

SOME MATHEMATICAL PROPERTIES OF WEIGHT-FOR-HEIGHT INDICES USED AS MEASURES OF ADIPOSITY

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A problem often encountered in large-scale epidemiological studies is the assessment of a person's relative adiposity from very limited data such as measurements of his height and weight only. There have been several papers in recent years, those of Billewicz, Kemsley and Thomson (1962), Kemsley, Billewicz and Thomson (1962), Khosla and Lowe (1967), Evans and Prior (1969), and Florey (1970), which have examined the properties of height-and-weight indices without agreeing on a single ideal index. The present paper develops a unified theory and gives some examples of its use.

CRITERIA OF OPTIMALITY

The following criteria are commonly proposed for a 'good' index of obesity: (1) it should be highly correlated with measures of relative adiposity; and (2) its distribution should be independent of height.

The first criterion is, of course, self-evident, though there is the question of what measure of relative adiposity (e.g., skinfold thicknesses, hydrostatic determinations of density, etc.) should be used to validate an index. The second criterion is a more disputed one and turns on the issue of whether relative adiposity is, or should be, conceived as being distributed independently of height. We will discuss this later, but let us for the moment accept the criterion as reasonable.

TYPES OF INDICES

There are two types of index in common use:

(1) Relative weight, which is the ratio of a person's weight to a standard of weight for persons of his sex and height. (Further standardizing variables such as age may be used.) This is the type of index favoured by Billewicz *et al.* (1962).

(2) Indices not specifically employing a standard, e.g., those of the form (weight)/(height)^p where *p* is some constant. *p* = 2 gives the index commonly called Quetelet's index (which was favoured by Khosla and Lowe (1967)), and *p* = 3 gives one

form of the ponderal index. We shall hereafter refer to this type of index as the power-type index.

STANDARDS OF WEIGHT FOR HEIGHT

Most standards of weight for height (be they mean or median weights for given heights) are linear functions, for example, the standards of Kemsley *et al.* (1962) for adult males and females (Fig. 1). Other examples collected from various sources will be found in Kemsley *et al.* (1962) and in Hathaway and Foard (1960). Assuming a linear relationship, it also happens that, if we plot the logarithms of the standards against log height, a linear standard also holds to a good approximation (Fig. 2). This is to be expected from mathematical theory, as will now be shown.

Given a standard of the form

$$W_s(H) = W_o + b(H - H_o)$$

where *H* is a person's height and *H*_o is some central measure of location of height, e.g., the mid-range of heights for which the standard is given, it may be shown by expressing log *W*_s(*H*) in terms of log *H* and expanding about log *H*_o that

$$\log W_s(H) \doteq \log W_o + p(\log H - \log H_o) \quad (1)$$

$$\text{where } p = \frac{b H_o}{W_o} \quad \dots (2)$$

Terms of order (log *H* - log *H*_o)² have been neglected, being unlikely to amount to more than 1% of the value of the standard.

$$\text{Hence, } W_s(H) \doteq H^p \frac{W_o}{H_o^p} \quad \dots (3)$$

So if we use this approximation to the standard to form a relative weight, we get an index of the power type multiplied by some constant (which, incidentally, makes the index independent of the units of measurement):

$$\frac{W}{W_s(H)} = \frac{W}{H^p} \cdot \frac{H_o^p}{W_o} \quad \dots (4)$$

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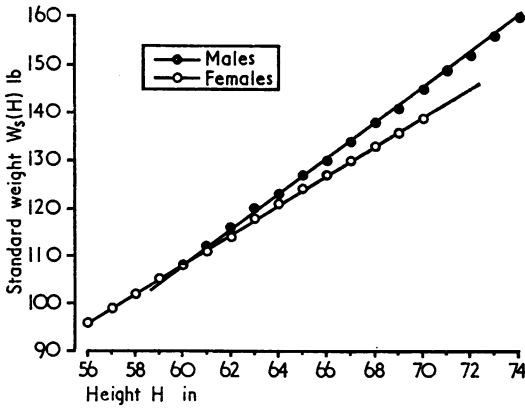


FIG. 1.—Standards of Kemsley *et al.* (1962). Median weights for height.

To test this in practice, note the standards of Kemsley *et al.* (1962) for males in Figure 1. We have $b = 3.66$ lb/in, the mid-range height $H_0 = 67$ in, and $W_0 = 134$ lb.

$$\text{Then } \frac{b H_0}{W_0} = \frac{3.66 \times 67}{134} = 1.83.$$

Compare this with the actual gradient of the line in Fig. 2, which is 1.77.

Similarly, for females we have

$$\frac{b H_0}{W_0} = \frac{3.11 \times 63}{118} = 1.66.$$

This compares very well with the actual gradient of 1.67.

So if we want a power-type index which is approximately equivalent to relative weight using Kemsley's male standards it would seem reasonable to take $p = 1.8$. The index $W/H^{1.8}$ should be proportional to the relative weight, and they should be equal if $W/H^{1.8}$ is multiplied by the scaling constant

$$\frac{H_0^p}{W_0} = \frac{67^{1.8}}{134} = 14.45 \text{ in}^{1.8} \text{ lb}^{-1}$$

Figure 3 shows the two indices plotted for a sample of 20 male executive grade civil servants (Ministry of Social Security) together with the line of equivalence, and it is obvious that they agree remarkably well across a wide range of values. In no case do the two indices differ by more than 0.004.

To summarize so far, we have demonstrated that, given a linear standard of weight for height, there exists an index of the power type which is equivalent (to a good approximation) to relative weight using

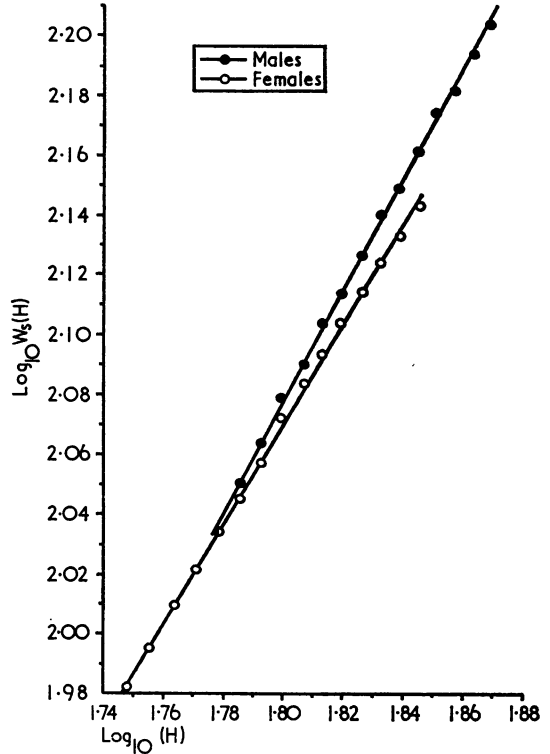


FIG. 2.—Standards of Kemsley *et al.*, log scales.

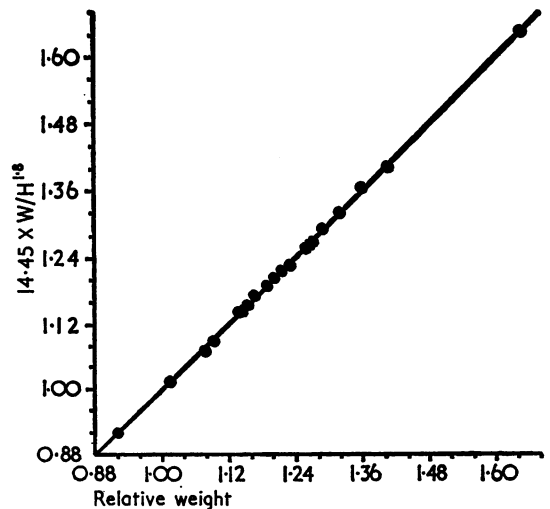


FIG. 3.—Relative weight using the standards of Kemsley *et al.* versus a power-type index.

this standard, and the exponent of height in the power-type index may be found by measuring the gradient of the linear standard when plotted on logarithmic scales, or by use of the formula (2).

Note that if we thus prove two indices to be equivalent, then they will be so for any group of persons to which we apply them. It is not even necessary that the standard be in any way appropriate to the group under study. Later, however, when we use this result (4) in determining which values of p satisfy the criteria of optimality, it will be necessary to stipulate that the standards are in some sense appropriate to the group under study, and so our results will not necessarily be applicable to all groups.

Note also that, although the standards of Kemsley *et al.* (1962), which we have used illustratively, are median weights for height, the result is equally applicable if mean weights or any other measure of location of the weight distributions are used as standards.

SATISFYING THE CRITERIA OF OPTIMALITY

Let us now consider what value, or values, of p will best satisfy our criteria. It will be easier to consider our second criterion first.

Let p_0 be the value of p such that W/H^p is distributed independently of H in a given population, assuming that such a value exists. If we are estimating p_0 from a sample, we can take p_0 to be the value of p such that W/H^p is uncorrelated with H : such a value will always exist since the correlation decreases continuously from positive to negative as p_0 increases. We could find p_0 by direct trial of various values of p , but we can get an approximation to it with less labour.

Billewicz *et al.* (1962) have proved that relative weights are distributed independently of height provided that the standards used represent the same location in each of the distributions of weight for height in the group under study. We can ensure this by fitting the linear regression of weight on height in the group and using this expression to give standards. So in (1) we will have $H_0 = \bar{H}$, $W_0 = \bar{W}$, and $b = r_{WH} \bar{S}_W / \bar{S}_H$ where \bar{H} and \bar{W} are the average heights and weights, \bar{S}_W and \bar{S}_H the estimated standard deviations, and r_{WH} the correlation of weight and height. Using (2) we get a value of p which makes W/H^p approximately equal to relative weight and hence approximately independent of height.

So our solution for p_0 , using a prime to indicate that it is a mathematical approximation, is

$$p_0' = \frac{\bar{H} \bar{S}_W}{\bar{W} \bar{S}_H} r_{WH} \quad \dots (5)$$

Now for the other criterion we must consider what value of p will maximize the correlation between W/H^p and some measure A of relative adiposity. It may be shown (see Appendix I) that the maximum correlation that can be achieved is approximately

$$r_{\max} = \sqrt{r_0^2 + r_{AH}^2} \quad \dots (6)$$

where r_0 is the correlation of A and W/H^p when $p = p_0'$, and r_{AH} is the correlation of A with height. r_0 is usually about 0.8 (as, for example, in London Transport bus crews, discussed in the next section), so unless r_{AH} is quite sizeable, say ± 0.1 or larger, it is obvious that r_{\max} represents very little increase over r_0 . So the use of $p = p_0'$ will not only give us an index distributed nearly independently of height, but also, provided the correlation of height and adiposity is around zero, it will give almost the best correlation with relative adiposity that it is possible to achieve with any p .

If we wish to estimate p_0 in a population by examining a sample, or if we wish to regard a group of people, however obtained, as being a random sample from a hypothetical population, it is obviously very valuable to know how p_0 will vary in sampling. It may be shown (see Appendix I for the proof) that the standard error of estimate of p_0 from a sample of N drawn from an effectively infinite population is approximately:

$$S.E. (p_0) \doteq \frac{1}{\sqrt{N-3}} \frac{\bar{H} \bar{S}_W}{\bar{W} \bar{S}_H} \sqrt{\frac{1 - r_{WH}^2}{1 + r_{WH}^2 \bar{S}_W^2 / \bar{W}^2}} \quad \dots (7)$$

APPLICATIONS OF THE FORMULAE

To test the usefulness of the foregoing results, let us examine two populations of London Transport busmen, drivers and conductors, respectively. As a measure of adiposity, we use the sum of three skinfolds, triceps, subscapular, and suprailiac, with a logarithmic transformation, based on that recommended by Edwards, Hammond, Healy, Tanner, and Whitehouse (1955), for obtaining a normal variate. We use

$$A = \log_{10} (\text{sum of 3 skinfolds in cm} - 0.54) \dots (8)$$

The constant term 0.54 cm is three times that used by Edwards *et al.* (1955) for single skinfolds, and is approximately the thickness of skin which is included with the fatty tissue in the skinfolds.

Table I gives the basic statistics for the population,

TABLE I

DISTRIBUTIONS OF HEIGHT, WEIGHT, AND TRANSFORMED SKINFOLD (A) IN LONDON BUS DRIVERS AND CONDUCTORS

	Drivers	Conductors
N	799	544
\bar{H} (cm)	172.31	170.06
\bar{W} (kg)	72.13	66.69
\bar{A}	0.4629	0.3681
S_H	5.562	5.382
S_W	10.740	9.745
S_A	0.2494	0.2536
r_{WH}	0.352	0.367
r_{AH}	-0.018	0.014
r_{AW}	0.755	0.760

TABLE II

EXACT (p_0) AND APPROXIMATE (p_0') VALUES OF p SUCH THAT W/H^p IS UNCORRELATED WITH H

	Drivers	Conductors
p_0	1.6208	1.7273
p_0'	1.6247	1.6927
$S.E.(p_0)$	0.1537	0.1859

and Table II the optimal, or approximately optimal, values of p . Figures 4 and 5 show the correlations of W/H^p with H and A respectively plotted against p . These graphs were obtained by calculating the correlations for values of p at intervals of 0.1 from 1.4 to 2.0. The points in Fig. 4 are well fitted by straight lines so that we may interpolate to find

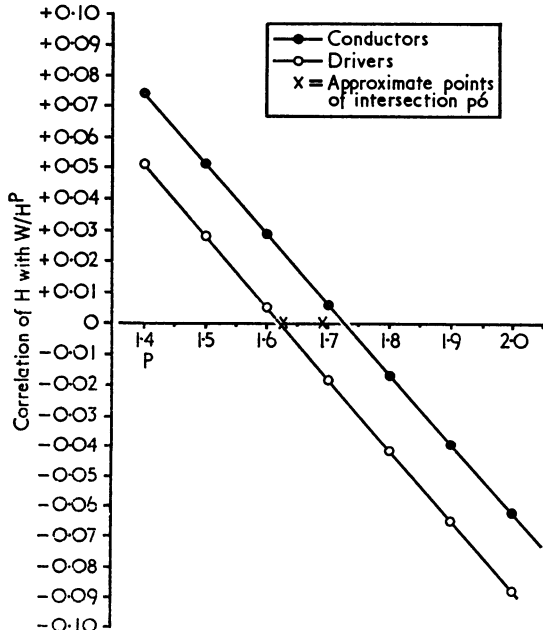


FIG. 4.—Correlations of height H with W/H^p for various values of p : London busmen.

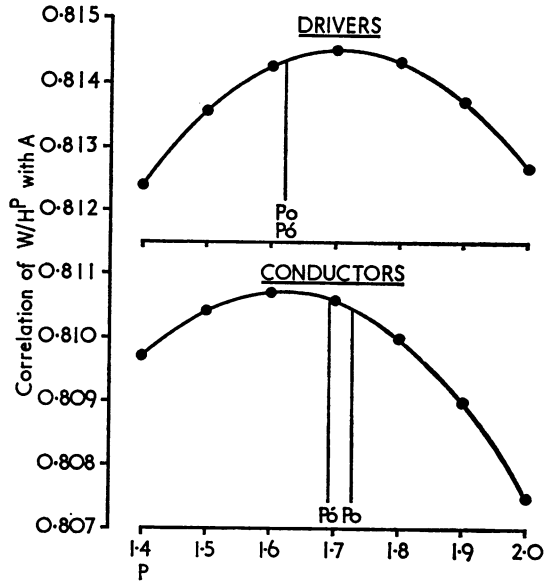


FIG. 5.—Correlations of log skinfold A with W/H^p for various values of p : London busmen.

the value of p_0 where the lines cut the horizontal axis. Similarly, the points in Fig. 5 are well fitted by quadratic curves, so again we can interpolate to find intermediate values.

The standard errors of p_0 given in Table II are obtained from formula (7). If, however, we merely assume that the standard error has the form $k/\sqrt{N-3}$, we may estimate k by drawing samples, computing p_0 (or p_0') for each sample and finding its standard deviation over samples. From (7) k has the value 4.32 for both drivers and conductors. From the sampling experiments (see Appendix II for full details of method and results) we get 4.33 for the drivers and 4.47 for the conductors. Similar values are obtained with the approximate estimate p_0' .

DISCUSSION

Our declared object of finding functions of height and weight to serve as indices of relative adiposity is, of course, capable of only limited fulfilment since relative weights (or equivalent functions) are affected not only by adiposity but by other factors such as skeletal width and muscular development. The fact that one can achieve correlations as high as about 0.8 between an index and a measure of adiposity means that some 64% (i.e., the square of the correlation) of the variance of the index is accounted for by adiposity and to this extent only the interpretation of such indices in terms of adiposity is justified. Florey (1970) does not consider them to be satisfactory measures of adiposity and recommends

that they be called merely indices of weight corrected for height.

It has been mentioned earlier that it is not universally agreed whether relative adiposity is actually, or should be defined as being, distributed independently of height, though Billewicz *et al.* (1962) and Khosla and Lowe (1967) regard it as proven and therefore say that an index designed to measure adiposity should have the same property. It appears from the present investigations that the process of getting an index independent of height also serves to maximize the correlation with adiposity unless there is a marked association between adiposity and height. In view of this and the fact already noted that none of these indices gives a 'pure' measure of adiposity, it seems best to concentrate attention on the height-independence criterion.

Some of the results in this paper have been published by others, but, so far as the author is aware, they have not been published before as a unified whole. Billewicz *et al.* (1962) examined power-type indices with the integer values 1, 2, and 3 for p . They concluded that 1 and 3 were unsatisfactory since the indices had marked biases with respect to height. They agreed, however, that $p = 2$ gave an index (usually known as Quetelet's) which had little or no such bias, but they considered it too tedious to compute for a large number of cases, and they also noted its disadvantage of being dependent on the units of measurement. They then considered a mathematical model for the distribution of weight at given heights and showed from this that relative weight, using a suitable standard, should be distributed independently of (not merely uncorrelated with) height and went on to demonstrate this (Kemsley *et al.*, 1962). However, they seem not to have realized that, by considering other values of p , including fractional ones, they could have obtained a power-type index equivalent to relative weight using their standards and, moreover, that this could be made independent of the units of measurement by multiplying by a suitable constant.

Khosla and Lowe (1967) adopted a slightly different approach. They showed that, for a power-type index to be distributed independently of height, it is a necessary condition that the exponent p be the gradient of the linear relationship between log standard weights and log heights. However, they did not give any rigorous proof that such an index can actually be found. The existence is, in fact, dependent on the log standard to log height relationship being linear (at any rate approximately), and this follows from the direct linear relationship between standards and height, as proved in this paper.

Using a population of some 5,000 men employed in an electrical engineering firm, Khosla and Lowe (1967) computed standard weights and fitted the gradient to the logarithmic relationship, getting a value of 1.94. (If they had used the approximate formula $b\bar{H}/\bar{W}$, they would have got 1.92 for slightly less effort.) They concluded that Quetelet's index should be suitable and showed that its distribution was, in fact, the same at all heights in their population. However, they also seem not to have realized that they had, in effect, proved an equivalence between relative weights and power-type indices, i.e., Quetelet's index should be approximately equivalent (under a suitable change of scale) to relative weight using Khosla and Lowe's standards.

Florey (1970) came to the crux of the matter by showing that the form of the height-independent index depends on the equation for standard weights. He considered only the integral values $p = 1, 2,$ and 3 (except that for $p = 3$ he used the ponderal index in its more common form H^3/\sqrt{W}) and found which of these had the smallest bias with respect to height for various artificially constructed linear standards, varying both the gradient and the intercept. He concluded that the best value of p increases as the gradient b increases, and decreases as the standard weight for average height increases.

The results he presents in his Table 5 are the same as those one would get by calculating $b\bar{H}/\bar{W}$ and rounding it to the nearest whole number, except that his values are in some cases one greater. Why these results are different is not wholly clear. His procedure was, given a linear standard $W_s(H)$, to calculate the indices W_s/H , W_s/H^2 and $H/\sqrt[3]{W_s}$ for H ranging from 54 to 76 in, and to determine which of the three has least variability over this range. Here, however, we have concentrated (explicitly or implicitly) on slightly narrower height ranges, e.g., 60 to 74 in for males and 56 to 70 in for females, using Kemsley's standards (Kemsley *et al.*, 1962). For wider ranges it becomes increasingly difficult to find a power-type index which is stable throughout the range. Common sense would, in any case, make us cautious in interpreting such an index for a man as short as 54 in, or a woman as tall as 76 in. It is possibly because Florey measures instability over such a wide range that his choice of index is sometimes different from that which would be indicated by $p = b\bar{H}/\bar{W}$.

Some readers may be suspicious of the amount of approximation that has been used in deriving the mathematical formulae. My defence for it is that it is a practical necessity and seems to work well enough in practice. Since relative weights and power-type indices are not exact linear functions

of weight and height, it would be very complicated to obtain exact formulae for their variances and correlations and it would require more assumptions about the joint distribution of weight, height, and adiposity than have been made here. The detailed justification of the approximations is given in Appendix I, but they depend essentially on height and relative weight having low coefficients of variation, which they do for adults. The author has not tested the formulae for children or adolescents and they probably would not work so well since their heights are more variable. In this context we may note the interesting work of Ehrenberg (1968) and Kpedekpo (1970) who propose a relationship $\log_{10} \bar{W} = 0.8 \bar{H} + 0.4$ (weight in kilogrammes, height in metres), which they claim is applicable to the average weights and heights of a wide range of groups of children, of both sexes, various ages, and from various backgrounds. However, despite a superficial similarity (compare equation (1)), they are really concerned with a different problem from ours: theirs is about 'between group' variation, ours is about 'within group'. It should be possible, however, to devise a model that would explain both types of variation.

The standard errors of p_0 are rather large, if our busmen are at all typical of humanity in general. We would need sample sizes of about 1,900 if we wanted to reduce the standard error to 0.1 in either of these groups. Since the sample sizes are nowhere near as big, no great reliance should be placed on the estimates obtained here: they are just by way of example. From a merely practical point of view, the large sampling variation is reassuring since it reflects the fact that there is a wide range of values of p which will satisfy our objectives adequately albeit not perfectly. Figure 4 shows that there is a sizeable range of values of p all of which give low correlations between W/H^p and H . Similarly, in Fig. 5 it is obvious from the figures on the vertical scale that the peaks of the curves are, in fact, very flat and there is little necessity to locate the exact positions of the summits. p_0' does indeed satisfy both criteria well enough for practical purposes.

It is an important question whether there exists an index which will be satisfactory for many populations. Florey examined standards from various sources to see which of the three indices weight/height ratio, Quetelet's index, and ponderal index was best and concluded that Quetelet's index was usually the best of the three for western men. For western women, Quetelet's index was preferable for some groups and the weight/height ratio for others. The ponderal index was seldom found to be appropriate. I have examined the values of $p = b \bar{H}/\bar{W}$

for the same groups and agree with these choices. However, if p falls about mid-way between two integer values, it is probable that neither will be wholly satisfactory.

CONCLUSIONS AND RECOMMENDATIONS

It seems advisable, whenever possible, to base one's choice of weight-for-height index on the actual group to be studied. One should estimate the linear regression $\bar{W} + b(H - \bar{H})$ of weight on height if there are sufficient numbers to do so reliably, and hence compute $p = b \bar{H}/\bar{W}$. One may then use relative weights with the linear regression as the standard, or with any other standard that gives a similar value of p . Alternatively, one may use W/H^p : p may be rounded to the nearest whole number to make computation easy, but if p is about mid-way between two integers it will probably be better to use relative weight instead. Note that the power-type index is independent of the units of measurement if multiplied by \bar{H}^p/\bar{W} . When there are insufficient numbers to estimate the linear regression one may use relative weight with any standard that is thought suitable, or if a power-type index is preferred, then Quetelet's will probably be as good as any if one is forced to choose blindly. As far as the numbers permit, one should check by direct examination that the chosen index does not have any marked association with height.

SUMMARY

It is demonstrated by theory and by an example that there is an equivalence between relative weight ratios and indices of the form (weight)/(height) ^{p} (called the power-type index). For any given linear standard of weight for height, a simple formula gives the value of the exponent p for a power-type index which is equivalent (to a good approximation) to relative weight using the standard.

Since relative weight is distributed independently of height when the population under study is used as its own reference standard, the same formula will determine what power-type index is distributed independently of height. A formula for the standard error of the estimated value of p is obtained.

Provided the correlation between height and relative adiposity does not differ too much from zero, the same index will have a correlation with relative adiposity very near the maximum that can be obtained with this type of index.

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APPENDIX I

MATHEMATICAL THEORY

Billewicz *et al.* (1962) used a model in which a person's weight was made up of a linear function of height multiplied by his relative weight, the relative weight being distributed independently of height. We may write such a model in the following form:

$$\left. \begin{aligned} H &= E[H](1 + X) \\ W &= (E[W] + \bar{b} E[H] X) (1 + Y) \end{aligned} \right\} \quad (A 1)$$

where X and Y are independently distributed with zero means. This model implies that weight has a linear regression on height with regression coefficient \bar{b} (the sign \sim being used to distinguish the population value from the sample estimate). X is height measured from, and in units of the mean height, and Y is relative weight minus one.

From A 1 we get

$$\left. \begin{aligned} \text{Var } H &= (E[H])^2 \text{Var } X \\ \text{Var } W &= (E[W])^2 \text{Var } Y + \bar{b}^2 (E[H])^2 \\ &\quad \text{Var } X (\text{Var } Y + 1) \\ \text{Cov}(H, W) &= \bar{b} (E[H])^2 \text{Var } X \end{aligned} \right\} \quad (A 2)$$

Hence

$$\left. \begin{aligned} \text{Var } X &= \frac{\text{Var } H}{(E[H])^2} \\ \text{Var } Y &= \frac{\text{Var } W - \bar{b}^2 \text{Var } H}{(E[W])^2 + \bar{b}^2 \text{Var } H} \\ \bar{b} &= \frac{\text{Cov}(H, W)}{\text{Var } H} \end{aligned} \right\} \quad (A 3)$$

Using A 1 we can express W/H^p by a Taylor expansion in X and Y , ignoring terms of order x^2 . We have

$$\frac{W}{H^p} \doteq \frac{E[W]}{(E[H])^p} \left\{ 1 - (p - \bar{p}_0') (X + XY) + Y \right\} \quad (A 4)$$

where we write \bar{p}_0' for $\bar{b}E[H]/E[W]$. X and Y have small standard errors—for example, they are 0.032 and 0.139 in London bus drivers. For this reason it seems justifiable to ignore terms in x^2 , and for some purposes XY also since they will be negligible for the great majority of people.

Now obviously from A 4 we have

$$\frac{W}{H^p} \doteq \frac{E[W]}{(E[H])^p} (1 + Y)$$

when $p = \bar{p}_0'$

Thus for this value of p the index is approximately proportional to relative weight $(1 + Y)$ and is, therefore, by our model distributed independently of height. Thus we verify the formula (4) in the main text.

Now to consider the sampling variation of the optimal value of p —we define p_0 to be the value such that the sample covariance of H and W/H^p is zero. Using A 1 and A 4 (ignoring the XY term), and writing SV and SC for sample variance and covariance, we get

$$SC(H, W/H^p) \doteq \frac{E[W]}{(E[H])^{p-1}} \left\{ SC(X, Y) - \frac{(p - \bar{p}_0')}{SV(X)} \right\} \quad (A 5)$$

Hence

$$0 \doteq SC(X, Y) - (p_0 - \bar{p}_0') SV(X)$$

$$\text{i.e. } p_0 \doteq \bar{p}_0' + \frac{SC(X, Y)}{SV(X)} \quad (A 6)$$

\bar{p}_0' is a population parameter and does not vary, so the random variation of p_0 is contained in

$SC(X, Y)/SV(X)$. Since X and Y are independent $SC(X, Y)$ has zero expectation conditional on any set x_1, x_2, \dots, x_N of sample values of X , so the ratio $SC(X, Y)/SV(X)$ has zero expectation unconditionally. Hence p_o is an approximately unbiased estimator of \bar{p}_o' . Now as regards its sampling variance, we have

$$\begin{aligned} Var(p_o) &\div Var\left(\frac{SC(X, Y)}{SV(X)}\right) \\ &= E\left[Var\left(\frac{\sum(x-\bar{x})Y}{\sum(x-\bar{x})^2} \middle| x_1, x_2, \dots, x_N\right)\right] \\ &= E\left[Var\left(\frac{\sum(x-\bar{x})Y}{\sum(x-\bar{x})^2}\right)\right] \\ &= Var Y \cdot E\left[\frac{\sum(x-\bar{x})^2}{(\sum(x-\bar{x})^2)^2}\right] \\ &= Var Y \cdot E\left[\frac{1}{\sum(x-\bar{x})^2}\right] \end{aligned}$$

Now if X (or, equivalently, height) is normally distributed, the corrected sample sum of squares $\sum(x-\bar{x})^2$ will be distributed as $Var X \cdot C^2_{N-1}$ where C^2_{N-1} is a chi-square variable with $N-1$ degrees of freedom. Even if the distribution of height is not normal, this will still be true approximately for moderately large N . It may be shown that such a chi-square variable has the expectation of its reciprocal equal to $1/(N-3)$.

Hence

$$Var(p_o) = \frac{1}{N-3} \cdot \frac{Var Y}{Var X}$$

So, using the expression in A 3, we get

$$\begin{aligned} Var(p_o) &\div \frac{1}{(N-3)} \cdot \frac{(E[H])^2 \sigma^2_W}{(E[W])^2 \sigma^2_H} \\ &\left\{ \frac{1 - \rho^2_{WH}}{1 + \rho^2_{WH} \sigma^2_W / (E[W])^2} \right\} \quad (A 7) \end{aligned}$$

Now to consider the correlation of W/H^p with relative adiposity, A . From A 4, again ignoring the XY term, we get

$$Var(W/H^p) \div \frac{(E[W])^2}{(E[H])^{2p}} \left\{ (p-\bar{p}_o')^2 Var X + Var Y \right\}$$

and

$$Cov(A, W/H^p) \div \frac{E[W]}{(E[H])^p} \left\{ Cov(A, Y) - (p-\bar{p}_o') Cov(A, X) \right\}$$

Hence we can form the correlation

$$\rho(A, W/H^p) \div \frac{Cov(A, Y) - (p-\bar{p}_o') Cov(A, X)}{\sqrt{\{Var A \cdot ((p-\bar{p}_o')^2 Var X + Var Y)\}}} \quad (A 8)$$

For $p = \bar{p}_o'$ we have

$$\rho(A, W/H^p) \div \rho_{AY} = \rho_o \text{ say}$$

Now differentiating A 8 with respect to p and equating the derivative to zero, we find that the correlation is a maximum when

$$p-\bar{p}_o' = \frac{-Var Y \cdot Cov(A, X)}{Var X \cdot Cov(A, Y)}$$

Substituting this back into A 8, we get the maximum correlation

$$\begin{aligned} \rho_{max} &\div \sqrt{\rho^2_{AY} + \rho^2_{AX}} \\ &= \sqrt{\rho_o^2 + \rho^2_{AH}} \end{aligned}$$

Thus since ρ_o is typically about 0.8, it will be only very slightly smaller than ρ_{max} when the correlation of adiposity and height is small in magnitude. Thus the use of \bar{p}_o' will satisfy both criteria adequately.

APPENDIX II

RANDOM SAMPLING ESTIMATES OF THE VARIANCES OF p_o AND p_o'

Given that p_o (or p_o') has a variance of the form $k^2/(N-3)$ we wish to estimate k from a number M of samples of size N . If the sampling variance is found to be s^2 , this should be equal to $k^2/(N-3)$, so $\sqrt{(N-3)s^2}$ is the obvious estimate for k . Moreover s^2 will be distributed approximately as $k^2 C^2_{M-1}/(M-1)(N-3)$ where C^2_{M-1} is a chi-square variable with $M-1$ degrees of freedom. Hence for a 95% confidence interval for k we have

$$\left(\sqrt{\frac{(M-1)(N-3)s^2}{P_{97.5}}}, \sqrt{\frac{(M-1)(N-3)s^2}{P_{2.5}}} \right)$$

where $P_{97.5}$ and $P_{2.5}$ are the upper and lower 2½% points of C^2_{M-1} .

From the 799 bus drivers, 450 samples were taken, each of 100 men drawn 'with replacement', using a pseudo-random number generator on an electronic computer. The estimates p_o and p_o' were computed

TABLE
ESTIMATES AND CONFIDENCE INTERVALS FOR k

Population	Exponent	Estimate of k	95% Confidence Interval for k
Drivers	p_o	4.33	4.07-4.64
	p_o'	4.21	3.95-4.51
Conductors	p_o	4.47	4.20-4.79
	p_o'	4.27	4.02-4.59

for each sample and the variances s^2 were computed using the 450 samples. The same procedure was followed with the 544 conductors. Using the above formulae for the estimates and confidence intervals for k , with $M = 450$ and $N = 100$, the results shown in the Table were obtained.