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Inequalities in Fourier Analysis on R^n

(Fourier transform/convolution/Hermite semigroup)

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ABSTRACT **This note describes two results: (*i*) a sharp Hausdorff-Young inequality for the Fourier transform on $L^p(R^n)$ which extends an earlier result of Babenko; and (*ii*) a sharp form of Young's inequality for the convolution of functions on R^n. That is, best possible constants are obtained for the following $L^p(R^n)$ inequalities: $\|\mathfrak{F}f\|_{p'} \leq C_p\|f\|_p$, $1 \leq p \leq 2$, and $1/p + 1/p' = 1$; and $\|f * g\|_r \leq C_{p,q,r}\|f\|_p\|g\|_q$, $1 \leq p,q,r \leq \infty$ with $1/r = 1/p + 1/q - 1$. $C_p = [p^{1/p}/p'^{1/p'}]^{n/2}$ and $C_{p,q,r} = C_pC_qC_{r'}$.**

Two classical inequalities in Fourier analysis are the Hausdorff-Young inequality for the Fourier transform

$$\|\mathfrak{F}f\|_{p'} \leq \|f\|_p \tag{1}$$

$f \in L^p(R^n)$, $1 \leq p \leq 2$, and $1/p + 1/p' = 1$, and Young's inequality for convolutions

$$\|f * g\|_r \leq \|f\|_p\|g\|_q \tag{2}$$

$f \in L^p(R^n)$, $g \in L^q(R^n)$, $1 \leq p,q,r \leq \infty$, and $1/r = 1/p + 1/q - 1$. Here the Fourier transform is defined for integrable functions by

$$(\mathfrak{F}f)(x) = \int_{R^n} \exp\,(2\pi ixy)f(y)dy \tag{3}$$

and

$$(f * g)(x) = \int_{R^n} f(x - y)g(y)dy \tag{4}$$

These inequalities have their origin in the efforts of W. H. Young in 1912 to generalize Parseval's theorem for Fourier series to other L^p-classes, and they extend naturally in the context of analysis on locally compact abelian groups. For the circle group $\mathbf{T} \approx \mathbf{R}/\mathbf{Z}$ the corresponding inequalities are sharp, but for the Fourier transform on $L^p(R)$, K. I. Babenko proved in 1961 that $\|\mathfrak{F}f\|_{p'} \leq A_p\|f\|_p$ with $A_p = [p^{1/p}/p'^{1/p'}]^{1/2}$ for the special case where the upper exponent was an even integer, i.e., $p' = 2k$ and $p = 2k/(2k - 1)$ (refs. 1 and 2). Babenko's proof used methods of entire functions and showed that gaussian functions $\exp(-ax^2)$, $a > 0$ were extremal functions. This result together with a convexity argument implied that in general inequalities [1] and [2] would have new sharp forms. The application of probabilistic methods to the study of the Hermite semigroup in recent work of Nelson (ref. 3) and Gross (ref. 4) suggested a different approach to this problem, and we have obtained the following theorem extending Babenko's result.

THEOREM 1. *For $f \in L^1(R^n)$ define $(\mathfrak{F}f)(x) = \int \exp(2\pi ixy)f(y)dy$. Then the Fourier transform $\mathfrak{F}$ extends to a bounded linear operator on $L^p(R^n)$ to $L^{p'}(R^n)$ with $1 < p \leq 2$, $1/p + 1/p' = 1$ and*

$$\|\mathfrak{F}f\|_{p'} \leq (A_p)^n\|f\|_p \tag{5}$$

$$A_p = [p^{1/p}/p'^{1/p'}]^{1/2} \tag{6}$$

We sketch some of the ideas involved in the proof of *Theorem 1* for one dimension. The n-dimensional result will follow from an application of *Lemma 2* below. Consider the operator T_ω defined by the Mehler kernel

$$T_\omega(x,y) = (1 - \omega^2)^{-1/2} \times \exp\left\{-\frac{\omega^2}{2(1 - \omega^2)}(x^2 + y^2) + \frac{\omega xy}{1 - \omega^2}\right\} \tag{7}$$

on $L^2(d\mu)$ with $d\mu(x) = (2\pi)^{-1/2}\exp(-x^2/2)\,dx$ and $|\omega| < 1$. The operator T_ω maps polynomials into polynomials, i.e., if H_m denotes the mth Hermite polynomial corresponding to the measure $d\mu$, then $T_\omega H_m = \omega^m H_m$. We first observe that, essentially by a change of variables argument, *Theorem 1* for $n = 1$ is equivalent to the following multiplier theorem for the Hermite semigroup.

THEOREM 2. *Let $g \in L^p(d\mu)$ and define $(T_\omega g)(x) = \int T_\omega(x,y)g(y)d\mu(y)$.* Then for $\omega = i\sqrt{p-1}$, $1 < p \leq 2$, *and $1/p + 1/p' = 1$, $T_\omega g \in L^{p'}(d\mu)$ and*

$$\|T_\omega g\|_{L^{p'}(d\mu)} \leq \|g\|_{L^p(d\mu)} \tag{8}$$

We prove this multiplier inequality for a dense set of functions, namely polynomials, and the crucial step is to obtain the gaussian measure $d\mu$ as a limiting probability distribution using the classical central limit theorem. Suppose we consider a sequence of Bernoulli trials; let $d\nu$ be the discrete probability measure with positive weight $1/2$ at the points $x = \pm 1$, and define $d\nu_n(x)$ as the n-fold convolution of $d\nu(\sqrt{n}x)$ with itself. Then $d\nu_n$ converges to $d\mu$ in $C_0(R)^*$, and in addition the moments of $d\nu_n$ converge to the moments of $d\mu$. Note that $\int h(x)d\nu_n(x) = \int h(x_1 + \cdots + x_n)\,d\nu(\sqrt{n}x_1)\cdots d\nu(\sqrt{n}x_n)$.

Consider the measure space of functions over the product measure $d\nu(\sqrt{n}x_1)\cdots d\nu(\sqrt{n}x_n)$, observe that all such functions are polynomials of degree at most one in each of the separate variables, and define operators $C_{n,k}: a + bx_k \to a + \omega bx_k$, where a and b are functions of the remaining $n - 1$ variables, and $K_n = C_{n,1} \cdots C_{n,n}$. First, we show that $C = C_{1,1}$ is a bounded linear operator on $L^p(d\nu)$ to $L^{p'}(d\nu)$ with norm 1, and then by a lemma on products of operators this "two-point inequality" will imply that K_n is a bounded linear operator with norm 1 on $L^p[d\nu(\sqrt{n}x_1)\cdots d\nu(\sqrt{n}x_n)]$ to $L^{p'}[d\nu(\sqrt{n}x_1)\cdots d\nu(\sqrt{n}x_n)]$. Clearly the restriction $\bar{K}_n$ of K_n to the subspace of functions symmetric in the n variables will also be a linear operator of norm 1. We denote this measure space of symmetric functions by X_n.

The functions $\varphi_{n,l}(x_1,\ldots,x_n) = l!\ \sigma_l(x_1,\ldots,x_n)$, $0 \le l \le n$, where the σ's are the elementary symmetric functions in n variables, form an orthogonal basis in $L^2[X_n]$, and, with respect to the limiting process $d\nu_n \to d\mu$, the functions $\varphi_{n,l}$ will "converge" in some sense to the Hermite polynomial H_l. In fact, $\varphi_{n,l}(x_1,\ldots,x_n) = K_l(x_1 + \cdots + x_n) +$ other terms, these additional terms "going to zero in norm over the measure space X_n." Note that the x_i assume only two values, $\pm 1/\sqrt{n}$. We remark that $\bar{K}_n\varphi_{n,l} = \omega^l\varphi_{n,l}$, the statement that $\bar{K}_n$ maps $L^p[X_n]$ into $L^{p'}[X_n]$ with norm 1, is an analogue of the multiplier inequality contained in *Theorem 2*, and a more detailed convergence argument using the relationship between $\varphi_{n,l}$ and H_l will show that the operators $\bar{K}_n$ having norm 1 implies that T_ω has norm 1 on $L^p(d\mu)$ to $L^{p'}(d\mu)$ where $\omega = i\sqrt{p-1}$, $1 < p \le 2$ and $1/p + 1/p' = 1$.

LEMMA 1. *$C: a + bx \to a + \omega bx$ is a bounded linear operator with norm one on $L^p(d\nu)$ to $L^{p'}(d\nu)$ with $\omega = i\sqrt{p-1}$, $1 < p \le 2$, and $1/p + 1/p' = 1$; that is, for all $a,b \in \mathbf{C}$*

$$\left\{\frac{|a+\omega b|^{p'} + |a-\omega b|^{p'}}{2}\right\}^{1/p'} \le \left\{\frac{|a+b|^p + |a-b|^p}{2}\right\}^{1/p} \quad [9]$$

By some algebraic manipulation and use of the classical Minkowski inequality, this inequality can be reduced to observing that for fixed $y > 0$ the function

$$\left\{\frac{\left|1 + \dfrac{y}{\sqrt{p-1}}\right|^p + \left|1 - \dfrac{y}{\sqrt{p-1}}\right|^p}{2}\right\}^{1/p} \quad [10]$$

is monotone decreasing as a function of p, $1 < p < \infty$.

LEMMA 2. *Consider two linear operators T_1 and T_2 defined by kernels; suppose*

$$T_1: L^p[d\rho_1] \to L^q[d\lambda_1] \qquad \|T_1\| \le 1$$
$$T_2: L^p[d\rho_2] \to L^q[d\lambda_2] \qquad \|T_2\| \le 1$$

then the product $T_1T_2: L^p[d\rho_1 \times d\rho_2] \to L^q[d\lambda_1 \times d\lambda_2]$ with $\|T_1T_2\| \le 1$ if $p \le q$.

With the addition of some measure-theoretic remarks about integration over product measures, the following steps that essentially contain the proof of this lemma can be made rigorous.

$$\begin{aligned}\Big\{\iint|(T_1T_2f)(x_1,x_2)|^q d\lambda_1(x_1)d\lambda_2(x_2)\Big\}^{1/q} &\le \Big\{\int d\lambda_2(x_2)\,[\int|(T_2f)(y_1,x_2)|^p d\rho_1(y_1)]^{q/p}\Big\}^{1/q}\\ &\le \Big\{\int d\rho_1(y_1)\,[\int|(T_2f)(y_1,x_2)|^q d\lambda_2(x_2)]^{p/q}\Big\}^{1/p}\\ &\le \Big\{\iint|f(y_1,y_2)|^p d\rho_1(y_1)d\rho_2(y_2)\Big\}^{1/p}\end{aligned}$$

Here we have used that both T_1 and T_2 are operators of norm one, and we have interchanged orders of integration using Minkowski's inequality for integrals.†

The relationship between the Fourier transform and convolution is basic to the study of harmonic analysis. Under the action of the Fourier transform convolution goes over to pointwise multiplication; that is, for integrable functions $\mathfrak{F}(f*g) = (\mathfrak{F}f)(\mathfrak{F}g)$. Because of this relationship there is a duality between the basic inequalities for these two operators. As a general result, we have obtained the following sharp form of Young's inequality for convolutions.

THEOREM 3. *For $f \in L^p(R^n)$, $g \in L^q(R^n)$, $1 \le p,q,r \le \infty$ and $1/r = 1/p + 1/q - 1$*

$$\|f*g\|_r \le (A_pA_qA_{r'})^n\|f\|_p\|g\|_q \quad [11]$$

$A_m = [m^{1/m}/m'^{1/m'}]^{1/2}$ and primes always denote dual exponents, $1/m + 1/m' = 1$.

But as a consequence of the sharp Hausdorff-Young inequality of *Theorem 1*, we can obtain immediately the following partial result.

THEOREM 3′. *For $1 \le p,q,r' \le 2$ and $1/r = 1/p + 1/q - 1$*

$$\|f*g\|_r \le (A_pA_qA_{r'})^n\|f\|_p\|g\|_q$$

Note that at least two of these exponents will always be ≤ 2. Consider $n = 1$ and observe that $\|f*g\|_r \le A_{r'}\|(\mathfrak{F}f)(\mathfrak{F}g)\|_{r'} \le A_{r'}\|\mathfrak{F}f\|_{p'}\|\mathfrak{F}g\|_{q'} \le A_{r'}(A_p\|f\|_p)(A_q\|g\|_q)$ for $1/r' = 1/p' + 1/q'$. In addition, the sharp Hausdorff-Young inequality of *Theorem 1* for the special case where p' is an even integer can be obtained directly from the sharp Young's inequality for convolutions in *Theorem 3*. *Theorems 1, 2,* and *3′* were obtained about a year ago. Influenced by these results, Brascamp and Lieb have recently obtained an independent proof of *Theorem 3*.

To obtain the general result of sharp convolution inequalities in *Theorem 3*, the basic problem is to calculate the one-dimensional norm

$$\mathfrak{C} = \sup\frac{\|f*g\|_r}{\|f\|_p\|g\|_q} = \sup\frac{\|f*g*h\|_\infty}{\|f\|_p\|g\|_q\|h\|_{r'}} \quad [12]$$

† See E. M. Stein (1970) *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, N.J.), p. 271.

with $1 \leq p,q,r \leq \infty$ and $1/r = 1/p + 1/q - 1$. Observe that on $\mathbf{R}^n$ the convolution operation has a product structure, in that it acts on the variables separately with respect to dimension, and it takes positive functions to positive functions.

LEMMA 3. *The convolution norm for n-dimensions will be $\mathfrak{C}^n$ where $\mathfrak{C}$ is the one-dimensional norm.*

Consider $n = 2$ and observe that for positive functions on $\mathbf{R}^2$

$$(f * g * h)(x) = \int f(x - y - z)g(y)h(z)dy\,dz$$
$$\leq \mathfrak{C}\int [\int |f(x_1 - y_1 - z_1,t)|^p dt]^{1/p} [\int |g(y_1,t)|^q dt]^{1/q}$$
$$\times [\int |h(z_1,t)|^{r'} dt]^{1/r'} dy_1 dz_1 \leq \mathfrak{C}^2 \|f\|_p \|g\|_q \|h\|_{r'}$$

But the two-dimensional norm is seen to be at least $\mathfrak{C}^2$ by considering products of functions in one variable. Note that the content of this lemma extends to the case where we consider the convolution of an arbitrary number of functions. In the general consideration of convolution inequalities the following lemma (refs. 5–7) allows a restriction to radial functions that are decreasing.

LEMMA (*Hardy and Littlewood, F. Riesz, Sobolev*)

$$\int_{R^n} h(x)(f_1 * \cdots * f_m)(x)dx \leq \int_{R^n} \bar{h}(x)(\bar{f}_1 * \cdots * \bar{f}_m)(x)dx \quad [13]$$

$\bar{f}$ denotes the equimeasurable symmetric decreasing rearrangement of f.

The original lemma was proved for rearrangements of three functions, but it is not difficult to extend this result to an arbitrary number of functions.‡

In a rough sense the interplay between these two lemmas would force an essentially unique extremal solution for which the maximum norm is attained to consist of gaussian functions. That is, for measurable functions on $\mathbf{R}^n$ the only way for a product of functions, each radial in separate variables, to also be radial in the variables jointly is for the functions to be gaussian. We modify the convolution problem in a natural way so that a smooth extremal solution will exist in two dimensions. In this modification, or regularization, we have retained the product structure of the convolution operation, which can then be used to show this smooth extremal solution must consist of gaussian functions. We then calculate the norm for the modified problem, and obtain through a limiting argument the norm for the original convolution inequality.

‡ Riesz remarks that this is an immediate consequence of the method used by Hardy and Littlewood for the rearrangements of series. For example, see the argument given on pp. 216–217 in G. Sampson (1971) "Sharp estimates of convolution transforms in terms of decreasing functions," *Pac. J. Math.* **38**, 213–231. This argument is independent of dimension.

We restrict our attention to symmetric decreasing functions in two dimensions. Let $k(x_1,x_2) = A \exp [-\alpha(x_1^2 + x_2^2)]$, $\alpha > 0$, and $\|k\|_{p_4} = 1$ be a fixed gaussian function, and consider the two-dimensional norm for the convolution of four functions with one fixed. That is,

$$\mathfrak{D}^2 = \sup \frac{\|f * g * h * k\|_\infty}{\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3}} \quad [14]$$

where $1 < p_1,p_2,p_3,p_4 < \infty$ and $1/p'_1 + 1/p'_2 + 1/p'_3 + 1/p'_4 = 1$. The fixed gaussian function k is smooth and of rapid decrease, and the symmetric decreasing functions f, g, h will have uniform majorizations on bounded sets, i.e., if $\|f\|_{p_1} \leq 1$, then $f(r) \leq [1/\pi r^2]^{1/p_1}$. These conditions insure by a weak compactness argument that a smooth extremal solution will exist so that the norm $\mathfrak{D}^2$ is attained. The gaussian function k splits into the product of one-dimensional functions, and the product structure of convolution allows us to consider this problem as the product of one-dimensional operations. By so doing we obtain relations to show that the extremal solution determined above must consist of gaussian functions. It is then easy to calculate the one-dimensional norm.

$$\mathfrak{D} = A_{p_1}A_{p_2}A_{p_3}A_{p_4} \qquad A_p = [p^{1/p}/p'^{1/p'}]^{1/2} \quad [15]$$

Let f, g, h be step functions, $k(x) = B \exp(-p'_4 x^2)$ with $\|k\|_{p_4} = 1$ and in the limit $p_4 \to 1$ we obtain

$$\mathfrak{C} = \sup \frac{\|f * g * h\|_\infty}{\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3}} \quad [16]$$

where now $1/p'_1 + 1/p'_2 + 1/p'_3 = 1$. More generally, we obtain the following sharp form of Young's inequality for convolutions.

THEOREM 4. *For $f_i \in L^{p_i}(R^n)$ with $1 < p_i < \infty$ and $1/p'_1 + \cdots + 1/p'_m = 1/r'$, $1 < r \leq \infty$, then*

$$\|f_1 * \cdots * f_m\|_r \leq [A_{p_1} \cdots A_{p_m} A_{r'}]^n \|f_1\|_{p_1} \cdots \|f_m\|_{p_m} \quad [17]$$

$A_p = [p^{1/p}/p'^{1/p'}]^{1/2}$ and in the limit $p \to 1$ or $p \to \infty$, $\lim A_p = 1$. As illustrated in our argument to obtain the sharp convolution inequalities, certain gaussian functions will be extremal functions on which the maximum norms are attained for *Theorems 1, 3,* and *4.*

We would like to mention here the relation between Nelson's inequality for the Hermite semigroup (ref. 3) and sharp convolution inequalities on the line. The notation is that used in *Theorem 2.*

LEMMA 4. *For real ω with $0 \leq \omega \leq [(p - 1)/(q - 1)]^{1/2}$ and $p \leq q$, the inequality*

$$\|T_\omega g\|_{L^q(d\mu)} \leq \|g\|_{L^p(d\mu)} \quad [18]$$

is equivalent to the following convolution inequality on the line

$$\|k * f\|_r \leq (A_{p_1}A_{p_2}A_{r'})\|f\|_{p_1}\|k\|_{p_2} \quad [19]$$

where k is a gaussian function, $f \in L^{p_1}(R)$ and $1 < p_1,p_2,r' < \infty$ with $1/r = 1/p_1 + 1/p_2 - 1$.

In addition, it is possible to write down a multiplier inequality on the Hermite semigroup that extends Nelson's inequality to several functions and is equivalent to the sharp convolution inequalities of *Theorem 4* for $n = 1$. This multiplier inequality can be expressed in terms of a kernel of several variables that generalizes the classical Mehler kernel, and the equivalence with *Theorem 4* for $n = 1$ reflects the same duality that exists between *Theorem 1* for $n = 1$ and *Theorem 2*.

These sharp results on Euclidean space also contain the usual inequalities for the torus; that is, it is easy to give a limiting argument so that the sharp inequalities for the Fourier transform and convolutions contained in *Theorems 1* and *4* will imply the usual sharp inequalities for the torus $\mathbf{T}^n \approx (\mathbf{R}/\mathbf{Z})^n$, namely inequalities [**1**] and [**2**].

Finally, we comment about the problem of sharp L^p inequalities for analysis on a locally compact abelian group. First, there is a remarkable structure theorem which states that any locally compact abelian group G is topologically isomorphic to a product $R^n \times G_0$, where G_0 is a locally compact abelian group which contains an open compact subgroup H_0, and the dimension n is an invariant of the group (ref. 8, Theorem 24.30). Essentially this product structure for a locally compact abelian group, together with *Lemma 2* above and the theorem of Hewitt and Hirschman for the Fourier transform on groups G_0 (ref. 8, Theorem 43.13) allows the natural extension of these sharp L^p inequalities on R^n to sharp L^p inequalities on locally compact abelian groups, thus affirming a conjecture of Hewitt and Ross (ref. 8, p. 630 in Vol. II).

Detailed proofs and discussion of these results will appear in the author's Princeton thesis.

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