Genus Ranges of Chord Diagrams

Jonathan Burns[*], **Nataša Jonoska**, and **Masahico Saito**

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

Abstract

A chord diagram consists of a circle, called the backbone, with line segments, called chords, whose endpoints are attached to distinct points on the circle. The genus of a chord diagram is the genus of the orientable surface obtained by thickening the backbone to an annulus and attaching bands to the inner boundary circle at the ends of each chord. Variations of this construction are considered here, where bands are possibly attached to the outer boundary circle of the annulus. The genus range of a chord diagram is the genus values over all such variations of surfaces thus obtained from a given chord diagram. Genus ranges of chord diagrams for a fixed number of chords are studied. Integer intervals that can be, and those that cannot be, realized as genus ranges are investigated. Computer calculations are presented, and play a key role in discovering and proving the properties of genus ranges.

Keywords

Chord diagrams; genus ranges; double-occurrence words

1. Introduction

A chord diagram is a circle (called the backbone) with line segments (called chords) attached at their endpoints. Chord diagrams have been extensively used in knot theory and its applications, as well as in physics and biology. They are main tools for finite type knot invariants [2], and are also used for describing RNA secondary structures [1], for example. A chord diagram is usually depicted as a circle in the plane with chords inside the circle. The chords may intersect in the circle, but such intersections are ignored (chords are regarded as pairwise disjoint).

The genus of a chord diagram is the genus of the orientable surface obtained by thickening the backbone to an annulus and attaching bands to the inner boundary circle at the ends of each chord, and it has been studied earlier in the context of knot theory. In [8], for example, it was pointed out that the genus of a chord diagram equals that of a surface obtained from the Seifert algorithm, a standard construction of orientable surfaces bounded by knots from diagrams. This fact was used in [8] to define the genus of virtual knots as the minimum of such genera over all virtual knot diagrams. Such a genus was used in [1] for the study of

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[*]*jtburns@mail.usf.edu.*

RNA foldings. Thickened chord diagrams were used for the study of DNA structures as well [5].

The genus of a chord diagram is defined by attaching bands at chord endpoints on the inner boundary circle of the annulus as mentioned above, but different surfaces could be obtained if some bands are allowed to be attached on the outer boundary circle of the annulus. It is, then, natural to ask which integers arise as genera of surfaces if such variants are allowed for thickened chord diagrams. Specifically, we consider the following questions.

Problem

For a given positive integer n, (1) determine which sets of integers appear as genus ranges of chord diagrams with n chords, and (2) characterize chord diagrams with n chords that have a specified genus range.

The genus range of graphs has been studied in topological graph theory [7]. Our focus in this paper is on a special class of trivalent graphs that arise as chord diagrams, and the behavior of their genus ranges for a fixed number of chords. The genus ranges of 4-regular rigid vertex graphs were studied in [3], where the embedding of rigid vertex graphs is required to preserve the given cyclic order of edges at every vertex.

The paper is organized as follows. Preliminary material is presented in Sec. 2. A method of computing the genus by the Euler characteristic is given in Sec. 3, where results of computer calculations are also presented. In Sec. 4, various properties of genus ranges are described, and some sets of integers are realized as genus ranges in Sec. 5. In Sec. 6, results from Secs. 4 and 5 are combined to summarize our findings on which sets of integers can and cannot be realized as genus ranges of chord diagrams for a fixed number of chords. We also list the sets for which realizability as the genus range of a chord diagram has yet to be determined, and end with some short concluding remarks.

2. Terminology and Preliminaries

This section contains the definitions of the concepts, their basic properties, and the notations used in this paper.

A *chord diagram* consists of a finite number of *chords*, that are closed arcs, with their endpoints attached to a circle, called the *backbone*. An example of a chord diagram is given in Fig. 1(a). For more details and the background of chord diagrams, see, for example, [2, 8].

A *double-occurrence* word w over an alphabet set is a word which contains each symbol of the alphabet set exactly 0 or 2 times. Double-occurrence words are also called (*unsigned*) *Gauss codes* in knot theory [6].

For a given chord diagram, we obtain a double-occurrence word as follows. If it has n chords, assign distinct labels (e.g. positive integers $\{1, \ldots, n\}$) to the chords. The endpoints of the chords lying on the backbone inherit the labels of the corresponding chords. Pick and fix a *base point* * on the backbone of a chord diagram. The sequence of endpoint labels obtained by tracing the backbone in one direction (say, clockwise) forms a double-

occurrence word corresponding to the chord diagram. Conversely, for a given double-occurrence word, a chord diagram corresponding to the word is obtained by choosing distinct points on a circle such that each point corresponds to a letter in the word in the order of their appearance, and then connecting each pair of points of the same letter by a chord. The chord diagram in Fig. 1(a) has the corresponding double-occurrence word 123132. Two double-occurrence words are equivalent if they are related by cyclic permutations, reversal, and/or symbol renaming. An equivalence relation on chord diagrams is defined accordingly.

Notation

Applying the above-mentioned correspondence between chord diagrams and double-occurrence words, in this paper a double-occurrence word *W* also represents the corresponding chord diagram.

A *thickened* chord diagram (or simply a *thickened diagram*) is a compact orientable surface obtained from a given chord diagram by thickening its backbone circle and chords as depicted in Figs. 1(b) and 1(c). The backbone is thickened to an annulus. A band corresponding to each chord is attached to one of two boundary circles of the annulus. In literature (e.g. [1, 8]), all bands are attached to the inner boundary of the thickened circle as in Fig. 1(b), and in this case we say that chords are *all-in*, or that the chord diagram is of *all-in*. For a chord diagram *D* we denote with F_D the all-in thickened chord diagram corresponding to *D*. In this paper, we consider thickened chord diagrams with band ends possibly attached to the outer boundary circle of the annulus, as is one of the ends of chord 1 in Fig. 1(c). Since each endpoint of a chord has two possibilities of band ends attachments (inner or outer), there are four possible band attachment cases for each chord, in total 4^n surfaces obtained from a chord diagram with *n* chords. To simplify exposition, we draw an endpoint of a chord attached to the outer side of the backbone as in Fig. 1(d) to indicate that the corresponding thickened diagram is obtained by attaching the corresponding band end to the outer boundary of the annulus. A band whose one end is connected to the outside curve of the annulus and the other is connected to the inside part of the curve is said to be a *one-in, one-out chord*.

Convention

We assume that all surfaces are orientable throughout the paper.

Definition 2.1—Let *g*(*F*) denote the genus of a compact orientable surface *F*. The *genus range* of a chord diagram *D* is the set of genera of thickened chord diagrams, and denoted by gr(*D*):

$$gr(D) = \{g(F) \,|\, F \text{ is a thickened chord diagram of } D\}.$$

The genus range defined here is different than the genus of virtual knots. In Fig. 2(a), two representations of flat virtual knots (see [4]) corresponding to the Gauss word 1212 are depicted, and both have genus 1. The genus range of the corresponding four-valent rigid vertex graph (see [3]) is {1}. In Fig. 2(b), the essential cases of chord connections for the chord diagram 1212 are depicted, providing the genus range {0, 1} for the word 1212.

We use the following terminology in the later sections. The portion of the backbone containing the base point, between the first and last chord endpoints, is called the *end edge*. Because the backbone and the chords are thickened to bands that constitute a thickened diagram, we regard that each backbone arc and each chord have two corresponding boundary curve segments, which may or may not belong to the same connected component of the boundary. In particular, the boundary curves corresponding to the end edge may belong to one or two boundary components, as depicted in Figs. 3(a) and 3(b), respectively. In each case, we say that the end edge is *traced by a single* (*respectively, double*) *boundary curve*(*s*).

3. Computing the Genus Range of a Chord Diagram

In this section we recall the Euler characteristic formula used to compute the genus ranges by counting the number of boundary components, and present outputs of computer calculations.

3.1. *Euler characteristic formula*

First we recall the well-known Euler characteristic formula, establishing the relation between the genus and the number of boundary components. The Euler characteristic $\chi(F)$ of a compact orientable surface F of genus $g(F)$ and the number of boundary components $b(F)$ of F are related by $\chi(F) = 2 - 2g(F) - b(F)$.

A thickened chord diagram F is a compact surface with the original chord diagram D as a deformation retract. If the number of chords is $n > 0$, $n \in \mathbb{Z}$, then there are $2n$ vertices in D and $3n$ edges (n chords and $2n$ arcs on the backbone), so that $\chi(F) = \chi(D) = 2n - 3n = -n$. Thus we obtain the following well-known formula, which we state as a lemma, as we will use it often in this paper.

Lemma 3.1—Let F be a thickened chord diagram of a chord diagram D. Let $g(F)$ *be the genus of F,* $b(F)$ be the number of boundary components of F, and n be the number of chords of D. Then we have $g(F) = (1/2)(n - b(F) + 2)$.

Thus we can compute the genus range from the set of the numbers of boundary components of thickened chord diagrams, $\{b(F) | F$ is a thickened chord diagram of $D\}$. Note that n and $b(F)$ have the same parity, as genera are integers.

3.2 *Computer calculations*

In [1], the genus of chord diagrams were defined (which is the genus of all-in chord diagrams), and an algorithm to compute the number of graphs with a given genus and n chords by means of cycle decompositions of permutations was presented. Our computer calculation is based on a modified version of their algorithm. The computational results are posted at http://knot.math.usf.edu/data/ under *Tables*.

The sets of all possible genus ranges of chord diagrams with n letters for $n = 1, \ldots, 7$ are shown in Table 1.

The following conjectures hold for all examples we computed.

Conjecture 3.2—For any n > 0, if a chord diagram with n chords has genus range consisting of two numbers, then the genus range is either {0, 1} or {1, 2}.

Conjecture 3.3—For any n ≥ 2, there is a unique (up to equivalence) double-occurrence word $11 \cdots nn$ that corresponds to a chord diagram with the genus range {0, 1}.

We note that there are two 2-letter words, 1122 and 1212, and both corresponding chord diagrams have the genus range {0, 1}.

Conjecture 3.4—For any n ≥ 4, there is a unique (up to equivalence) chord diagram with genus range {1, 2}, and it is $(123123)(44 \cdots nn)$.

There are several more chord diagrams for $n = 4$ with genus range {1, 2}.

4. Properties of Genus Ranges

The following is standard for cellular embeddings of general graphs [7], and also known for 4-regular rigid vertex graphs [3]. Below we state the property for chord diagrams.

Proposition 4.1

The genus range of any chord diagram consists of consecutive integers.

By Proposition 4.1 the genus ranges of chord diagrams are integer intervals, therefore in the rest of the paper we use the notation $[a, b] = \{k \in \mathbb{Z} | a \leq k \leq b\}$.

Lemma 4.2

There does not exist a chord diagram whose genus range consists only of a singleton.

Proof—Since all-in thickened diagram F_D for a chord diagram D has an outside boundary component and some inside ones, any chord diagram has a thickened chord diagram with more than one boundary component. Let n be the number of boundary components of F_D, one of which is the outside circle.

Let c be a chord in D. Then removing the corresponding band (thickened chord) from F_D either increases or decreases the number of boundary components of a thickened diagram by exactly one. If c is traced by a single boundary component, then its band removal splits the component in two parts, and if c is traced by two components, then the removal of its band connects the two components as a single one. Suppose that when a band for c is removed from F_D, the number of boundary components increases by one. Let D' be the chord diagram with c removed from D, and consider $F' = F_{D'}$, the all-in thickened diagram of D'. Then the number of boundary components of F' is $n + 1$. In this case, adding the band of c back to F' to obtain F_D will connect two inside boundary components of F'. Instead, connecting both ends of the band of c to the outside boundary circle of F' increases the

number of boundary components by one, and gives rise to a thickened diagram of D with n + 2 boundary components, and with genus $g(F_D) + 1$. Hence, gr(D) is not a singleton.

We repeat a similar argument for the case when the number of boundary components of F_D decreases by one when a band of c is removed. Again, let D' be the chord diagram with c removed from D and $F' = F_{D'}$ be the all-in thickened diagram of D', then the number of boundary components of F' is $n - 1$. Adding to F' a one-in, one-out chord for c connects the original inside boundary with the outside curve and decreases the number of boundary components by one, giving rise to a thickened diagram of D with $n - 2$ boundary components with genus $g(F_D) - 1$.

The connected sum of two chord diagrams with base points is defined in a manner similar to the connected sum of knots, see Fig. 4. A band is attached at the base points preserving orientations to obtain a new chord diagram. In the figure, the left and right chord diagrams, respectively, before taking connected sum are represented by double-occurrence words $W_1 =$ 123123 and $W_2 = 123132$, respectively, and after the connected sum, it is represented by W = 123123456465, after renaming W_2. We use the notation $W = W_1 W_2$ to represent the word thus obtained, by renaming and concatenation.

Lemma 4.3

Let W_1 and W_2 be chord diagrams such that the genus ranges of corresponding chord diagrams are $[g_1, g_1']$ and $[g_2, g_2']$, respectively. Let e_1, e_2 be the end edges of W_1, W_2, respectively. Then the genus range of the chord diagram corresponding to $W = W_1 W_2$ is $[g_1 + g_2 - \in, g_1' + g_2' - \in']$ for some, $\in, \in' \in \{0, 1$ where $, \in, \in'$ are determined as follows.

(E_0) $\in = 0$ if and only if at least one of the end edges (say, the end edge e_1 of W_1) has the following property: any thickened graph of genus g_1 traces e_1 by two boundary curves.

(E_1) $\in = 1$ if and only if both end edges e_1 and e_2 have the following property: there exist thickened graphs of genus g_1 and g_2, respectively, that trace both e_1 and e_2 by a single boundary curve.

(E_0') $\in' = 0$ if and only if at least one of the end edges (say, e_1 of W_1) has the following property: there exists a thickened graph of genus g_1' that traces e_1 by two boundary curves.

(E_1') $\in' = 1$ if and only if both end edges e_1 and e_2 of W_1 and W_2 have the following property: any thickened graphs of genus g_1' and g_2', respectively, trace e and e' by a single boundary curve.

Proof—This is proved by a case-by-case analysis of the number of boundary components and by using Lemma 3.1. A similar argument is found in [3, 5]. Let n_1, n_2 be the number of chords of chord diagrams corresponding to W_1, W_2, respectively. The number of chords of $W = W_1 W_2$ is $n = n_1 + n_2$. Let b_1 and b_2 be the number of boundary components of thickened chord diagrams for W_1 and W_2, respectively. The number of boundary component b of a thickened diagram D of $W = W_1 W_2$ after taking the connected sum equals $b_1 + b_2 - a$ where $a = 0$ or $a = 2$. If both end edges e_1 and e_2 are traced by a single component (the situation as

in Fig. 5(a)) then $a = 0$. If at least one end edge e_1 or e_2 is traced by two components (the situations Figs. 5(b) and 5(c)), then $a = 2$. Then Lemma 3.1 implies

$$g = \frac{1}{2}(n - b + 2) = \frac{1}{2}[(n_1 + n_2) - (b_1 + b_2 - \alpha) + 2] = g_1 + g_2 + \frac{\alpha}{2} - 1,$$

where g_1, g_2, and g are genera of W_1, W_2, and W, respectively.

For statement (E_1), there are thickened diagrams with minimal genus of W_1 and W_2 for which $a = 0$ and whose connected sum preserves the number of boundary components (Fig. 5(a)), hence the statement follows. The other cases are proved by similar arguments.

Since we often refer to the number of boundary components tracing the end edge, we define the following notation. Let e be the end edge of a chord diagram corresponding to a double-occurrence word W. Let c be 1 or 2. We say that W satisfies the condition $A(\min, c)$ (respectively, $A(\max, c)$) if any thickened diagram of minimum (respectively, maximum) genus g traces e by a single boundary curve for $c = 1$, and by two boundary curves for $c = 2$. Similarly, we say that W satisfies the condition $E(\min, c)$ (respectively, $E(\max, c)$) if there exists a thickened diagram of minimum (respectively, maximum) genus g that traces e by a single boundary curve for $c = 1$, and by two boundary curves for $c = 2$. We also simply say W is (of) $A(\min, c)$ etc. Then Lemma 4.3 is summarized as follows:

Cases	W_1, W_2	$W_1 W_2$
(E_0)	one $A(\min, 2)$	$\varepsilon = 0$
(E_1)	both $E(\min, 1)$	$\varepsilon = 1$
$\left(E_0'\right)$	one $E(\max, 2)$	$\varepsilon' = 0$
$\left(E_1'\right)$	both $A(\max, 1)$	$\varepsilon' = 1$

If a chord diagram D' is obtained from D by removing some chords, then D' is called a *sub-chord diagram* of D. The following lemma covers a large family of chord diagrams that support Conjecture 3.2.

Lemma 4.4

If a chord diagram has a sub-chord diagram corresponding to the double-occurrence word 123321, *then its genus range contains more than 2 integers*.

Proof—Consider the surface F obtained by thickening the chord diagram such that the three parallel chords represented by 1, 2 and 3 are all-in, and the other chords are all-out. Then it has four inside boundary curves and at least one outside, total at least 5. Refer to Fig. 6, where other chords are not depicted. Move one end of chord 1 from inside to outside, keeping the other inside. Then the total number of boundary curves decreases by 2. This is seen as follows. Regard this operation in two steps: (1) remove a band corresponding to chord 1 from F, and (2) add a band corresponding to 1 with one-in and one-out ends. The

step (1) joins the two inside boundary curves to a single curve, thus reduces the number of boundary curves by 1. In step (2), the new one-in, one-out band joins the newly formed inside curve with one of the outside curves, reducing the boundary curve by 1 again. Hence replacing an all-in chord 1 in F with one-in, one-out chord decreases the number of boundary curves by 2. Performing the same procedure for the chord labeled 3, further decreases the number of components by 2. Therefore, the genus range consists of at least 3 numbers.

5. Realizations of Genus Ranges

We use the following notations for respective double-occurrence words and corresponding chord diagrams:

$$
\begin{aligned}
U_n &= 1122 \cdots nn \quad (=U_1^n), \\
R_n &= 12 \cdots n12 \cdots n, \\
G_\ell &= (1212)(3434) \cdots ((2\ell-1)2\ell(2\ell-1)2\ell) \quad (=R_2^\ell).
\end{aligned}
$$

Lemma 5.1

For the chord diagram corresponding to $R_n = 12 \cdots n12 \cdots n$, where $n = 2m$ or $n = 2m - 1$ and $n > 2$, we have $gr(R_n) = [1, m]$.

Proof—For an even $n = 2m$ ($m \geq 1$), consider the all-in thickened diagram F_{Rn}. Then F_{Rn} has exactly two boundary components: One inside curve, tracing chords in successive order (see Fig. 7(a)), and one outside. Hence F_{Rn} achieves the maximum genus m. By adding a one-in, one-out chord, the two curves are joined to a single component, therefore for an odd n, the resulting surface has the maximum genus.

Consider a thickened diagram for R_n where every chord is one-in, one-out (see Fig. 7(b)). Each boundary curve traces a single side of two chords. Then the resulting surface has n boundary components, and genus 1.

Since the chord diagram D for $R_3 = 123123$ is isomorphic (as a graph) to the bipartite graph $K_{3,3}$, D is non-planar and the genus range of any chord diagram that has R_3 as a sub-chord diagram does not contain 0. The result follows from Lemma 4.1.

Lemma 5.2

(1) For any $n > 0$, there exists a chord diagram of n chords with genus range [0, 1]. (2) For any $n > 2$, there exists a chord diagram of n chords with genus range [1, 2].

Proof—The chord diagram $U_1 = 11$ has genus range [0, 1] and also has the properties $A(\min, 2)$ and $A(\max, 1)$. By Lemma 4.3 (cases (E_0) and (E_1')), the chord diagram of $U_2 = U_1^2$ has genus range [0, 1], and its end edge retains the conditions $A(\min, 2)$ and $A(\max, 1)$. Inductively, U_n has genus range [0, 1] for any n.

Recall that the chord diagram corresponding to R_3 is non-planar. The chord diagram of R_3 has genus range [1, 2] (Lemma 5.1), and has property $A(\max, 1)$. Then the chord diagram of

R_3U_1 has genus range $[1, 2]$ by Lemma 4.3 (cases (E_0) and (E_1')), and retains the condition $A(\max, 1)$. Inductively, R_3U_m has genus range $[1, 2]$ for any $m \geq 0$, hence for any $n = m + 3 > 2$.

Lemma 5.3

For any chord diagram W with $gr(W) = [g, g']$, we have $gr(R_2W) = [g, g' + 1]$.

Proof—The chord diagram R_2 has genus range $[0, 1]$ and is of $A(\min, 2)$ and $A(\max, 2)$, so it is $E(\max, 2)$. By Lemma 4.3 (cases (E_0) and (E_0')), we obtain the result.

Lemma 5.4

For $G_m = (1212)(3434) \cdots ((2m - 1)2m(2m - 1)2m)$, we have $gr(G_m) = [0, m]$ for any $m > 0$.

Proof—This follows from Lemma 5.3 by induction.

Lemma 5.5

For any $k, m > 0$, we have $gr(G_mU_k) = [0, m + 1]$.

Proof—The chord diagram G_m is of $E(\max, 2)$. By Lemma 4.3 (case (E_0')), we have $gr(G_nU_1) = [0, m + 1]$. The chord diagrams U_1 and G_mU_k for $k > 0$ are of $A(\max, 1)$, hence by Lemma 4.3 (case (E_1')) the statement holds by induction.

We use the notation $X = 12341342$.

Lemma 5.6

For any $k > 0$, $k \in Z$, there exists a chord diagram with n chords, where $n = 4k - 1$ or $4k$, having genus range $[k, 2k]$.

Proof—Computer calculation shows that the chord diagram D corresponding to $W = 12312345674675$ has genus range $[2, 4]$. The word W is the concatenation of $R_3 = 123123$ and $X = 12341342$. Computer calculation also shows that $gr(X) = [1, 2]$. By Lemma 5.1, we also have $gr(R_3) = [1, 2]$.

The diagram of R_3 is of $E(\min, 1)$ as depicted in Fig. 8. This implies that $X = 12341342$ is of $A(\min, 2)$. (Otherwise R_3X has minimum genus 1 by Lemma 4.3 (case (E_0)).) A connected sum of two chord diagrams of $A(\min, 2)$ is again a diagram of $A(\min, 2)$ (case (c) in Fig. 5). By Lemma 4.3 (case (E_0)) again, inductively, the minimum genus of X^k is k, and X^k is of $A(\min, 2)$.

Any thickened diagram of R_3 with genus 2 must have a single boundary component, and therefore, every edge is singly traced, hence it is $A(\max, 1)$. Since $gr(W) = [2, 4]$, Lemma 4.3 (case (E_0')) implies that X is of $E(\max, 2)$. Note that the end edge e of X^k for any k is of $E(\max, 2)$ (case (c) in Fig. 5). By using Lemma 4.3 (case (E_0')) inductively, we obtain that the maximum genus of X^k is $2k$.

Hence the diagram for X^k has genus range $[k, 2k]$ and $4k$ chords. The diagram corresponding to $X^{k-1} R_3$ has $n = 4k - 1$ chords, the minimum genus $(k-1)+1 = k$, the maximum genus $2(k-1) + 2 = 2k$, by Lemma 4.3 as desired.

We note here that computer calculation was critical for this proof, since it would otherwise be difficult to determine the genus range of $R_3 X = 12312345674675$.

The proof of Lemma 5.6 shows that X^k is of $A(\min, 2)$ and $E(\max, 2)$ for every $k > 0$.

Lemma 5.7

For any $h > 0$ and $k \geq 0$, there is a chord diagram $W (h, k)$ of $n = 4k + h$ chords with genus range $[k, 2k + 1]$.

Proof—Let $W (h, k) = U_h X^k$ which has $n = 4k + h$ chords. The diagram U_1 is of $A(\min, 2)$, and inductively, so is U_h for any $h > 0$. Lemma 4.3 (case (E_0)) implies that $W (h, k)$ has the minimum genus k.

We have that U_h has genus range $[0, 1]$ and is of $A(\max, 1)$, and X^k is of $E(\max, 2)$ (Proof of Lemma 5.6). Hence $W (h, k)$ has highest genus $2k + 1$. The statement holds for $k = 0$ as well, from the proof of Lemma 5.2(1).

Lemma 5.8

For any $h > 0$ and $k, \ell \geq 0$, there is a chord diagram $V (h, k, \ell)$ with $n = 4k + 2\ell + h$ chords such that $\mathrm{gr}(V (h, k, \ell)) = [k, 2k + \ell + 1]$. In the case $h = 0$, for any $k, \ell \geq 0$, there is a chord diagram $V (0, k, \ell)$ with $n = 4k + 2\ell$ chords such that $\mathrm{gr}(V (0, k, \ell)) = [k, 2k + \ell]$.

Proof—We consider $V(h, k, \ell) = W (h, k)G_\ell = U_h X^k G_\ell$. We see that G_ℓ is of $A(\min, 2)$ and $E(\max, 2)$ inductively from the proof of Lemma 5.3. Hence the minimum genus of $V (h, k, \ell)$ is k for any $h, k, \ell \geq 0$ by Lemma 4.3 (case (E_0)).

Recall that X^k is of $E(\max, 2)$ and G_ℓ is of $E(\max, 2)$ for any $k, \ell > 0$. Hence $V (0, k, \ell)$ is of $E(\max, 2)$ and $\mathrm{gr}(V (0, k, \ell)) = \mathrm{gr}(X^k G_\ell) = [k, 2k + \ell]$ by Lemma 4.3 (case (E_0')), proving the second statement of the lemma. Because $V (0, k, \ell)$ is of $E(\max, 2)$ and $\mathrm{gr}(U_h) = [0, 1]$, Lemma 4.3 (case (E_0')) implies $V (h, k, \ell) = [k, 2k + \ell + 1]$ for $h > 0, k, \ell \geq 0$.

Proposition 5.9

For any g, g' such that $g' \geq 2g$ there is a chord diagram with genus range $[g, g']$.

Proof—For $g' > 2g$, we set $\ell = g' - 2g$. Then the chord diagram $V (0, k, \ell)$ in Lemma 5.8 has genus range $[g, g']$. If $g' = 2g > 0$ then Lemma 5.6 provides a desired chord diagram.

6. Towards Characterizing Genus Ranges

In this section we state and prove the main theorem. Recall from Lemma 3.1 that for any chord diagram of n chords, the genus g of a thickened diagram is at most $\lceil n/2 \rceil$.

Theorem 6.1

There exists a chord diagram with n chords and genus range [g, g′] whenever g, g′ satisfy one of the following conditions: (1) $g' = 2g$ and either $g = 1$ or $g' = \lceil n/2 \rceil$, or (2) $0 \leq 2g < g' \leq \lceil n/2 \rceil$.

Proof—Let $m = \lceil n/2 \rceil$. Case (1): The case of [g, g′] = [1, 2] follows from Lemma 5.2(2). In the case of $0 < g' = 2g = m$, setting $k = g$ in Lemma 5.6, we obtain a chord diagram with genus range [k, 2k] = [g, g′] with $n = 2m = 4k$ or $n = 2m - 1 = 4k - 1$ as desired.

Case (2): Suppose $0 \leq 2g < g' \leq m$. First we consider the case $n = 2m - 1$. Set $\ell = g' - 2g - 1 \geq 0$ and $h = 2m - 2g' + 1 > 0$. Then $V(h, g, \ell)$ in Lemma 5.8 has genus range [g, 2g + ℓ + 1] = [g, g′] and the number of chords is $4g + 2\ell + h = 4g + 2(g' - 2g - 1) + (2m - 2g' + 1) = 2m - 1 = n$, as desired.

Next we consider the case $n = 2m$. If $m = g'$, set $\ell = g' - 2g > 0$. Then the chord diagram $V(0, g, \ell)$ in Lemma 5.8 has genus range [g, 2g + ℓ] = [g, g′] and the number of chords is $4g + 2\ell = 2g' = 2m = n$, as desired. If $m > g'$, set $\ell = g' - 2g - 1$ and $h' = m - g' = m - 2g - \ell \geq 0$. Then the chord diagram corresponding to $V(2h', g, \ell)$ in Lemma 5.8 has genus range [g, 2g + ℓ + 1] = [g, g′] and the number of chords is $4g + 2\ell + 2h' = 2g' + 2h' = 2m = n$, as desired.

The situation of the theorem is represented in the graph of Fig. 9. Each lattice point of coordinate (*a, b*) represents the genus range [*a, b*]. A black dot represents that there is a chord diagram of the corresponding genus range. A circle with backslash inside, located on the line $b = a$, represents that there is no singleton genus range by Lemma 4.2. White dots between two lines $b = a$ and $b = 2a$, and those on the line $b = 2a$, denote the cases for which we do not know whether there are diagrams of those ranges. Note that we know only two points on the lines $b = 2a$ and $b = a+1$, respectively, that appear in the genus ranges of chord diagrams. Other points on the integer lattice, not indicated in the figure, are excluded by the Euler characteristic formula (Lemma 3.1).

7. Concluding Remarks

In this paper, we studied sets of genus values, called the genus ranges, for thickened chord diagrams. Various surfaces occur when ends of bands corresponding to chords are attached to inside or outside boundary circles of the backbone annulus. For a fixed number of chords we investigated which ranges can or cannot occur, but for many integer intervals it remains undetermined. Computer calculations motivated the constructive methods used to prove the results and prompted several conjectures on characterizing double-occurrence words for certain genus ranges. However due to the exponential growth of the number of chord diagrams and attachment cases, additional computer calculations seem difficult. Further studies are desirable.

Acknowledgments

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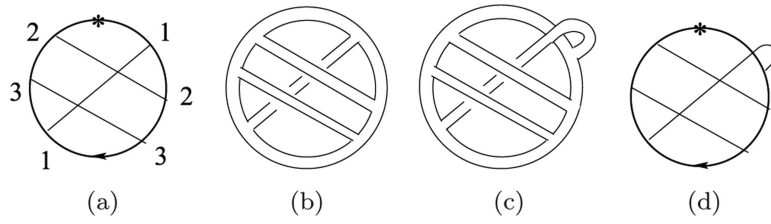
Fig. 1.
(a) An example of a chord diagram 123132, * indicates the base point; (b) and (c) Two examples of thickened chord diagrams corresponding to the chord diagram in (a); (d) Schematic representation of the thickened diagram in (c).

(a)

(b)
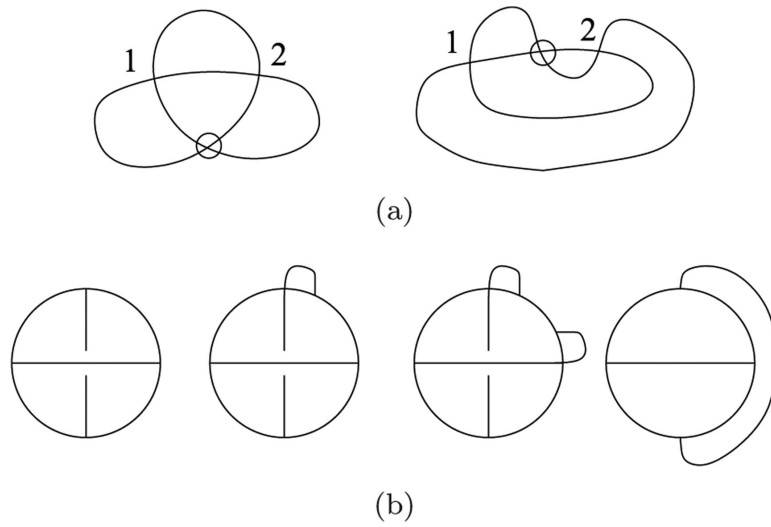
Fig. 2.
(a) Flat virtual knot diagrams corresponding to the Gauss word 1212; (b) Four possible representations of the word 1212 as a chord diagram.
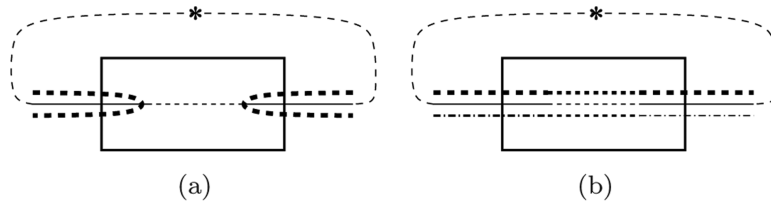
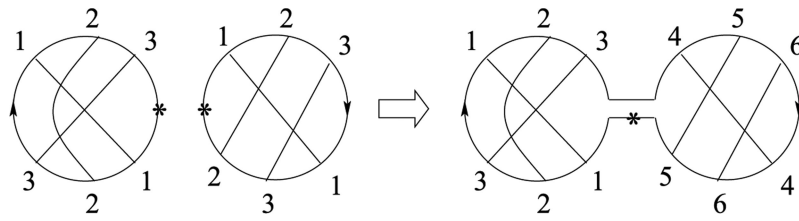(a) (b)

Fig. 3.
End edges traced by (a) single and (b) two boundary curve(s).

Fig. 4.
Connected sum of two chord diagrams.
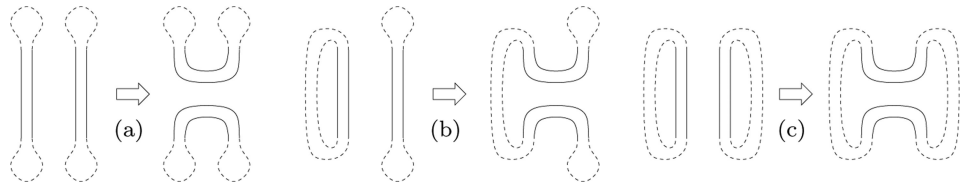
Fig. 5.
Connected sum and boundary curves: (a) both end edges are traced by a single boundary curve; (b) one end edge is traced by a single boundary curve but the other is traced by two boundary curves; (c) both end edges are traced by two boundary curves.
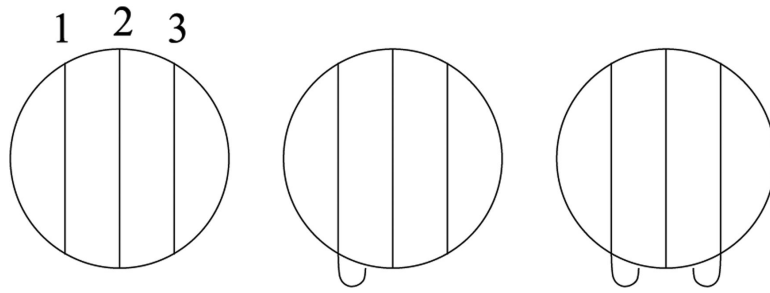
Fig. 6.
Three thickened surfaces with distinct genera for a chord diagram containing the sub-chord diagram 123321. All other chords (not pictured) are all-out.
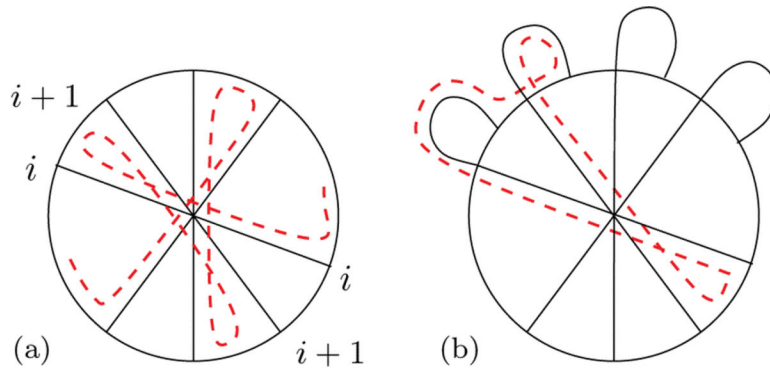
Fig. 7.
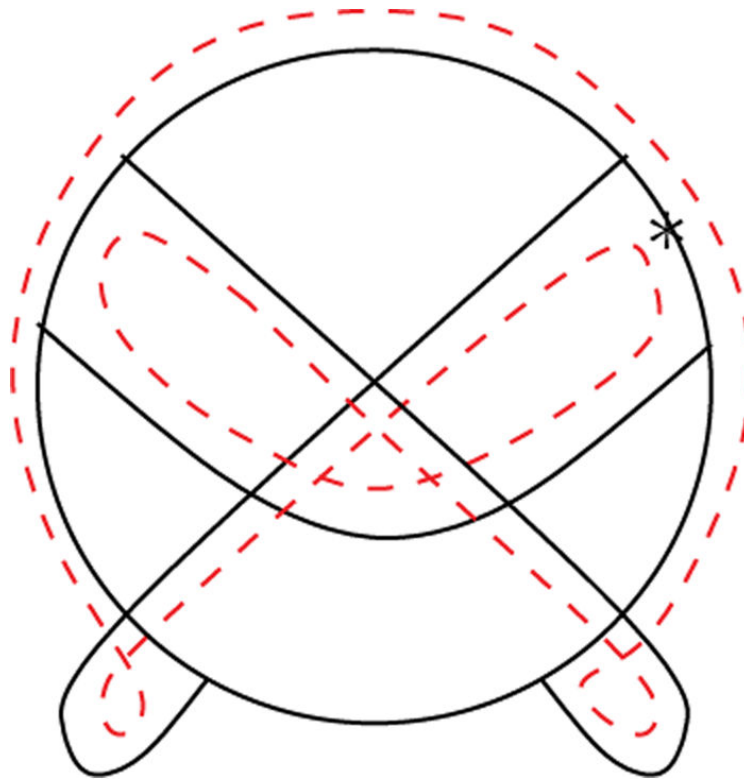Thickened diagrams of R_n with the maximum genus (a) and the minimum genus (b).

Fig. 8.
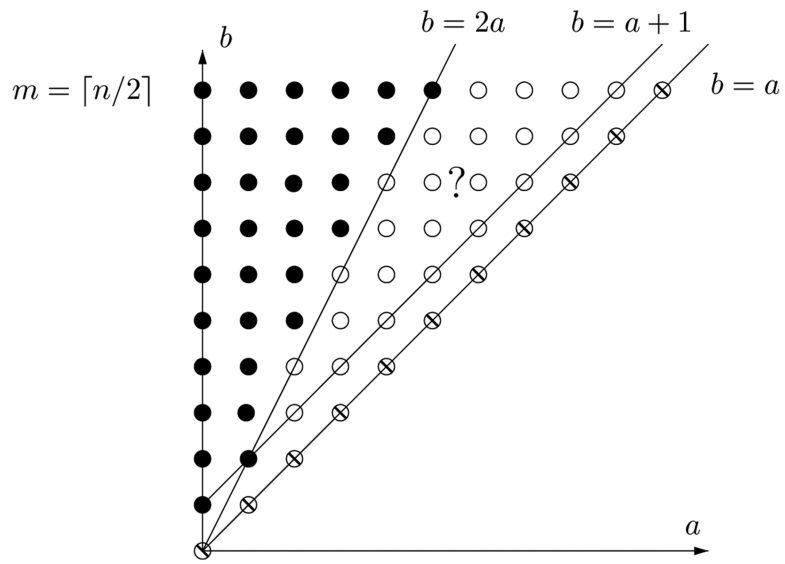Edges traced by a single boundary curve of a genus 1 surface.

Fig. 9.
Realizing genus ranges for chord diagrams with *n* chords.

Table 1

Genus ranges for chord diagrams with up to 7 chords.

$n = 1, 2 : \{0, 1\}$

$n = 3, 4 : \{0, 1\}, \{0, 1, 2\}, \{1, 2\}$

$n = 5, 6 : \{0, 1\}, \{0, 1, 2\}, \{1, 2\}, \{0, 1, 2, 3\}, \{1, 2, 3\}$

$n = 7 : \{0, 1\}, \{0, 1, 2\}, \{1, 2\}, \{0, 1, 2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{2, 3, 4\}$