u is a constant.

Combining theorems 2.4 and 2.7, we conclude that the one-dimensional CS hydrodynamics (2.2) has global strong solutions for a suitable set of *subcritical initial conditions*.

Corollary 2.9 (one-dimensional global strong solution). *Consider the one-dimensional CS system* (2.2), *then there exist thresholds* $\sigma_\pm$ *such that the following holds.*

If the initial configuration satisfies both (2.3) *and* (2.4), *then there exists a strong solution* $\rho \in L^\infty([0,+\infty), L^1(\mathbb{R}))$ *and* $u \in L^\infty([0,+\infty), W^{1,\infty}(\mathbb{R}))$. *Moreover, the solution converges to a flock in the sense of definition* 2.1.

If the initial configuration satisfies $d_0 < \sigma_-(V_0)$, *then the corresponding solution* (ρ, u) *will blow-up at a finite time.*

(c) Critical thresholds for the two-dimensional Cucker–Smale model

We extend our main result to two-space dimensions

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{2.5a}$$

and

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \int_{\mathbb{R}^2} \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y},t)-\mathbf{u}(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}, \quad \mathbf{x}\in\mathbb{R}^2,\ \ t\geq 0, \tag{2.5b}$$

where the critical threshold is captured in terms of V_0 and its initial divergence $d_0 := \inf_{\mathbf{x}\in \operatorname{supp}(\rho_0)} \operatorname{div}\mathbf{u}_0(\mathbf{x})$. The main difficulty here is to control the remaining terms in $\nabla\mathbf{u}$, beyond just div $\mathbf{u}$ itself. We measure the size of those additional terms, setting

$$B_0 := \sup_{x\in\operatorname{supp}(\rho_0)} \max\{2|\partial_{x_1}u_{02}|, 2|\partial_{x_2}u_{01}|, |\partial_{x_1}u_{01} - \partial_{x_2}u_{02}|\},$$

and we prove that if B_0 is sufficiently small then, in fact, all terms of $\nabla\mathbf{u}$, except for div $\mathbf{u}$, remain equally small.

Theorem 2.10 (two-dimensional critical thresholds in a non-vacuum region). *Consider the two-dimensional CS system* (2.5). *There exist upper threshold functions* σ_+, ζ *such that, if* $d_0 > \sigma_+(V_0)$ *and* $B_0 < \zeta(V_0)$ *at* $t=0$, *then* $\nabla_x \boldsymbol{u}(\boldsymbol{x},t)$ *remains bounded for all* $(\mathbf{x},t)\in \operatorname{supp}(\rho)$.

On the other hand, there exists a lower threshold function σ_- *such that, if* $d_0 < \sigma_-(V_0)$, $|\partial_{x_1}u_{02}|$ *and* $|\partial_{x_2}u_{01}|$ *are large enough at* $t=0$, *then there exists a finite-time blow-up at* $T_c > 0$, *where* $\inf_{\boldsymbol{x}\in \operatorname{supp}(\rho(\cdot,t))} \operatorname{div}\boldsymbol{u}(\boldsymbol{x},t) \to -\infty$ *as* $t \to T_c-$.

Remark 2.11. The smallness assumption on B_0 guarantees that the terms in $\nabla\,\mathbf{u}_0$ remain small relative to div $\mathbf{u}(\cdot,t)$. Put differently, theorem 2.10 states that if the *vorticity*, ω_0 and the *spectral gap*, η_0, are small enough at $t=0$, then they will remain small for all time (the result has the same flavour of the critical threshold in the two-dimensional *restricted* Euler–Poisson equations expressed in terms of (ω_0,η_0); [19]). Consult theorem 5.7 for all details.

Theorem 2.10 is restricted to the non-vacuum part of the dynamics. For vacuum, it is easy to derive a result analogous to the one-dimensional set-up in theorem 2.7. We omit the details.

(d) Critical thresholds for the Motsch–Tadmor system

We extend the main results to the macroscopic MT system, $\mathbf{x}\in\mathbb{R}^n$, $t\geq 0$ in $n=1,2$ spatial dimensions

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \tag{2.6a}$$

and

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \int_{\mathbb{R}} \frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\Phi(\mathbf{x},t)}(u(\mathbf{y},t)-u(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y},$$

$$\Phi(\mathbf{x},t) := \int_{\mathbb{R}} \phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}, \tag{2.6b}$$

subjected to the same initial condition (1.1*c*). A main difference from the CS system is the lack of conservation of momentum. As an example, we prove that the analogues of CS non-vacuum threshold dynamics in one and two dimensions hold for (2.6) (albeit with a different choice of critical $\sigma_\pm$).

Theorem 2.12 (critical thresholds for the Motsch–Tadmor system). *Consider the MT hydrodynamics* (2.6). *There exists threshold functions $\sigma_+ > \sigma_-$ and ζ, such that the conclusions of theorem 2.4* (*in $n=1$ dimension*) *and theorem 2.10* (*in $n=2$ dimensions*) *hold.*

3. Strong solutions must flock

Here, we prove theorem 2.2: any global strong solution of (1.1) converges to a flock with fast alignment. The key idea following [3] is to measure the decay of $S(t)$ and $V(t)$ dynamically.

Proposition 3.1 (decay estimates towards flocking). *A strong solution $(\rho, \mathbf{u})$ of the CS or MT models* (1.1) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t) \leq V(t) \tag{3.1a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) \leq -m\phi(S(t))V(t). \tag{3.1b}$$

Proof. Consider two characteristics $\dot{X}(t) = \mathbf{u}(X,t)$, $\dot{Y}(t) = \mathbf{u}(Y,t)$ subjected to initial conditions $X(0) = \mathbf{x}$, $Y(0) = \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathrm{supp}(\rho_0)$. To simplify the notations, we omit the time variable throughout the proof. First, we compute $(\mathrm{d}/\mathrm{d}t)|Y - X|^2 = 2\langle Y - X, \mathbf{u}(Y) - \mathbf{u}(X)\rangle \leq 2SV$. Taking the supreme of the left-hand side, for all $\mathbf{x}, \mathbf{y} \in \mathrm{supp}(\rho_0)$, yields the inequality (3.1*a*).

Next, define

$$b(\mathbf{x},\mathbf{y}) := \frac{1}{m} a(\mathbf{x},\mathbf{y})\rho(\mathbf{y}) + \left(1 - \frac{1}{m}\int_{\mathbb{R}^n} a(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\right)\delta_0(\mathbf{x}-\mathbf{y}), \quad a(\mathbf{x},\mathbf{y}) = \frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\Phi(\mathbf{x},t)};$$

note that because the MT model has the 'correct' scaling (with $m = 1$) then the second term involving the Dirac mass δ_0 drops out. We leave it to the reader to verify that $b(\cdot,\cdot)$ satisfies the two properties:

(P1) $\int_{\mathbb{R}^n} b(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} = 1$, for all $\mathbf{x} \in \mathrm{supp}(\rho)$.
(P2) $\int_{\mathbb{R}^n} b(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{y} = (1/m)\int_{\mathbb{R}^n} a(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}$.

To prove (3.1*b*), we use the momentum equation (1.1*b*)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}(Y) - \mathbf{u}(X)|^2 = 2\Big\langle \mathbf{u}(Y) - \mathbf{u}(X), \int_{\mathbb{R}^n} [a(Y,\mathbf{z})(\mathbf{u}(\mathbf{z}) - \mathbf{u}(Y)) \\ - a(X,\mathbf{z})(\mathbf{u}(\mathbf{z}) - \mathbf{u}(X))]\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\Big\rangle.\end{aligned}$$

We expand the second expression on the right in terms of $\eta_{X,Y}(\mathbf{z}) = \eta(\mathbf{z}) := \min\{b(X,\mathbf{z}), b(Y,\mathbf{z})\}$:

$$\begin{aligned}&\int_{\mathbb{R}^n} [a(Y,\mathbf{z})(\mathbf{u}(\mathbf{z}) - \mathbf{u}(Y)) - a(X,\mathbf{z})(\mathbf{u}(\mathbf{z}) - \mathbf{u}(X))]\rho(\mathbf{z})\,\mathrm{d}\mathbf{z} \\ &\stackrel{(P2)}{=} m\int_{\mathbb{R}^n} [b(Y,\mathbf{z})(\mathbf{u}(\mathbf{z}) - \mathbf{u}(Y)) - b(X,\mathbf{z})(\mathbf{u}(\mathbf{z}) - \mathbf{u}(X))]\,\mathrm{d}\mathbf{z} \\ &\stackrel{(P1)}{=} m\int_{\mathbb{R}^n} (b(Y,\mathbf{z}) - b(X,\mathbf{z}))\mathbf{u}(\mathbf{z})\,\mathrm{d}\mathbf{z} - m(\mathbf{u}(Y) - \mathbf{u}(X)) \\ &= m\int_{\mathbb{R}^n} (b(Y,\mathbf{z}) - \eta(\mathbf{z}))\mathbf{u}(\mathbf{z})\,\mathrm{d}\mathbf{z} - m\int_{\mathbb{R}^n} (b(X,\mathbf{z}) - \eta(\mathbf{z}))\mathbf{u}(\mathbf{z})\,\mathrm{d}\mathbf{z} - m(\mathbf{u}(Y) - \mathbf{u}(X)).\end{aligned}$$

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Set $c(\mathbf{z}) := b(Y,\mathbf{z}) - \eta(\mathbf{z}) \geq 0$ and $\mathrm{d}(\mathbf{z}) := b(X,\mathbf{z}) - \eta(\mathbf{z}) \geq 0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}(Y) - \mathbf{u}(X)|^2 \leq 2m \int_{\mathbb{R}^n} c(\mathbf{z})\,\mathrm{d}\mathbf{z} \times \max_{\mathbf{z}}\langle \mathbf{u}(Y) - \mathbf{u}(X), \mathbf{u}(\mathbf{z})\rangle$$
$$- 2m \int_{\mathbb{R}^n} \mathrm{d}(\mathbf{z})\,\mathrm{d}\mathbf{z} \times \min_{\mathbf{z}}\langle \mathbf{u}(Y) - \mathbf{u}(X), \mathbf{u}(\mathbf{z})\rangle - 2m|\mathbf{u}(Y) - \mathbf{u}(X)|^2,$$

and because by (P1), $\int c(\mathbf{z})\,\mathrm{d}\mathbf{z} = \int \mathrm{d}(\mathbf{z})\,\mathrm{d}\mathbf{z} = 1 - \int \eta(\mathbf{z})\,\mathrm{d}\mathbf{z}$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}(Y) - \mathbf{u}(X)|^2 \leq 2m\left(1 - \int \eta(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)\max_{\mathbf{x},\mathbf{y}}|\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})|^2 - 2m|\mathbf{u}(Y) - \mathbf{u}(X)|^2.$$

Because the support of ρ is compact, we can take the two maximal characteristics Y and X which realize the diameter $V = |\mathbf{u}(Y) - \mathbf{u}(X)|$. We conclude the decay estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}V^2 \leq -2m\left(\int \eta(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)V^2, \quad \eta(\mathbf{z}) = \min\{b(X,\mathbf{z}), b(Y,\mathbf{z})\}.$$

At the heart of the matter is the decay factor $\int \eta(\mathbf{z})\,\mathrm{d}\mathbf{z}$: we compute its lower bound for the CS model

$$\int_{\mathbb{R}^n} \min\{b(X,\mathbf{z}), b(Y,\mathbf{z})\}\,\mathrm{d}\mathbf{z} \geq \frac{1}{m}\phi(S)\int_{\mathbb{R}^n} \rho(\mathbf{z},t)\,\mathrm{d}\mathbf{z} = \phi(S), \quad X, Y \in \mathrm{supp}(\rho);$$

similarly, for the MT model (where $m = 1$ and $\Phi(X) \leq \int \rho(\mathbf{z})\,\mathrm{d}\mathbf{z}$), we have

$$\int_{\mathbb{R}^n} \min\{b(X,\mathbf{z}), b(Y,\mathbf{z})\}\,\mathrm{d}\mathbf{z} \geq \frac{\phi(S)}{\max_{X,Y}\{\Phi(X), \Phi(Y)\}}\int_{\mathbb{R}^n} \rho(\mathbf{z})\,\mathrm{d}\mathbf{z} \geq \phi(S), \quad X, Y \in \mathrm{supp}(\rho).$$

The result (3.1*b*) follows from the last three bounds. ■

Equipped with the decay estimates (3.1), we use the technique introduced in [16] to prove the flocking behaviour of (1.1).

Proof of theorem 2.2. Consider free energy $\mathcal{E} := V + m\int_0^S \phi(s)\,\mathrm{d}s$. The decay estimates (3.1) imply $(\mathrm{d}/\mathrm{d}t)\mathcal{E} \leq 0$ and hence $V(t) - V_0 \leq -m\int_{S_0}^{S(t)} \phi(s)\,\mathrm{d}s$. By assumption (2.1), there exists a finite number D (depending on $\phi, \rho_0, \mathbf{u}_0$), such that

$$D := \psi^{-1}(V_0 + \psi(S_0)), \quad \text{where } \psi(t) = m\int_0^t \phi(s)\,\mathrm{d}s, \tag{3.2}$$

for which $V_0 = m\int_{S_0}^D \phi(s)\,\mathrm{d}s$. Hence, we have $0 \leq V(t) \leq m\int_{S(t)}^D \phi(s)\,\mathrm{d}s$. In particular, it yields that $S(t) \leq D < \infty$, and because ϕ is monotonically decreasing, (3.1*b*) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) \leq -m\phi(D)V(t) \quad \rightsquigarrow \quad V(t) \leq V_0\,\mathrm{e}^{-m\phi(D)t} \to 0, \quad \text{as } t \to +\infty.$$

We conclude that $(\rho, \mathbf{u})$ converges to a flock with fast alignment. ■

4. Strong solutions exist for subcritical non-vacuum initial data

(a) General considerations

Here, we discuss the existence of global strong solutions of the alignment system (1.1). The goal is to control $\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}$ for all time.

Let $M \equiv M(\mathbf{x},t) = \nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x},t)$ be the gradient velocity matrix. Apply the gradient operator on both sides of (2.5*b*) to obtain

$$M_t + \mathbf{u}\cdot\nabla M + M^2 = \int_{\mathbb{R}^n} \nabla_{\mathbf{x}}\phi(|\mathbf{x}-\mathbf{y}|)\otimes(\mathbf{u}(\mathbf{y},t)-\mathbf{u}(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} - M\int_{\mathbb{R}^n}\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}.$$

Let $' = \partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}$ denote differentiation along the particle path $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x},t)$, then the above system reads

$$M' + M^2 = \int_{\mathbb{R}^n} \nabla_{\mathbf{x}}\phi(|\mathbf{x}-\mathbf{y}|)\otimes(\mathbf{u}(\mathbf{y},t)-\mathbf{u}(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} - M\int_{\mathbb{R}^n}\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}, \tag{4.1}$$

subjected to initial data $M(\mathbf{x},0) = \nabla\mathbf{u}_0(\mathbf{x})$.

Instead of working with the specific system (4.1) directly, we consider the following *majorant system*:

$$M' = -M^2 - pM + Q, \quad \text{where } 0 < \gamma \le p \le \Gamma \quad \text{and} \quad |Q_{ij}| \le c,\ i,j = 1,\dots,n. \tag{4.2}$$

Here, $p(\cdot,t)$ and the matrix $Q(\cdot,t)$ are uniformly bounded in terms of the constants γ, Γ and $\pm c$. As an example, proposition 4.1 shows that system (4.1) admits a majorant of the type (4.2), with

$$\gamma = \phi(D)m, \quad \Gamma = m, \quad c = V_0\|\phi\|_{\dot{W}^{1,\infty}}m. \tag{4.3}$$

Proposition 4.1. *Suppose* $(\rho,\boldsymbol{u})$ *is a solution of the CS system* (2.5). *Then, for any* $\boldsymbol{x}\in supp(\rho(t))$,

$$\left|\int_{\mathbb{R}^n}\partial_{x_j}\phi(|\boldsymbol{x}-\boldsymbol{y}|)(u_i(\boldsymbol{y},t)-u_i(\boldsymbol{x},t))\rho(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}\right| \le V_0\|\phi\|_{\dot{W}^{1,\infty}}m, \quad i,j=1,\dots,n,$$

$$\phi(D)m \le \int_{\mathbb{R}^n}\phi(|\boldsymbol{x}-\boldsymbol{y}|)\rho(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y} \le m.$$

Proof. For the first inequality,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\partial_{x_j}\phi(|\mathbf{x}-\mathbf{y}|)(u_i(\mathbf{y},t)-u_i(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\right|\\ &\quad\le\int_{\mathbb{R}^n}|u_i(\mathbf{y},t)-u_i(\mathbf{x},t)|\,|\partial_{x_j}\phi(|\mathbf{x}-\mathbf{y}|)|\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\\ &\quad\le V(t)\|\phi\|_{\dot{W}^{1,\infty}}\int_{\mathbb{R}^n}\rho(y,t)\,\mathrm{d}\mathbf{y} \le V_0\|\phi\|_{\dot{W}^{1,\infty}}m = V_0\|\phi\|_{\dot{W}^{1,\infty}}m.\end{aligned}$$

One half of the second inequality is straightforward $\int_{\mathbb{R}^n}\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \le \|\phi\|_{L^\infty}m = m$. For the other half, recall that the flocking behaviour of $(\rho,\mathbf{u})$ implies the uniform bound $S(t)\le D$, and hence

$$\int_{\mathbb{R}^n}\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} = \int_{\mathbf{y}\in\mathrm{supp}(\rho(t))}\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \ge \phi(D)\int_{\mathbb{R}^n}\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} = \phi(D)m,$$

for all $\mathbf{x}\in\mathrm{supp}(\rho(t))$. ∎

We proceed to discuss the regularity of the CS model in view of its majorant system (4.2).

(b) Flocking in one-dimensional Cucker–Smale hydrodynamics

We study the majorant system (4.2) for dimension $n=1$; in that case, $M=u_x$ is a scalar. Denoting $d(t)=u_x(t)$, we end up with a Riccati-type scalar equation along the particle path

$$d' = -d^2 - pd + Q, \quad \text{where } p \in [\gamma, \Gamma], \ \ Q \in [-c, c], \tag{4.4}$$

for which we have the following conditional stability; consult [22] for details.

Proposition 4.2 (critical threshold for the Riccati-type majorant). *Consider the initial value problem of* (4.4). *We have the following*:

— *If* $\gamma^2 - 4c \geq 0$ *and* $d(0) \geq -\left(\gamma + \sqrt{\gamma^2 - 4c}\right)/2$, *then* $d(t)$ *is bounded for all time* $t \geq 0$.
— *If* $d(0) < -\left(\Gamma + \sqrt{\Gamma^2 + 4c}\right)/2$, *then* $d(t) \to -\infty$ *in finite time.*

Applying proposition 4.2 for the CS majorant equation (4.4) with γ, Γ and c given in (4.3) we derive the following critical thresholds for the one-dimensional CS in the non-vacuum region.

Theorem 4.3 (one-dimensional critical thresholds). *Consider one-dimensional CS system* (2.2). *If the initial configuration satisfies*

$$V_0 \leq \frac{\phi^2(D)m}{4\|\phi\|_{\dot{W}^{1,\infty}}} \quad \textit{and} \quad d_0 \geq -\frac{1}{2}\left(\phi(D)m + \sqrt{\phi^2(D)m^2 - 4V_0\|\phi\|_{\dot{W}^{1,\infty}} m}\right),$$

then $u_x(x,t)$ *remains uniformly bounded for all* $(x,t) \in supp(\rho)$. *On the one hand, if* $d_0 < -\frac{1}{2}\left(m + \sqrt{m^2 + 4V_0\|\phi\|_{\dot{W}^{1,\infty}} m}\right)$, *then there is a finite-time blow-up at* $t = T_c$, *where*

$$\inf_{x \in \mathrm{supp}(\rho(\cdot,t))} u_x(x,t) \to -\infty \quad \textit{as } t \to T_c - .$$

Remark 4.4. The thresholds in theorem 4.3 correspond to darker areas in §5*a*, taking into account the additional fast alignment property.

(c) Flocking in two-dimensional Cucker–Smale hydrodynamics

We extend the result to two dimensions. Instead of being a scaler, M is a 2×2 matrix. The dynamics of M in (4.2) reads

$$\begin{aligned}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}' = &-\begin{bmatrix} M_{11}^2 + M_{12}M_{21} & M_{12}(M_{11}+M_{12}) \\ M_{21}(M_{11}+M_{12}) & M_{22}^2 + M_{12}M_{21} \end{bmatrix} \\ &- \begin{bmatrix} pM_{11} & pM_{12} \\ pM_{21} & pM_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},\end{aligned} \tag{4.5}$$

where $p \in [\gamma, \Gamma]$ and $|Q_{ij}| \leq c$ for $i, j = 1, 2$. For CS system (2.2), the constants are given in (4.3).

To bound the entries of M_{ij}, it is natural to employ $d := \mathrm{div}\, \mathbf{u}$ which will play the role that u_x had in the one-dimension set-up. Note that $d = \mathrm{div}\, \mathbf{u}$ is the trace $d = \mathrm{tr}(M) = \lambda_1 + \lambda_2$, where $\lambda_{1,2}$ are two eigenvalues of M. The remaining difficulty is to bound the other entries of M, namely $q := M_{11} - M_{22}$, $r := M_{12}$ and $s := M_{21}$. Expressed in terms of (d, q, r, s), system (4.5) reads

$$d' + \frac{d^2 + \eta^2}{2} = -pd + (Q_{11} + Q_{22}), \tag{4.6a}$$

$$q' + q(d+p) = Q_{11} - Q_{22}, \tag{4.6b}$$

$$r' + r(d+p) = Q_{12} \tag{4.6c}$$

and

$$s' + s(d+p) = Q_{21}. \tag{4.6d}$$

Here, the dynamics of d in (4.6*a*) involves the *spectral gap* $\eta := \lambda_1 - \lambda_2$ introduced in [19] to characterize the critical thresholds of two-dimensional restricted Euler–Poisson equations.

Observe that if η is uniformly bounded in time, $\eta(t) \le \tilde{c}$, then (4.6*a*) yields

$$d' = -\frac{d^2}{2} - pd + \tilde{Q}, \quad p \in [\gamma, \Gamma], \quad \tilde{Q} := Q_{11} + Q_{22} - \frac{\eta^2}{2} \in \left[-2c - \frac{\tilde{c}}{2}, 2c + \frac{\tilde{c}}{2}\right], \tag{4.7}$$

which is a one-dimensional equation of the type (4.2). Therefore, as argued in [19], a bound on η is at the heart of the matter for the two-dimensional critical threshold. To this end, we note the relation $\eta^2 \equiv q^2 + 4rs$ and hence, if (q, r, s) are bounded, so is the spectral gap, η. This is the content of lemma 4.5.

Lemma 4.5 (uniform bound for the spectral gap). *Suppose (q, r, s) are bounded initially by*

$$\max\{|q(0)|, 2|r(0)|, 2|s(0)|\} \le B. \tag{4.8}$$

If $d(t) \ge -\gamma + 2cB^{-1}$ for $t \in [0, T]$, then the (q, r, s) remain bounded

$$\max\{|q(t)|, 2|r(t)|, 2|s(t)|\} \le B, \quad \textit{for } t \in [0, T].$$

Moreover, the spectral gap $|\eta(t)| \le \sqrt{2}B$ is also bounded for $t \in [0, T]$.

Proof. We prove the result for q by contradiction. Suppose there exists a (smallest) $t_0 \in [0, T]$ such that $|q(t)| > B$ for $t \in (t_0, t_0 + \delta)$. By continuity, $|q(t_0)| = B$. There are two cases.

— $q(t_0) = B$, $q'(t_0) > 0$. Then $q'(t_0) + q(t_0)(d(t_0) + p) > 0 + B(2cB^{-1}) = 2c$. This contradicts (4.6*b*) as $Q_{11} - Q_{22} \le 2c$.
— $q(t_0) = -B$, $q'(t_0) < 0$. Then $q'(t_0) + q(t_0)(d(t_0) + p) < 0 - B(2cB^{-1}) = -2c$. This also contradicts (4.6*b*) as $Q_{11} - Q_{22} \ge -2c$.

Therefore, $|q(t)| \le B$ for $t \in [0, T]$. The same argument yields the boundedness of r and s. Finally, $|\eta(t)| = \sqrt{|q^2(t) + 4r(t)s(t)|} \le \sqrt{2}B$, for $t \in [0, T]$. ■

Lemma 4.5 tells us that the spectral gap η is bounded as long as the divergence d is not too negative. Under this assumption, the majorant equation (4.7) holds with $\tilde{Q} \in [-2c - B^2, 2c + B^2]$, and proposition 4.2 then yields the following result.

Proposition 4.6. *Let B denote the bound of (4.8) and assume $d(0) \ge -\gamma + \sqrt{\gamma^2 - 4c - 2B^2} \ge -\gamma + 2cB^{-1}$. Then, M_{ij} are uniformly bounded for all time.*

Proof. We claim that $d(t) \ge -\gamma + \sqrt{\gamma^2 - 4c - 2B^2} \ge -\gamma + 2cB^{-1}$ and $\max\{|q(t)|, 2|r(t)|, 2|s(t)|\} \le B$. Indeed, violation of the first condition contradicts proposition 4.2. Violation of the second condition contradicts lemma 4.5. ■

Remark 4.7. We can rewrite the assumption in proposition 4.6 as follows:

$$d(0) \ge -\gamma + \sqrt{\gamma^2 - 4c - 2B^2}, \quad B \le \tfrac{1}{2}\sqrt{(\gamma^2 - 4c) + \sqrt{(\gamma^2 - 4c)^2 - 32c^2}}.$$

Therefore, to ensure boundedness of M in all time, we need $d(0)$ not too negative, and $q(0)$, $r(0)$, $s(0)$ small.

We now combine these estimates across the fan of all particle paths. With CS set-up (4.3), we conclude the following theorem.

Theorem 4.8 (two-dimensional critical thresholds). *Consider the two-dimensional CS system (2.5). If the initial configuration satisfies the following three estimates:*

$$V_0 \le \frac{\left(\sqrt{2} - 1\right)\phi^2(D)m}{4\|\phi\|_{\dot{W}^{1,\infty}}}, \quad d_0 \ge -\frac{1}{2}\left(\phi(D)m + \sqrt{\phi^2(D)m^2 - 4V_0\|\phi\|_{\dot{W}^{1,\infty}}m - 2B_0^2}\right),$$

$$B_0 \le \frac{1}{2}\sqrt{\phi^2(D)m^2 - 4V_0\|\phi\|_{\dot{W}^{1,\infty}}m + \sqrt{(\phi^2(D)m^2 - 4V_0\|\phi\|_{\dot{W}^{1,\infty}}m)^2 - 32V_0^2\|\phi\|^2_{\dot{W}^{1,\infty}}m^2}},$$

then $\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x}, t)$ remains uniformly bounded for all $(\mathbf{x}, t) \in supp(\rho)$.

On the other hand, if

$$d_0 < -\frac{1}{2}\left(m + \sqrt{m^2 + 4V_0\|\phi\|_{\dot{W}^{1,\infty}} m}\right),$$

$$|\partial_{x_2} u_{01}|,\ |\partial_{x_1} u_{02}| \geq \frac{V_0\|\phi\|_{\dot{W}^{1,\infty}} m}{\sqrt{m^2 + 4V_0\|\phi\|_{\dot{W}^{1,\infty}} m}}, \quad \text{and} \quad \partial_{x_2} u_{01} \cdot \partial_{x_1} u_{02} > 0,$$

then there is a finite-time blow-up $t = T_c > 0$ *such that* $\inf_{x\in \mathrm{supp}(\rho(\cdot,t))} \mathrm{div}\, \boldsymbol{u}(x,t) \to -\infty$ *as* $t \to T_c-$.

Remark 4.9. Note that, if $B_0 = 0$, the above result is reduced to the one-dimensional case. In general, the bound on B (and hence on the spectral gap) restricts the range of subcritical $d(0)$, while still keeping the relevant range to include negative initial divergence.

Remark 4.10. We provide a critical threshold of the initial profile which leads to a finite-time break down. The idea and the result are similar to the one-dimensional case. The additional assumptions on $\partial_{x_2} u_{01}$ and $\partial_{x_1} u_{02}$ made in the second part of the theorem guarantee that the spectral gap $\eta(\cdot, t)$ is real for all time; we omit the proof, as it does not prevent d from a finite-time blow-up.

5. Fast alignment in Cucker–Smale hydrodynamics

(a) General considerations

Here, we introduce a new prototype of problems to characterize the dynamics of M, taking advantage of the fast alignment property.

From the proof of proposition 4.1, we have

$$\left|\int_{\mathbb{R}^n} \partial_{x_j}\phi(|\mathbf{x}-\mathbf{y}|)(u_i(\mathbf{y},t) - u_i(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\right| \leq V(t)\|\phi\|_{\dot{W}^{1,\infty}} m, \quad i,j = 1,\ldots,n.$$

Instead of taking the rough maximum principle bound $V(t) \leq V(0) = V_0$, we make use of the much stronger fast alignment property $(\mathrm{d}/\mathrm{d}t)V(t) \leq -m\phi(D)V(t)$. It leads to the following prototype of the majorant system:

$$M' = -M^2 - pM + VQ, \quad 0 < \gamma \leq p \leq \Gamma, \quad |Q_{ij}| \leq c, \ i,j = 1,\ldots,n, \tag{5.1a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} V \leq -GV. \tag{5.1b}$$

Such a majorant system holds for CS equations (2.5), with

$$\gamma = \phi(D)m, \quad \Gamma = m, \quad C = \|\phi\|_{\dot{W}^{1,\infty}} m, \quad G = \phi(D)m. \tag{5.2}$$

We now couple the dynamics of M to the dynamics of V. Because of fast alignment, the ('bad') term VQ has an exponentially decaying change on M, which enables a larger set of subcritical initial configurations which ensure the boundedness of M (illustrated in figure 1).

(b) One-dimensional flocking with fast alignment

We study the evolution of system (5.1) in one dimension, where $d = M$ is a scalar. The 2×2 system reads

$$d' = -d^2 - pd + cV, \quad p \in [\gamma, \Gamma], \quad c \in [-C, C] \tag{5.3a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} V \leq -GV. \tag{5.3b}$$

Theorem 5.1 characterizes the dynamics of (d, V).

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Theorem 5.1. *Consider the* 2×2 *majorant system* (5.3). *There exists an upper threshold function* $\sigma_+ : \mathbb{R}^+ \to [-\gamma, +\infty)$, *defined implicitly as*

$$\sigma_+(0) = -\gamma, \quad \sigma_+'(x) = \begin{cases} \dfrac{C}{\gamma + G} & x \to 0+ \\ \dfrac{-\sigma_+^2(x) - \gamma\sigma_+(x) - Cx}{-Gx} & \text{if } \sigma_+(x) < 0 \\ \dfrac{-\sigma_+^2(x) - \Gamma\sigma_+(x) - Cx}{-Gx} & \text{if } \sigma_+(x) \geq 0, \end{cases} \tag{5.4}$$

such that, if $d(0) > \sigma_+(V_0)$ *for all* x, *i.e. if* $(V_0, d(0))$ *lies above* σ_+, *then* $(V(t), d(t))$ *remain bounded for all time, and* $d(t) \to 0$, $V(t) \to 0$ *as* $t \to \infty$.

On the other hand, there exists a lower threshold function $\sigma_- : \mathbb{R}^+ \to (-\infty, -\Gamma]$,

$$\sigma_-(0) = -\Gamma \quad \sigma_-'(x) = \begin{cases} -\dfrac{C}{\Gamma + G} & x \to 0+ \\ \dfrac{-\sigma_-^2(x) - \Gamma\sigma_-(x) + Cx}{-Gx} & x > 0, \end{cases} \tag{5.5}$$

such that, if $d(0) < \sigma_-(V_0)$, *i.e.* $(V_0, d(0))$ *lies below* σ_-, *then* $d(t) \to -\infty$ *at a finite time.*

Apply theorem 5.1 to system (2.5) by plugging in the values of the constants given in (5.2) and combine across the fan of all particle paths. Theorem 2.4 follows with

$$\sigma_+(0) = -\phi(D)m, \quad \sigma_+'(x) = \begin{cases} \dfrac{\|\phi\|_{\dot{W}^{1,\infty}}}{2\phi(D)} & x \to 0+ \\ \dfrac{-\sigma_+^2(x) - \phi(D)m\sigma_+(x) - \|\phi\|_{\dot{W}^{1,\infty}}mx}{-\phi(D)mx} & \text{if } \sigma_+(x) < 0 \\ \dfrac{-\sigma_+^2(x) - m\sigma_+(x) - \|\phi\|_{\dot{W}^{1,\infty}}mx}{-\phi(D)mx} & \text{if } \sigma_+(x) \geq 0 \end{cases} \qquad \text{(Cucker–Smale: } \sigma_+)$$

and

$$\sigma_-(0) = -m, \quad \sigma_-'(x) = \begin{cases} -\dfrac{\|\phi\|_{\dot{W}^{1,\infty}}}{1 + \phi(D)} & x \to 0+ \\ \dfrac{-\sigma_-^2(x) - m\sigma_-(x) + \|\phi\|_{\dot{W}^{1,\infty}}mx}{-\phi(D)mx} & x > 0 \end{cases} \qquad \text{(Cucker–Smale: } \sigma_-).$$

Remark 5.2. Comparing theorem 4.3 and theorem 2.4, we see that the additional fast alignment property enables us to establish a much larger area of subcritical (V_0, d_0) for which u_x remains bounded in the non-vacuum area. In particular, an upper bound of V_0 is not required any more.

(c) Proof of theorem 5.1

The proof of the theorem can be separated into two parts. First, we discuss the evolution of the majorant system

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega = -\omega^2 - E\omega + F\eta \tag{5.6a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = -G\eta, \tag{5.6b}$$

where $E > 0$, $F \in \mathbb{R}$, $G > 0$ are constant coefficients. Then, we state a comparison principle, comparing (d, V) with (ω, η), and derive critical thresholds for the evolution of the inequality system (5.3).

Proposition 5.3 (critical threshold for the majorant system (5.6)). *Suppose* $(\eta(t), \omega(t))$ *satisfy* (5.6) *with initial condition* $\omega(0) = \omega_0$, $\eta(0) = \eta_0 > 0$. *Then, there exists a separatrix curve* $f(\cdot)$ *such that*

— *If* $\omega_0 > f(\eta_0)$, *i.e.* (η_0, ω_0) *lies above* f, *we have* $\omega(t) \to 0$, $\eta(t) \to 0$ *as* $t \to \infty$.

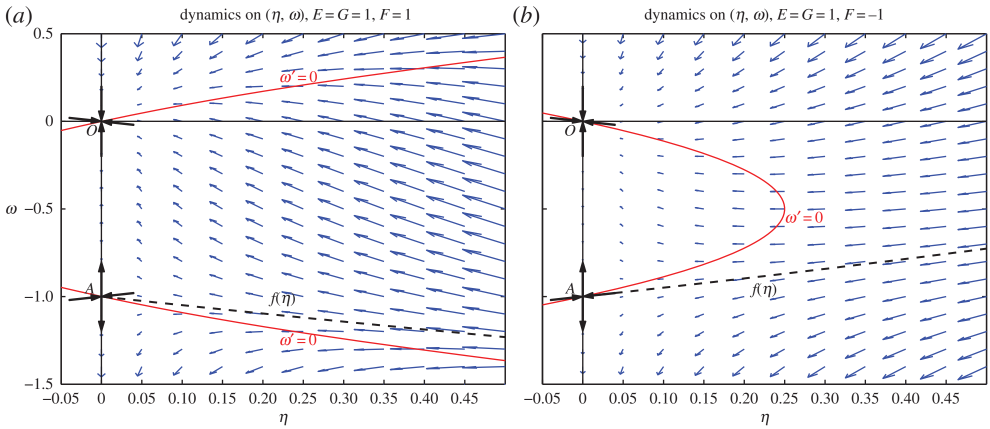

Figure 2. Phase plane of (η, ω) and critical thresholds for (*a*) $F > 0$ and (*b*) $F < 0$. (Online version in colour.)

— *If* $\omega_0 = f(\eta_0)$, *i.e.* (η_0, ω_0) *lies on* f, *we have* $\omega(t) \to -E$, $\eta(t) \to 0$ *as* $t \to \infty$.
— *If* $\omega_0 < f(\eta_0)$, *i.e.* (η_0, ω_0) *lies below* f, *we have* $\omega(t) \to -\infty$ *as* $t \to \infty$.

The separatrix f *is implicitly defined below,*

$$f(0) = -E, \quad f'(0) = -\frac{F}{E+G}, \quad f'(x) = \frac{-f^2(x) - Ef(x) + Fx}{-Gx} \quad \textit{for } x \in (0, +\infty). \tag{5.7}$$

Proof. The linearized system associated with (5.6) has two stationary points—a stable point at $O(0, 0)$ and a saddle at $A(0, -E)$. Figure 2 shows the phase plane of (η, ω). A critical curve f, starting from A and travelling along the vector field, divides the plane $\mathbb{R}^+ \times \mathbb{R}$ into two parts. Flows starting above f converge to the stable point O, whereas flows starting below f will diverge.

Along the line, clearly, we have

$$f'(x) = \frac{\mathrm{d}\omega}{\mathrm{d}\eta} = \frac{(\mathrm{d}/\mathrm{d}t)\omega}{(\mathrm{d}/\mathrm{d}t)\eta} = \frac{-\omega^2 - E\omega + F\eta}{-G\eta} = \frac{-f^2(x) - Ef(x) + Fx}{-Gx}.$$

When $x \to 0$, we have $f'(0) = \lim_{x\to 0}((-f^2(x) - Ef(x) + Fx)/-Gx) = (-Ef'(0) - F)/G$ and (5.7) follows. ■

Lemma 5.4 states the relationship between the solution of the equality system (5.6) and the inequality system (5.3). It allows us to extend the critical thresholds result to the inequality system.

Lemma 5.4 (a comparison principle). *Let* (d, V) *satisfy the inequalities* (5.3) *which involve the parameters range* $[\gamma, \Gamma]$ *and* $[-C, C]$, *and let* (ω, η) *satisfy the corresponding ODEs* (5.6) *with parameters* E, F *to be specified below.*

(*i*) *Suppose* $\omega(t) \ge 0$, *for* $t \in [t_0, T]$.

(1a) *Let* $E = \gamma$, $F = C$. *If* $\begin{cases} d(t_0) \le \omega(t_0) \\ V(t_0) \le \eta(t_0) \end{cases}$ *then* $\begin{cases} d(t) \le \omega(t) \\ V(t) \le \eta(t) \end{cases}$ *for* $t \in [t_0, T]$.

(1b) *Let* $E = \Gamma$, $F = -C$. *If* $\begin{cases} d(t_0) \ge \omega(t_0) \\ V(t_0) \le \eta(t_0) \end{cases}$ *then* $\begin{cases} d(t) \ge \omega(t) \\ V(t) \le \eta(t) \end{cases}$ *for* $t \in [t_0, T]$.

(*ii*) *Suppose* $\omega(t) \le 0$, *for* $t \in [t_0, T]$.

(2a) *Let* $E = \Gamma$, $F = C$. *If* $\begin{cases} d(t_0) \le \omega(t_0) \\ V(t_0) \le \eta(t_0) \end{cases}$ *then* $\begin{cases} d(t) \le \omega(t) \\ V(t) \le \eta(t) \end{cases}$ *for* $t \in [t_0, T]$.

(2b) *Let* $E=\gamma$, $F=-C$. *If* $\begin{cases} d(t_0)\geq\omega(t_0) \\ V(t_0)\leq\eta(t_0)\end{cases}$ *then* $\begin{cases} d(t)\geq\omega(t) \\ V(t)\leq\eta(t)\end{cases}$ *for* $t\in[t_0,T]$.

Proof. We prove only (1a); the other cases are similar. Subtracting (5.3) with (5.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega-d)\geq-(\omega+d)(\omega-d)-p(\omega-d)+C(\eta-V),\quad \frac{\mathrm{d}}{\mathrm{d}t}(\eta-V)\geq-G(\eta-V).$$

Suppose by contradiction $V(t)>\eta(t)$ for some $t\in(t_0,T)$. As V,η are continuous, there exists $\tau\in(t_0,t)$ such that $\eta(\tau)-V(\tau)=0$ and $(\mathrm{d}/\mathrm{d}t)(\eta(\tau)-V(\tau))<0$. This violates the second inequality. So, $V(t)\leq\eta(t)$ for all $t\in[t_0,T]$. Similarly, suppose by contradiction $d(t)>\omega(t)$ for some $t\in(t_0,T)$. As V,η are continuous, there exists $tau\in(t_0,t)$ such that $d(tau)-\omega(tau)=0$ and $(\mathrm{d}/\mathrm{d}t)(d(\tau)-\omega(\tau))<0$. Meanwhile, $V(\tau)\leq\eta(\tau)$. This violates the first inequality. So, $d(t)\leq\omega(t)$ for all $t\in[t_0,T]$. ∎

We can use this comparison principle to verify theorem 5.1. First, d remains bounded from above. Indeed, suppose by contradiction that $d(t)\to+\infty$ as $t\to T$. Then, there exists a $t_0\in[0,T)$ such that $d(t_0)>0$. Construct (ω,η) by (5.6) with $E=\gamma$, $F=C$ and with initial values $\omega(t_0)=d(t_0)>0$, $\eta(t_0)=V(t_0)$. But according to proposition 5.3, $\omega(t)$ is bounded from above and comparison principle (1a) implies that $d(t)\leq\omega(t)$ is also upper bounded. To prove that d is bounded from below, we apply the same comparison argument. If $\sigma_+(x)<0$, we use (2b). If $\sigma_+(x)\geq0$, we use (1b). Details are left to the reader. For lower threshold σ_-, we prove that $d(t)\to-\infty$ in a finite time using comparison principle (2a). Again, we omit the details.

(d) Two-dimensional flock with fast alignment

Here, we invoke the fast alignment property to derive critical thresholds determined by initial quantities (V_0,d_0,B_0), which are more relaxed than those in theorem 4.8.

First, rewrite system (4.6) coupled with fast decay property (5.1*b*).

$$d'+\frac{d^2+\eta^2}{2}=-pd+(Q_{11}+Q_{22}),\tag{5.8a}$$

$$q'+q(d+p)=(Q_{11}-Q_{22})V,\tag{5.8b}$$

$$r'+r(d+p)=Q_{12}V,\tag{5.8c}$$

$$s'+s(d+p)=Q_{21}V\tag{5.8d}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}V=-GV,\tag{5.8e}$$

where $p\in[\gamma,\Gamma]$ and $|Q_{ij}|\leq c$ for $i,j=1,2$. We now state the uniform boundedness result for the spectral gap η.

Lemma 5.5. *Let* $b_0=\max\{|q(0)|,2|r(0)|,2|s(0)|\}$. *Suppose there exists a positive constant* δ *such that* $d(t)\geq-\gamma+\delta$ *for all* $t\geq0$. *Then, there exists a threshold* $\zeta=\zeta(V_0;\delta,B)$ *such that if* $b_0\leq\zeta$, *then* (q,r,s) *are uniformly bounded*, $\max\{|q(0)|,2|r(0)|,2|s(0)|\}\leq B$.

The details of the function ζ are given below

$$\zeta(x;\delta,B)=\begin{cases} B & x\in\left[0,\dfrac{\delta B}{2C}\right] \\ \dfrac{B}{\delta-G}\left[-G\left(\dfrac{2C}{\delta B}x\right)^{\delta/G}+\dfrac{2C}{\delta B}x\right] & x\in\left[\dfrac{\delta B}{2C},\left(\dfrac{\delta}{G}\right)^{G/(\delta-G)}\dfrac{\delta B}{2C}\right],\ \delta\neq G\,. \\ \dfrac{2C}{\delta}\left(1-\log\left(\dfrac{2C}{\delta B}x\right)\right)x & x\in\left[\dfrac{\delta B}{2C},\dfrac{\delta Be}{2C}\right],\ \delta=G\end{cases}\tag{5.9}$$

Lemma 5.5 provides a region in phase space of $(b_0,\ V_0)$ such that the spectral gap is uniformly bounded in all time. From the definition of ζ, we observe that, to guarantee a uniform upper

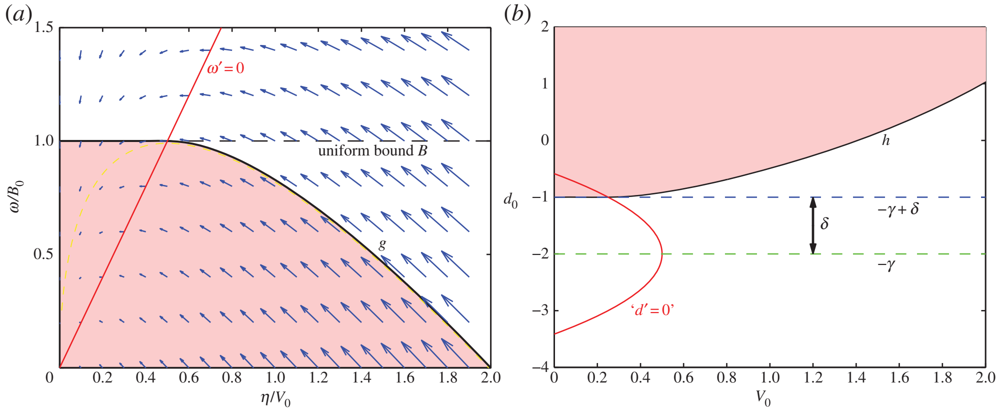

Figure 3. (*a*) Dynamics of the system (η, ω); (*b*) dynamics of the system (V, d). (Online version in colour.)

bound, B, V_0 cannot be too large. Given δ and B, the upper bound of V_0 is $(\delta/G)^{G/(\delta-G)}(\delta B/2C)$, independent of the choice of b_0.

Proof. We prove the result for q. Consider the coupled system (5.8*b*) and (5.8*e*). The corresponding majorant system reads $\omega' = -\delta\omega + 2C\eta$, $\eta' = -G\eta$. This system can be easily solved. Figure 3*a* shows the dynamics of (η, ω). The filled area includes all initial conditions such that $\omega(t) \le B$ for all $t \ge 0$. The area is governed by a function g. A simple computation yields an explicit expression of g, which is stated in (5.9).

A comparison argument enables us to connect the equality system with the inequality system, which says

$$\text{If} \begin{cases} |q(0)| \le \omega(0) \\ V_0 \le \eta(0) \end{cases} \quad \text{then} \begin{cases} |q(t)| \le \omega(t) \\ V(t) \le \eta(t) \end{cases}, \quad \text{for all } t \ge 0.$$

Therefore, $|q|$ is bounded by B uniformly in time as long as $(V_0, |q(0)|)$ lies inside the area, i.e. $|q(0)| \le g(V_0)$. Similarly, we prove for r and s, which ends the proof. ■

Next, for given δ and B, we consider the coupled system (5.8*a*) and (5.8*e*) and find the region of $(V_0, d(0))$ such that $d(t) \ge -\gamma + \delta$.

Proposition 5.6. *Suppose there exists a* B *such that* $|\eta(t)| \le B \le \gamma/\sqrt{2}$ *for* $t \ge 0$ *and let* $\delta \in \left(0, \sqrt{\gamma^2 - 2B^2}\right]$. *Then, there exists a threshold* $\sigma_+ = \sigma_+(V_0; \delta, B)$ *such that if* $d(0) \ge \sigma_+$, *then* $d(t)$ *remains uniformly bounded in time, and* $d(t) \ge -\gamma + \delta$ *for* $t \ge 0$. *The upper threshold* σ_+ *is defined implicitly*

$$\sigma_+(x; \delta, B) = -\gamma + \delta, \quad x \in \left[0, \frac{\gamma^2 - \delta^2 - 2B^2}{4C}\right) \tag{5.10a}$$

and

$$\sigma'_+(x; \delta, B) = \begin{cases} \dfrac{\sigma_+^2(x) + 2\gamma\sigma_+(x) + 4Cx + 2B^2}{2Gx} & \text{if } \sigma_+(x) < 0, \\ \dfrac{\sigma_+^2(x) + 2\Gamma\sigma_+(x) + 4Cx + 2B^2}{2Gx} & \text{if } \sigma_+(x) \ge 0, \end{cases} \quad x \in \left[\frac{\gamma^2 - \delta^2 - 2B^2}{4C}, +\infty\right). \tag{5.10b}$$

Similar to the one-dimensional case, proposition 5.6 can be easily proved by analysis of the equality system and a comparison rule. Figure 3*b* shows the area of $(V_0, d(0))$ such that $d(t)$ is lower bounded by $-\gamma + \delta$ for all time. The area is governed by h defined in (5.10). We omit the details of the proof.

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Theorem 5.7 (two-dimensional critical thresholds). *Consider the two-dimensional CS system* (2.5) *with majorant systems involving the constants* (γ, Γ, C, G) *given in* (5.2). *If there exists* (δ, B) *such that* $\delta^2 + 2B^2 \leq \gamma^2$, *and the initial profiles* (V_0, d_0, B_0) *satisfy*

(*i*) $B_0 \leq \zeta(V_0; \delta, B)$, *where* ζ *is defined in* (5.9), *and* (ii) $d_0 \geq \sigma_+(V_0; \delta, B)$, *where* σ_+ *is defined in* (5.10), *then* $|\nabla \mathbf{u}(x,t)|$ *remains bounded all* $(x,t) \in supp(\rho)$.

Remark 5.8. The theorem guarantees the boundedness of $\nabla \mathbf{u}$ provided B_0 is not too large and d_0 is not too negative.

6. Strong solutions in the presence of vacuum

Here, we discuss the boundedness of $\nabla \mathbf{u}$ when $(\mathbf{x}, t) \notin \mathrm{supp}(\rho)$. The result allows us to study the system in whole space, without worrying about the free boundary. It also extends the global existence result to the initial density which is supported over disconnected blobs. In the case of standard local models, lack of any relaxation inside the vacuum enables the solution to form shock discontinuities in finite time. In the present set-up, however, non-local alignment prevents the formation of shock discontinuities.

(a) Dynamics inside the vacuum

Consider the dynamics of $\nabla \mathbf{u}$ (4.2) for $\mathbf{x} \notin \mathrm{supp}(\rho_0)$. Define the maximum diameter of the velocity field between a point in the whole space and a point in the non-vacuum area

$$V^{\infty}(t) := \sup\{|\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{y},t)|,\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{y} \in \mathrm{supp}(\rho(\cdot,t))\},$$

and the distance between $\mathbf{x}$ and the non-vacuum region $L(\mathbf{x},t) := \mathrm{dist}(\mathbf{x},\ \mathrm{supp}(\rho(\cdot,t)))$. We have the following bounds (to be compared with those in proposition 4.1).

Proposition 6.1 (bounds inside the vacuum). *Suppose* $(\rho, \mathbf{u})$ *is a solution of system* (1.1). *Then, for any* $\mathbf{x} \notin supp(\rho(t))$,

$$\left| \int_{\mathbb{R}^n} \partial_{x_j} \phi(|\boldsymbol{x} - \boldsymbol{y}|)(u_i(\boldsymbol{y},t) - u_i(\boldsymbol{x},t))\rho(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y} \right| \leq V^{\infty}(0)|\phi'(L(\boldsymbol{x},t))|m, \quad i,j = 1,\ldots,n,$$

$$\phi(L(\boldsymbol{x},t) + D)m \leq \int_{\mathbb{R}^n} \phi(|\boldsymbol{x} - \boldsymbol{y}|)\rho(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y} \leq m.$$

Proof. For the first inequality,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \partial_{x_j} \phi(|\mathbf{x} - \mathbf{y}|)(u_i(\mathbf{y},t) - u_i(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \right| \\
&\quad \leq \int_{\mathrm{supp}(\rho(t))} |u_i(\mathbf{y},t) - u_i(\mathbf{x},t)|\,|\partial_{x_j}\phi(|\mathbf{x} - \mathbf{y}|)|\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \\
&\quad \leq V^{\infty}(t)|\phi'(L(\mathbf{x},t))| \int_{\mathrm{supp}(\rho(t))} \rho(y,t)\,\mathrm{d}\mathbf{y} \leq V^{\infty}(0)|\phi'(L(\mathbf{x},t))|m.
\end{aligned}$$

The last inequality is valid owing to the maximum principle. For the second inequality, $\int_{\mathbb{R}^n} \phi(|\mathbf{x} - \mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \leq \|\phi\|_{L^\infty} m = m$. On the other hand, as $(\rho, \mathbf{u})$ converges to a flock, $S(t)$ is uniformly bounded by D, defined in (3.2). Hence, $\max_{y \in \mathrm{supp}(\rho(t))} \mathrm{dist}(\mathbf{x}, \mathbf{y}) \leq L(x,t) + D$. It yields

$$\int_{\mathbb{R}^n} \phi(|\mathbf{x} - \mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} = \int_{\mathrm{supp}(\rho(t))} \phi(|\mathbf{x} - \mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \geq \phi(L(x,t) + D)m.$$

■

Remark 6.2. The key estimate in proposition 6.1, which distinguishes itself from the local system, is the positive lower bound of $\phi \star \rho$.

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Next, we turn to discuss the criterion to guarantee the boundedness of $\|\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x},t)\|_{L^\infty}$ in whole space, using a similar technique to that in §4. For the sake of simplicity, we focus on the one-dimensional case. A similar result can be easily established for two dimensions.

Lemma 6.3. *Assume the initial configuration is such that* $V^\infty(0)$ *and* u_{0x} *satisfy*

$$V^\infty(0) \le \inf_{r\ge 0}\left[\frac{m\phi^2(r+D)}{4|\phi'(r)| + 2|\phi'(r+D)|}\right] \tag{6.1a}$$

and

$$u_{0x}(x) \ge -\frac{m}{2}\phi(L(x,0)+D), \quad \text{for } x \notin \operatorname{supp}(\rho_0). \tag{6.1b}$$

Then, $u_x(x,t)$ *remains bounded for all time for* $(x,t) \notin \operatorname{supp}(\rho)$.

Remark 6.4. Condition (6.1*a*) has two aspects. (i) Slow decay at infinity. Suppose $\phi(r) \approx r^{-\alpha}$ as $r\to\infty$. The right-hand side is proportional to $r^{1-\alpha}$. If ϕ decays fast with $\alpha > 1$, i.e. (H) is violated, the right-hand side goes to 0. The condition cannot be achieved unless u_0 is a constant. A slow decay assumption on ϕ is needed to make sure the condition is meaningful. (ii) Behaviour of V^∞ near the origin. Take $r=0$, the condition reads $V^\infty(0) \le m\phi^2(D)/(4\|\phi\|_{\dot W^{1,\infty}} + 2|\phi'(D)|)$. This is equivalent to the thresholds of V_0 in proposition 4.2, assuming $V^\infty(0) \lesssim V_0$.

As for condition (6.1*b*)—it is satisfied automatically for large $|x|$. As a matter of fact, when $|x|\to\infty$, (6.1*b*) says that $u_{0x}(x) \gtrsim -|x|^{-\alpha}$. This is a consequence of $u_0 \in L^\infty(\mathbb{R})$ and the fact that $\alpha \le 1$.

Proof of lemma 6.3. Consider $(x,t)\notin\operatorname{supp}(\rho)$. It belongs to a characteristic starting from $(x_0,0)$, where $x_0 \notin \operatorname{supp}(\rho_0)$, as long as u_x is bounded. At this point, we have $d' = -d^2 - pd + Q$, with $p\in[\phi(L(x,t)+D)m, m]$ and $|Q| \le V^\infty(0)|\phi'(L(x,t))|m$.

It is then sufficient to discuss the following majorant equation and use the comparison principle to draw the desired conclusion on d,

$$\omega' = -\omega^2 - \phi(L(x,t)+D)m\omega - V^\infty(0)|\phi'(L(x,t))|m.$$

Condition (6.1*a*) ensures that the right-hand side has two distinguished solutions. In particular, if we pick $\omega = -\frac{1}{2}\phi(L(x,t)+D)m$, then

$$\omega' = \tfrac{1}{4}\phi^2(L(x,t)+D)m^2 - V^\infty(0)|\phi'(L(x,t))|m \overset{(6.1a)}{\ge} \tfrac{1}{2}|\phi'(L(x,t)+D)|V^\infty(0)m > 0.$$

Let $A(x_0)$ denote the area where $\omega \ge -\frac{1}{2}\phi(L(x,t)+D)m$, and $(x,t)=(X(t),t)$ is a point on the characteristic starting from $(x_0,0)$, namely

$$A(x_0) := \{(z,t) | z \ge -\tfrac{1}{2}(\phi(L(X(t),t)+D)m),\ t\ge 0\}.$$

Its boundary $\partial A(x_0)$ reads $\partial A(x_0) = \{(\gamma(t),t) | t\ge 0\}$, where $\gamma(t) = -\frac{1}{2}\phi(L(X(t),t)+D)m$.

Criterion (6.1*b*) implies $(\omega(0),0)\in A(x_0)$. We are left to show that $(\omega(t),t)$ stays in $A(x_0)$ for all $t\ge 0$. As $A(x_0)$ is uniformly bounded from below in z, it implies ω is lower bounded in all time.

Finally, we prove that $(\omega(t),t)\in A(x_0)$ for $t\ge 0$ by contradiction.

Suppose there exist $t>0$ such that $(\omega(t),t)\in\partial A(x_0)$, and $(\omega(t+\delta),t+\delta)\notin A(x_0)$. It means that $\omega(t)=\gamma(t)$ and $\omega'(t) < \gamma'(t)$. On the other hand, we compute

$$\gamma'(t) = \frac{m}{2}\phi'(L(X(t),t)+D)\frac{\mathrm{d}}{\mathrm{d}t}L(X(t),t) \le \frac{1}{2}|\phi'(L(x,t)+D)|V^\infty(0)m.$$

The last inequality is true as both $\partial(\operatorname{supp}(\rho))$ and X are travelling with a speed between $u_{\min}$ and $u_{\max}$. It yields $(\mathrm{d}/\mathrm{d}t)L(X(t),t) \le V^\infty(t) \le V^\infty(0)$. Combined with the estimate on ω', we conclude that $\omega'(t)\ge\gamma'(t)$, which leads to a contradiction. ■

(b) Fast alignment property inside the vacuum

In §4*a*, we derived a much larger region of critical threshold for $(x,t)\in\operatorname{supp}(\rho)$, assuming a fast alignment property. Here, we extend the enhanced result to the vacuum area. We start by showing

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a fast alignment property where vacuum is involved. As the strength of viscosity at point $(\mathbf{x}, t)$ is determined by $L(\mathbf{x}, t)$, it is natural to introduce the following definitions.

Definition 6.5 (level sets). For any level $\lambda \geq 0$, define

$$\Omega^\lambda(t) = \left\{ X(t) \,\middle|\, \begin{cases} \dot{X}(t) = \mathbf{u}(X, t) \\ X(0) = \mathbf{x} \end{cases} \quad L(\mathbf{x}, 0) \leq \lambda \right\},$$

$$S^\lambda(t) = \sup\{|\mathbf{x} - \mathbf{y}|,\ \mathbf{x} \in \Omega^\lambda(t),\ \mathbf{y} \in \Omega^0(t)\},$$

$$V^\lambda(t) = \sup\{|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{y}, t)|,\ \mathbf{x} \in \Omega^\lambda(t),\ \mathbf{y} \in \Omega^0(t)\}.$$

If $\lambda = 0$, $\Omega^0(t) = \mathrm{supp}(\rho(t))$, and $S^0(t)$, $V^0(t)$ coincides with $S(t)$, $V(t)$ respectively. Moreover, $S^\lambda(0) = S_0 + \lambda$. If $\lambda = \infty$, $V^\infty(t)$ coincide with the definition before.

Theorem 6.6 (fast alignment on Ω^λ). *Let $(\rho, \mathbf{u})$ be a global strong solution of system (1.1). Suppose the influence function ϕ satisfies $m\int_{S_0}^\infty \phi(r)\,\mathrm{d}r > V^\lambda(0)$. Then, there exists a finite number D^λ (given by $D^\lambda = \psi^{-1}(V^\lambda(0) + \psi(S_0 + \lambda)$, $\psi(t) := m\int_0^t \phi(s)\,\mathrm{d}s)$, such that $\sup_{t\geq 0} S^\lambda(t) \leq D^\lambda$ and $V^\lambda(t) \leq V^\lambda(0)\,\mathrm{e}^{-m\phi(D^\lambda)t}$.*

Remark 6.7. The proof of theorem 6.6 follows the same idea as in proposition 3.1 and theorem 2.2 by considering X as a characteristic starting from $\mathbf{x} \in \Omega^\lambda(0)$. We observe that $V^\lambda(t)$ still has an exponential decay in time, with rate $m\phi(D^\lambda)$. When λ becomes larger, the rate becomes smaller. However, as long as λ is finite, we always have fast alignment.

We are now ready to derive an improvement of lemma 6.3 using a fast alignment property.

Proof of theorem 2.7. We repeat the proof of lemma 6.3 using a better bound on the term Q which reads $|Q| \leq V^{L(x_0,0)}(t)|\phi'(L(x,t))|m$. In addition, we use a better bound on $(\mathrm{d}/\mathrm{d}t)L(X(t), t) \leq V^{L(x_0,0)}(t)$. It yields the following modified condition:

$$V^{L(x_0,0)}(t) \leq \frac{m\phi^2(L(x,t) + D)}{4|\phi'(L(x,t))| + 2\phi'(L(x,t) + D)|},$$

for all x_0 and t, with $(x, t) = (X(t), t)$ being a point on the characteristics starting from $(x_0, 0)$.

When $t = 0$, let $\lambda = L(x_0, 0)$, we obtain the condition (2.4*a*) stated in the theorem, i.e.

$$V^\lambda(0) \leq \frac{m\phi^2(\lambda + D)}{4|\phi'(\lambda)| + 2\phi'(\lambda + D)|}.$$

Finally, we prove that if (2.4*a*) holds, then the modified condition automatically holds for all $t > 0$. Take $\lambda = L(x, t)$; it suffices to prove that $V^{L(x_0,0)}(t) \leq V^\lambda(0)$. Applying theorem 6.6, we are left to prove $V^{L(x_0,0)}(0)\,\mathrm{e}^{-m\phi(D^{L(x_0,0)})t} \leq V^{L(x,t)}(0)$. This is true if V^λ grows slower than the exponential rate in λ, which is the case for the finite V^∞. ■

7. Critical thresholds for the macroscopic Motsch–Tadmor model

Here, we briefly discuss the critical thresholds phenomenon of the MT system stated in theorem 2.12. The evolution of the gradient velocity matrix $M = \nabla\mathbf{u}$ reads

$$M' + M^2 = \int_{\mathbb{R}^n} \nabla_{\mathbf{x}} \left(\frac{\phi(|\mathbf{x} - \mathbf{y}|)}{\Phi(\mathbf{x}, t)} \right) \otimes (\mathbf{u}(\mathbf{y}, t) - \mathbf{u}(\mathbf{x}, t))\rho(\mathbf{y}, t)\,\mathrm{d}\mathbf{y} - M. \tag{7.1}$$

Proposition 7.1. shows that (7.1) admits a majorant system of the type (5.1), with

$$\gamma = \Gamma = 1, \quad c = \frac{2\|\phi\|_{\dot{W}^{1,\infty}}}{\phi(D)}, \quad G = \phi(D),$$

and the existence of critical thresholds for MT hydrodynamics follows along the lines of theorems 2.4, 5.1 in the $n = 1$ dimension and theorems 2.10 and 5.7 in the $n = 2$ dimensions.

Proposition 7.1. *Suppose* $(\rho, \mathbf{u})$ *is a solution of system* (2.6). *Then, for any* $x \in supp(\rho(t))$,

$$\left|\int_{\mathbb{R}^n} \partial_{x_j}\left(\frac{\phi(|x-y|)}{\Phi(x,t)}\right)(u_i(y,t)-u_i(x,t))\rho(y,t)\,\mathrm{d}y\right| \le \frac{2\|\phi\|_{\dot{W}^{1,\infty}}}{\phi(D)} V(t), \quad i,j=1,\ldots,n.$$

Proof. We begin with the estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \partial_{x_j}\left(\frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\Phi(\mathbf{x},t)}\right)(u_i(\mathbf{y},t)-u_i(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\right| \\
&\quad\le \int_{\mathbb{R}^n} |u_i(\mathbf{y},t)-u_i(\mathbf{x},t)| \left|\partial_{x_j}\left(\frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\Phi(\mathbf{x},t)}\right)\rho(\mathbf{y},t)\right| \mathrm{d}\mathbf{y} \\
&\quad\le V(t)\int_{\mathbb{R}^n} \frac{|\partial_{x_j}\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\Phi(\mathbf{x},t)-\phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\partial_{x_j}\Phi(\mathbf{x},t)|}{\Phi^2(\mathbf{x},t)}\,\mathrm{d}\mathbf{y} \\
&\quad\le V(t)\left[\frac{\int_{\mathbb{R}^n}|\partial_{x_j}\phi(|\mathbf{x}-\mathbf{y}|)|\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}+|\partial_{x_j}\Phi(\mathbf{x},t)|}{\Phi(\mathbf{x},t)}\right] \le \frac{2V(t)\int_{\mathbb{R}^n}|\partial_{x_j}\phi(|\mathbf{x}-\mathbf{y}|)|\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}}{\Phi(\mathbf{x},t)}.
\end{aligned}$$

Because by theorem 2.2 $S(t) \le D$, then $\Phi(\mathbf{x},t) = \int_{\mathbb{R}^n} \phi(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \ge \phi(D)\int_{\mathbb{R}^n}\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}$ for all $\mathbf{x} \in \mathrm{supp}(\rho(t))$. Hence, $|\int_{\mathbb{R}^n} \partial_{x_j}(\phi(|\mathbf{x}-\mathbf{y}|)/\Phi(\mathbf{x},t))(u_i(\mathbf{y},t)-u_i(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}| \le (2\|\phi\|_{\dot{W}^{1,\infty}}/\phi(D))V(t)$. ■

Using the same argument, we claim that theorem 2.12 holds with the following thresholds functions:

$$\sigma_\pm(0) = -1, \quad \sigma'_\pm(x) = \begin{cases} \dfrac{2\|\phi\|_{\dot{W}^{1,\infty}}}{(1+\phi(D))\phi(D)} & x \to 0+ \\[2ex] \dfrac{-\phi(D)\sigma_\pm^2(x)-\phi(D)\sigma_\pm(x) \mp 2\|\phi\|_{\dot{W}^{1,\infty}} x}{-\phi^2(D)x} & x > 0 \end{cases} \quad \text{(Motsch–Tadmor: } \sigma_\pm).$$

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