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Quantum Bayesianism as the basis of general theory of decision-making

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We discuss the subjective probability interpretation of the quantum-like approach to decision making and more generally to cognition. Our aim is to adopt the subjective probability interpretation of quantum mechanics, quantum Bayesianism (QBism), to serve quantum-like modelling and applications of quantum probability outside of physics. We analyse the classical and quantum probabilistic schemes of probability update, learning and decision-making and emphasize the role of Jeffrey conditioning and its quantum generalizations. Classically, this type of conditioning and corresponding probability update is based on the formula of total probability—one the basic laws of classical probability theory.

1. Introduction

The recent revolution in quantum information has not as yet given birth to promised quantum computers which would beat the present 'classical computers', but it has made a number of important contributions to quantum foundations. *quantum Bayesianism* (QBism), the subjective interpretation of a quantum state [1–6] (wave function) and the corresponding probabilities given by Born's rule, is probably one of the most important (and unexpected!) outputs of this revolution. (At the same time, a few authors criticized some principles of QBism, starting with the author's paper [7] against non-objective treatment of quantum probability (QP) to the recent paper of Marchildon [8].)

In 2001, when one of the first Växjö conferences [9] on quantum information and foundations of quantum mechanics (QM) took place, we (organizers and participants) strongly believed that the revolution in quantum information would soon lead to the great foundational revolution. Unfortunately, these dreams did not come true.

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Nevertheless, the atmosphere stimulated new views and led to a series of debates, in which the basic problems of QM were not fully solved but still considerably clarified. This especially concerns the problem of the interpretation of a quantum state. Here, besides QBism (which won its recognition in Växjö), we can mention the *Växjö interpretation* of QM (statistical realist and contextual) [7,10], the derivation of the QM-formalism from simple operational principles, D'Ariano and co-workers [11–14], the *statistical Copenhagen interpretation* (statistical non-realist) which final formulation was presented at the Växjö-15 conference by Plotnitsky—it is a result of his long studies on Bohr's views and the interpretation of probability [15–17]. Finally, we point to derivation of the quantum formalism from general principles of logical inference by De Raedt *et al.* [18].

All these interpretational contributions to the quantum foundations are of an informational nature (even the realist Växjö interpretation—through the issue of contextuality). We remark that information processing is not a characteristic of solely physical systems. (There is no doubt that just biological (brain, cell) and technical systems (computers) should be on the top of the list of information processing systems.) Therefore, one may try to apply the novel approaches of Fuchs, Schack, Mermin, Caves, D'Ariano, Plotnitsky and the author of this paper to other information processors, e.g. various biosystems, from cells to brains. In the framework of the Växjö interpretation, such a possibility was investigated [19–22] by the author of this paper with the emphasis on the role of contextuality where the latter is treated very generally, in the spirit of Niels Bohr, as taking into account the whole experimental arrangement.[1] There was found one of the key non-classical elements of the quantum theory of probability update (PU), namely the *formula of total probability* (*FTP*) *with the interference term*—an additive perturbation of the classical FTP [10,23]. It became clear that this generalized FTP can be used to describe the probabilistic information processing not only by quantum physical systems. Biosystems form an important and very special class of information processors. They definitely exhibit some features which are not covered by classical probability (CP) and information—in particular, their behaviour is fundamentally contextual and adaptive. Therefore, one can try to apply QP and information theory to model behaviour of these systems.[2] This was done (keeping the Växjö interpretation) in a series of works [19–23,25–29]; QP was used to describe statistical data collected in cognitive psychology experiments on decision-making (DM), where the data were found to violate classical FTP.

Another stream of applications of QP outside physics started from within the cognition and psychology community as an attempt to describe known statistical data which, as was found by Busemeyer *et al.* [30–33], violate classical FTP indicating violation of the savage sure thing principle [34] and the disjunction effect [35–39]. While the author of this paper took interest in cognitive quantum-like models when looking for novel QP applications, Busemeyer did so when looking for a novel mathematical apparatus to model DM in psychology. In this paper, we are not able to discuss the variety of contributions of other authors in the rapidly growing domain of applications of QP to mathematical modelling of DM, cognition, psychology, finances, politics, see the basic monographs [22,32,40–42] and, e.g., papers [43–52].

The *statistical interpretation of probability* is one of the distinguishing features of the Växjö interpretation. In my opinion, this is the most adequate interpretation of QP in quantum physics. Here, experimenters collect statistical data in huge ensembles of quantum systems, e.g. photons, prepared in an (approximately) identical state ψ.

This interpretation of probability also matches modelling of statistics of DM by ensembles of cognitive systems, the probabilities of the answers to questions asked to groups of participants (typically students, mostly of departments of psychology and cognitive science). Thus, here the

[1]The modern theory of quantum information explores the notion of contextuality as dependence of the output of a measurement of one observable A on joint measurement of a compatible observable B. This is a very special case of 'Bohrian contextuality'. For Bohr, even measurement of any single observable A is contextual.

[2]QP is considered as one possible alternative to CP. In principle, there is no reason to believe that QP covers completely biology, see [24] for the detailed discussion.

ψ-function (a quantum state) is used to represent statistical features of an ensemble of systems—physical, biological or cognitive: photons, electrons, cells or humans. In quantum physics, this was the interpretation of both Einstein and Bohr as well as Pauli [15–17].

Now, we are interested not in the description (and prediction) of the outputs of experiments for groups of people, but in mathematical modelling of the intrinsic structure of the process of DM by humans. For such a problem, the statistical (ensemble) interpretation seems to be improper, because it is not about the probability of an individual event (the concrete decision), but about 'mass phenomenon, or a repetitive event, or simply a long sequence of observations', as was emphasized by von Mises [53]. Moreover, one of the main successes of quantum-like DM is modelling of violations of the savage sure thing principle [34] which was formulated in the subjective probability framework. Therefore, it is natural to use the corresponding subjective probability interpretation of a quantum state.

To solve this problem in the quantum-like framework, we are looking for a *subjective probability interpretation of the quantum state* and the corresponding probabilities for outputs of measurements. As we know, there is only one such consistent and logically structured interpretation, the one given by QBism. The main views of QBists on probability were presented by Fuchs & Schack [6] as follows:[3]

> *The fundamental primitive of QBism is the concept of experience. According to QBism, quantum mechanics is a theory that any agent can use to evaluate his expectations for the content of his personal experience.*
> *QBism adopts the personalist Bayesian probability theory pioneered by Ramsey* [54] *and de Finetti* [55] *and put in modern form by Savage* [34] *and Bernardo & Smith* [56] *among others. This means that QBism interprets all probabilities, in particular those that occur in quantum mechanics, as an agent's personal, subjective degrees of belief. This includes the case of certainty—even probabilities 0 or 1 are degrees of belief….*
> *In QBism, a measurement is an action an agent takes to elicit an experience. The measurement outcome is the experience so elicited. The measurement outcome is thus personal to the agent who takes the measurement action. In this sense, quantum mechanics, like probability theory, is a single user theory. A measurement does not reveal a pre-existing value. Rather, the measurement outcome is created in the measurement action.*

This declaration of QBism supports strongly the use of quantum-like models for DM; this is even more than one may expect from some subjective probability interpretation of QM. QBists not only interpret probability subjectively, but they also emphasize the role of 'personal experience' and interpret QM as 'a single user theory'. Experts in cognition, DM, psychology and psychophysics could not even expect that such a cognition friendly interpretation might be elaborated in quantum *physics.* (Of course, one has to take into account the contribution of the aforementioned quantum information revolution that changed the face of QM.)[4] In this paper, we would like to highlight the role of QBism in quantum-like modelling of cognition and DM in a rather general sense, i.e. not only regarding possible DM results, but about all DM-activity of humans. The QBist DM-scheme is based on generalization of classical FTP based on PU of information given by an *information complete measurement.* This formula differs from the aforementioned additive generalization of FTP representing quantum interference and is used in the Växjö interpretation of the quantum formalism as a PU scheme.

We remark that, in fact, generalized 'Växjö-FTP' must not be rigidly coupled to the statistical interpretation of the quantum state and QP. Following QBists, one can also use the subjective interpretation.

Then, our final proposal is to proceed with the QBist subjective probability and private experience ideology, but endowed with another PU rule—the one based on the Växjö version

[3]It is always better to cite explicitly the creators of theory.

[4]Of course, QBism is a grandson of Copenhagenists and the Copenhagen interpretation played the fundamental role in establishing QBism. However, this private agent perspective is an invention of QBists, not of Copenhagenists, see the paper of Mermin [57] for an excellent explanation of differences between QBism and Copenhagen.

of generalized FTP. This can be considered as a new version of QBism, because both generalized FTPs, owing to Fuchs & Schack [2] and my own [10], lead to the same result.

One can pose a deeper question: Is one of these generalized FTPs really realized by brain's 'hardware' to perform DM and PU? For the moment, this is a very speculative question. But who knows?

2. Subjectivist model of learning: classical (Boolean) probability framework

Here, we use some parts of the presentation from [58] on the subjectivist approach to PU and learning. We complete this presentation with remarks on the role of CP and Boolean logic. The probability of a hypothesis H conditional on a collected data E is the ratio of the unconditional probability of the conjunction of the hypothesis with the data to the unconditional probability of the data alone. This is the famous *Bayes' formula* (in fact, serving as the definition of conditional probability):

$$p(H|E) = \frac{p(H\&E)}{p(E)}, \tag{2.1}$$

provided that both terms of this ratio exist[5] and $p(E) > 0$.

Bayes' theorem relates the 'direct' probability of a hypothesis conditional on the data, $p(H|E)$, to the 'inverse' probability of the data conditional on the hypothesis, $p(E|H)$,

$$p(H|E) = \left[\frac{p(H)}{p(E)}\right] p(E|H). \tag{2.2}$$

This possibility to 'invert' probability $p(H|E)$, i.e. express it through the probability $p(E|H)$ is based on the use of Boolean logic, the basis to the modern probability theory and Bayesian reasoning. It may be even better to consider Bayes' theorem as an exhibition of commutativity of the operation of conjunction in Boolean logic:

$$H\&E = E\&H. \tag{2.3}$$

Quantum-like models of subjectivist PU, see §5, are based on non-classical (non-Boolean) logic. Hence, we cannot expect that it would be possible to invert the $p(H|E)$. It seems that Bayes' theorem is a purely classical (Boolean) statement. We remark that, in spite of the great value assigned to Bayes' theorem in subjectivist approach to PU and learning on the basis of new evidences, this approach is not reduced to the theorem, i.e. one can proceed as a consistent subjectivist, but without any reference to the theorem.

Subjectivists think of learning as a process of belief revision in which a prior subjective probability p is replaced by a posterior probability q that incorporates newly acquired information. This process proceeds in two stages. First, some of the subject's probabilities are directly altered by experience, intuition, memory or some other non-inferential learning process. Second, the subject 'updates' the rest of her opinions to bring them into line with her newly acquired knowledge.

The simplest learning experiences are those in which the learner becomes certain of the truth of some proposition E about which she was previously uncertain. Here, the constraint is that all hypotheses inconsistent with E must be assigned probability zero. Subjectivists model this sort of learning as simple conditioning, the process in which the prior probability of each proposition H is replaced by a posterior that coincides with the prior probability of H conditional on E.

Simple conditioning. If a person with a prior such that $0 < p(E) < 1$ has a learning experience whose sole immediate effect is to raise her subjective probability for E to 1, then her post-learning posterior for any proposition H should be

$$q(H) = p(H|E). \tag{2.4}$$

We remark once again that if one proceeds without Bayes' theorem as the basic tool of PU and learning, then she does not need to use Bayes' formula to define conditional probability (nor to

[5] This remark on existence of these terms is very important in quantum-like modelling of subjectivist PU.

use even Boolean logic); other non-Bayesian definitions of conditioning can be considered; one of them is quantum conditional probability, §5.

Although useful as an ideal, simple conditioning is not widely applicable because it requires the learner to become absolutely certain of E's truth. As Jeffrey has argued [59,60], the evidence we receive is often sufficient only to assign some probabilities to occurrence of E. Here, the direct effect of a learning experience will be to alter the subjective probability of some proposition without raising it to 1 or lowering it to 0. Experiences of this sort are appropriately modelled by what has come to be called *Jeffrey conditioning.*

Jeffrey conditioning. If a person with a prior such that $0 < p(E) < 1$ has a learning experience whose sole immediate effect is to change her subjective probability for E to $q(E)$, then her post-learning posterior for any H should be given by FTP:

$$q(H) = q(E)p(H|E) + (1 - q(E))p(H|\bar{E}), \tag{2.5}$$

where, for any proposition F, the symbol $\bar{F}$ denotes negation of the proposition F. Obviously, Jeffrey conditioning reduces to simple conditioning when $q(E) = 1$. We remark that FTP is also based on Bayes' formula as well as additivity of CP (Boolean logic is exploited twice, we use commutativity of conjunction and distributivity law for disjunction and conjunction).

In the process of DM, one computes odds; for simple conditioning, it is given by

$$O(H) = \frac{p(H|E)}{p(\bar{H}|E)}; \tag{2.6}$$

for Jeffrey conditioning,

$$O(H) = \frac{q(E)p(H|E) + (1 - q(E))p(H|\bar{E})}{q(E)p(\bar{H}|E) + (1 - q(E))p(\bar{H}|\bar{E})}. \tag{2.7}$$

If $O(H) > 1$, the collected data E can be treated as an evidence in favour of H, in the opposite case, in favour of $\bar{H}$.

Jeffrey conditioning can be generalized to the case of a collection of hypotheses and pieces of data represented mathematically as the disjoint partitions of the space of elementary events Ω, (H_j) and (E_i), where $H_i \& H_j = \emptyset,\ E_i \& E_j = \emptyset,\ i \neq j$:

$$q(H_j) = \sum_i q(E_i)p(H_j|E_i), \tag{2.8}$$

with odds:

$$O(H_j) = \frac{\sum_i q(E_i)p(H_j|E_i)}{\sum_{i\neq j} q(E_i)p(H_j|E_i)}. \tag{2.9}$$

3. QBist generalization of formula of total probability

QBists strive to treat the quantum formalism in the purely probabilistic terms, as a machinery for PU, cf. with the Växjö interpretation, §4 and appendix A. Similar to the latter they interpret QP as a generalization of classical PU, in particular, the Born rule is coupled with FTP.

In QBism, quantum states are represented by density operators ρ in a Hilbert space assumed to be *finite dimensional.* A measurement (an action taken by the agent) is described by a POVM $H = (H_j)$, where j labels the potential outcomes experienced by the agent. The agent's personal probability $q(H_j)$ of experiencing outcome j is given by the *Born rule*

$$q(H_j) = \operatorname{Tr} \rho H_j. \tag{3.1}$$

The main achievement of QBism is the demonstration that this operational rule (which was postulated by Born and which origin remains one the main mysteries of QM) can be treated as a generalization of classical PU.[6]

[6] In fact, this is generalization of Jeffrey conditioning [59,60], see §2, the transformation (2.5). But it seems that QBists have never paid attention to coupling of QBism with Jeffrey's conditioning.

One of the basic points of QBism is that the agent's reference measurement is an arbitrary *informationally complete POVM*, $E=(E_i)$, such that each E_i is of rank 1, i.e. is proportional to a one-dimensional projector Π_i.[7] Because the reference measurement is informationally complete, any state ρ corresponds to a unique vector of probabilities $q(E_i)=\mathrm{Tr}\,\rho E_i$, and any POVM $F=(F_j)$ corresponds to a unique matrix of conditional probabilities $p(H_j|E_i)=\mathrm{Tr}\,H_j\Pi_i$. QBists formulated the following statement of a high intrepretational value: the Born rule can be interpreted as one special form of transformation of probabilities:

$$q(H_j)=f(q(E_i),p(H_j|E_i)). \tag{3.2}$$

The same statement also plays the fundamental role in the Växjö interpretation of QM [7,10] and §4. Thus, these two interpretations have a very important common point. However, otherwise they differ crucially; not only in the interpretation of probability, but (what is more crucial for formal considerations) also in the mathematical formulations of the transformation law (3.2), cf. (3.3) and (5.3).

We remark that the classical FTP used in Jeffrey conditioning, see (2.8), has the form $q(H_j)=\sum_i q(E_i)p(H_j|E_i)$. As was emphasized above, in QBism *the Born rule is treated as one of the generalizations of FTP.* In the special case when the reference measurement is SIC-POVM (symmetric informationally complete POVM [2,61]), the functional relationship f takes the simple form [2]:

$$q(H_j)=\mathrm{Tr}\,\rho_i E_j=\sum_i\left((d+1)q(E_i)-\frac{1}{d}\right)p(H_j|E_i). \tag{3.3}$$

Thus, a QBist says that QP-calculus is an operational representation (in complex Hilbert space, by using linear operators) of the probabilistic calculus based on a new FTP given by (3.3). Of course, it would be a great achievement of QBists if they were able to proceed another way around, i.e. to reconstruct the formalism of QM starting directly from this generalized FTP. Really, this problem is extremely complex mathematically.

4. Generalization of the formula of total probability: the Växjö version

The Växjö interpretation of QM is a statistical realist contextual interpretation, see appendix A for brief presentation. We now proceed to derivation of the Växjö version of generalized FTP matching the Born rule [10].

Consider two observables of the von Neumann–Lüders type given spectral decompositions of the Hermitian operators $A=\sum_i a_iA_i$ and $B=\sum_j b_jB_j$, where A_i and B_j are orthogonal projectors on eigenspaces corresponding to eigenvalues a_i and b_j. In the case of infinite-dimensional state space, we restrict our consideration to operators with purely discrete spectra. For the sake of simplicity, here we proceed with pure states only; generalization to the case of an arbitrary quantum state represented by a density operator is straightforward, but less visualizable. By using Born's rule, we obtain

$$\begin{aligned} p(B=b_j)&=\|B_j\psi\|^2=\sum_k\sum_m\langle B_jA_k\psi|B_jA_m\psi\rangle\\ &=\sum_{k=m}(\ldots)+\sum_{k\neq m}(\ldots)=p_{\mathrm{diag}}(B=b_j)+p_{\mathrm{ndiag}}(B=b_j). \end{aligned} \tag{4.1}$$

The term p_{diag} can be treated as the classical counterpart of the quantum generalization of FTP and the term p_{ndiag} as the non-classical counterpart, the interference term.

[7]Such measurements exist for any finite Hilbert-space dimension! Thus, the assumption that the state space is finite-dimensional plays the important role in QBist derivation of generalized FTP.

Set $\psi_{a_k} = A_k\psi/\|A_k\psi\|$. This is the output state of the A-measurement with the fixed result $A = a_k$ (for the input state ψ). For p_{diag}, we have

$$p_{\mathrm{diag}}(B = b_j) = \sum_k \|B_j\psi_{a_k}\|^2 \|A_k\psi\|^2 = \sum_k p(b = b_j|a = a_k)p(a = a_k). \tag{4.2}$$

Thus, formally (4.2) coincides with classical FTP, see (2.8). Now, we turn to analysis of the interference term. We set $\psi_{b_j|a_k} = B_j\psi_{a_k}/\|B_j\psi_{a_k}\|$. This is the output state of the B-measurement with the fixed result $B = b_j$ (for the input state ψ_{a_k}).

By representing the elements of p_{ndiag} in the Euler form with the phases denoted as $\gamma_{j;k,m}$, we obtain the following representation:

$$\begin{aligned}
p_{\mathrm{ndiag}}(B = b_j) &= \sum_{k<m} |\langle B_jA_k\psi|B_jA_m\psi\rangle|[\mathrm{e}^{\mathrm{i}\gamma_{j;k,m}} + \mathrm{e}^{-\mathrm{i}\gamma_{j;k,m}}] \\
&= 2\sum_{k<m} |\langle\psi_{b_j|a_k}|\psi_{b_j|a_m}\rangle| \|B_j\psi_{a_k}\| \|B_j\psi_{a_m}\| \|A_k\psi\| \|A_m\psi\| \cos\gamma_{j;k,m} \\
&= 2\sum_{k<m} \cos\gamma_{j;k,m} |\langle\psi_{b_j|a_k}|\psi_{b_j|a_m}\rangle| \\
&\quad \times \sqrt{p(B = b_j|A = a_k)p(A = a_k)p(B = b_j|A = a_m)p(A = a_m)}.
\end{aligned} \tag{4.3}$$

Now, we discuss the probability interpretation of the term

$$p(b_j|a_k, b_j|a_m) = |\langle\psi_{b_j|a_k}|\psi_{b_j|a_m}\rangle|^2.$$

This is the probability of transition between the two states $\phi = \psi_{b_j|a_k}$ and $\xi = \psi_{b_j|a_m}$. In the purely PU-framework (i.e. without explicit relation to physics), this term can be treated as the coefficient of correlation between two conditional measurements of B with the same result $b = b_j$: one is conditioned on the result $a = a_k$ and another on the result $a = a_m$ of the A-measurement. Thus, finally, we obtain the following generalized FTP:

$$\begin{aligned}
p(B = b_j) &= \sum_k p(B = b_j|A = a_k)p(A = a_k) + 2\sum_{k<m} \cos\gamma_{j;k,m} \sqrt{p(b_j|a_k, b_j|a_m)} \\
&\quad \times \sqrt{p(B = b_j|A = a_k)p(A = a_k)p(B = b_j|A = a_m)p(A = a_m)}.
\end{aligned} \tag{4.4}$$

Consider now the case of the B-observable represented by an operator $\hat{B}$ with non-degenerate spectrum. Here, the states $\psi_{b_j|a_k}$ and $\psi_{b_j|a_m}$ coincide (the eigenvector of $\hat{B}$ corresponding to the eigenvalue b_j). Thus, the transition probability $p(b_j|a_k, b_j|a_m) = 1$, and the quantum FTP is simpler:

$$\begin{aligned}
p(B = b_j) &= \sum_k p(B = b_j|A = a_k)p(A = a_k) \\
&\quad + 2\sum_{k<m} \cos\gamma_{j;k,m} \sqrt{p(B = b_j|A = a_k)p(A = a_k)p(B = b_j|A = a_m)p(A = a_m)}.
\end{aligned} \tag{4.5}$$

We remark that by representing the product $\cos\gamma_{j;k,m}\sqrt{p(b_j|a_k, b_j|a_m)}$ as a cosine of new angle, we can write even (4.4) in simpler way as (4.5).

5. Quantum-like models of decision-making

We now borrow methods of QM to use them for DM and cognition modelling. Thus, the reader must not project the coming considerations to quantum physics. In particular, we hope that QBists will not be annoyed too much by a possible misuse of their formalism and ideology. (For example, we suspect that the reference to the conscious–unconscious interrelation in the process of DM would not be accepted by a QBist-physicist.) In the same way, those who keep to the statistical interpretation of QP might be displeased that we explore the methods developed in the Växjö approach (elaborated for justification the possibility of the realist interpretation of QM, statistical and contextual) for PU of subjective probabilities of decision-makers.

(a) A prior state version of a prior probability

In classical subjectivist approach to DM, an agent, say Alice, assigns to a hypothesis H, a prior probability which is mathematically represented by a probability measure p. Then, she updates $p(H)$ on the basis of gained information about the data E as the conditional probability $p(H|E)$ (simple conditioning). To determine this conditional probability, Alice uses a prior probability $p(H)$ and conditional probability $p(E|H)$ and Bayes' theorem, see (2.2). More generally, she proceeds with Jeffrey conditioning (2.5), but conditional probability $p(H|E)$ is still determined with the aid of (2.2). Probability $q(E)$ (more generally, probabilities $q(E_i), i = 1, \ldots, n$) can be determined by context of PU, context of gaining new information about the degree of truthfulness of the hypothesis H. It depends on the prior probability $p(E)$ (the prior probabilities $p(E_i)$).

In the quantum-like approach to DM, the basic initial entity is not the prior subjective probability of H, but the prior belief state ρ, encoding complete context of decision-making. This belief state encodes not only information about the hypothesis H, but also all available information about the data $E = (E_i)$ which can be used to make decision regarding the degree of truthfulness of H. From this state ρ, Alice extracts the probabilities $q(E_i), , i = 1, \ldots, n$, and conditional probabilities $p(H|E_i)$. Thus, in general,

$$q(E_i) \equiv q_\rho(E_i), p(H|E_i) \equiv p_\rho(H|E_i).$$

The probabilities $q(E_i)$ are her subjective probabilities assigned to pieces of data E_i which are encoded in ρ and extracted during self-measurement, see below. These quantities are extracted 'directly' from ρ, i.e. without the inversion procedure (2.2) which is impossible in the quantum framework.

We emphasize the crucial difference between the notions of conditional probability in classical and quantum frameworks. We repeat that classically the probability of a hypothesis H conditional on a collected data E is the ratio of the unconditional probability of the conjunction of the hypothesis with the data to the unconditional probability of the data alone. In the quantum case, conjunction of H and E is not well defined.[8] Therefore, quantum(-like) conditioning is based on a different approach. Here we are interested in the feedback $E = (E_i)$ based on measurement of the belief state ρ.

This is a delicate point of our considerations. The E-measurement is performed by Alice on the unconscious level. This is a *self-measurement* performed by Alice; its aim is to determine probabilities $q(E_i)$. If such measurement outputs the data E_i, then the initial belief state ρ is transformed to the new state ρ_i. It is important to point out that Alice does not feel this update of the belief-state consciously. She makes unconscious reasoning of the following type. Suppose the correct piece of data (the true state) is E_i.

'What (subjective) probability shall I (Alice) assign to it on the basis on my personal belief state ρ?'

By assigning the concrete value of probability $q(E_i)$ (still unconsciously) she changes her belief state $\rho \to \rho_i$, Then, for this belief state ρ_i, Alice assign the subjective probability $p(H|E_i)$, representing the degree of her belief in truthfulness of the hypothesis H for the belief state ρ_i. This self-measurement E performed on the unconscious level differs crucially from the quantum physical measurements: it does not destroy the initial belief state ρ. Alice can repeat her reasoning for another $E_k, k \neq i$, and assign $q(E_k)$ and $p(H|E_k)$. In this way, she collects all probabilities $(q(E_i), p(H|E_i), i = 1, \ldots, n)$.

[8]To be more precise, one can say that we can consider 'ordered conjunctions' first H and then E and first E and then H. They do not coincide and they generate different probability distributions which (in general) cannot be coupled.

(b) Quantum(-like) Jeffrey conditioning

Finally, she uses some generalization of FTP (in this paper, we consider two of them, QBism-FTP and Växjö-FTP) to perform generalized Jeffrey-like conditioning. This step is also realized at the unconscious level of information processing. Only the output of this step, $q(H)$, is transferred to consciousness. This procedure differs crucially from the classical Bayesian PU and learning. The only coupling to the Bayesian scheme is the use of a generalized FTP. (Its classical version is derived by using the Bayes formula for conditional probability). Therefore, the terminology related to Bayesian PU and learning is not very acceptable anymore. Generalized FTPs are derived without the Bayes formula. Such FTPs are the cornerstones of the quantum scheme. I would prefer to speak about *quantum-like Jeffrey conditioning and learning*.[9]

Instead of one hypothesis H, we can consider a complete group of mutually exclusive hypotheses $(H_j, j=1,\dots,k)$. In the quantum model, they are represented by a POVM $H=(H_j)$.

(i) QBism

For example, we can guess that Alice's brain uses QBist FTP (3.3). We now briefly discuss its features. One of the most important points concerns informationally complete (self-)measurement. This implies that determination of the probabilities $q(E_i)$ is equivalent to complete determination of the belief state ρ which represents the context of (unconscious) learning about the probability $q(H_j)$. Thus, by PU based on QBist FTP (3.3), Alice uses complete information about ρ.

Of course, this completeness of information gain consumes a lot of computational resources, POVM $E=(E_i)$ consists of $n=d^2$ elements, where d is the dimension of the Hilbert space. In some DM-situation this question, about consumption of computational resources, can be critical. In such situations, it can be more useful to proceed with E-measurement which provides incomplete information about ρ, cf. with the Växjö scheme for PU.

For me personally, the main problem of QBist FTP (3.3) is that it does not have the form of a perturbation of the classical FTP (2.8).

(ii) QBism with Växjö flavour

Another possibility is to use the Växjö version of generalized FTP. However, we have to change the interpretation of probability from statistical to subjective. Thus, we adopt the basic mathematical formula of the Växjö interpretation of QM, endowing it with a new interpretation borrowed from QBism. This section can be considered as (an unexpected, cf. [7]) mixture of QBism and 'Växjöism'. First, we write Växjö-FTP in PU friendly notations, $A \to E=(E_i), B \to (H_j)$, where components of POVMs are orthogonal projectors and $E_i \perp E_j, H_i \perp H_j, i \neq j$ and $\sum_i E_i = I, \sum_j H_j = I$, and FTP has the form:

$$q(H_j) = \sum_k p(H_j|E_k)q(E_k) + 2\sum_{k<m} \cos\gamma_{j;k,m}\sqrt{p(H_j|E_k)q(E_k)p(H_j|E_k)q(E_m)}. \tag{5.1}$$

First, we point to the main advantage of this type of quantum Jeffrey conditioning. This generalized FTP matches well the classical FTP (2.8), i.e. coincides with it if the interference term

[9] We remark that in the quantum formalism the output probability $q(H)$ can be written directly with the aid of Born's rule. Thus, one can speculate that Alice's brain assigns the probability $q(H)$ without using the presented quantum-like Jeffrey conditioning. For example, the brain might really represent belief states in complex Hilbert space and use directly Born's rule. Of course, one has to distinguish between a mathematical and physical models. On the physical level, the complex Hilbert space can appear in quantum-like models of brain's functioning based on the representation of belief states by the classical electromagnetic field in the brain, see [42]. However, in this paper, we want to explore the subjective probability framework. Here, quantum-like Jeffrey conditioning provides the most consistent scheme of PU and learning. Finally, we remark that here PU has to be treated not so straightforwardly as in the classical Bayesian approach where PU is the update of a prior probability to a posterior probability. In the quantum-like model, this 'update' is, in fact, the assignment of subjective probability to the hypothesis H on the basis of the reference self-measurement on the belief state ρ.

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in (5.1) goes to zero. Thus, new Jeffrey conditioning differs from 'conventional one' by appearance of additional terms (of the interference type) correcting the classical conditioning. We also remark that, in contrast to QBist FTP (3.3), our FTP is valid even for infinite-dimensional state spaces.

The main problem of FTP (5.1) is to present a proper interpretation of the interference term.

The most consistent interpretation is that Alice's brain really constructs a kind of Hilbert space representation for probabilities, i.e. it operates (unconsciously) with complex amplitudes (to be precise, with physical carries of such amplitudes).

However, by presenting the general scheme of DM and learning based on FTP (5.1), we do not couple it to any concrete quantum or subquantum physical model. Therefore, we treat 'phases' $\gamma_{j;k,m}$ as adjustment parameters which are used by Alice's brain in the process of PU, i.e. Alice uses not only probabilities, $q(E_k), p(H_j|E_k)$, but also some updating parameters that depend on the belief state ρ, the data-observable E representing self-measurement to extract information from ρ, and the hypothesis observable H.

There are two possibilities, either these phases are produced in each act of PU and DM or they are outputs of learning based on the previous experience (they were memorized in Alice's brain). When Alice recognizes the concrete belief state ρ, she takes these parameters from her memory. We can assume that some representations of the observables E and H were also created from the previous experience and were memorized.

In this framework, Alice operates just with the collection of probabilities $q(E_k), p(H_j|E_k)$ and the phases $\gamma_{j;k,m}$ or simply adjustment parameters $\lambda_{j;k,m} = \cos\gamma_{j;k,m}$. We remark that

$$|\lambda_{j;k,m}| \leq 1. \tag{5.2}$$

She takes these probabilities and parameters and transfer them to

$$q(H_j) = \sum_k p(H_j|E_k)q(E_k) + 2\sum_{k<m} \lambda_{j;k,m}\sqrt{p(H_j|E_k)q(E_k)p(H_j|E_m)q(E_m)}. \tag{5.3}$$

We remark that, for any pair of vectors of probabilities $(p(H_j|E_k), q(E_k))$, we have $\sum_j \sum_k p(H_j|E_k)q(E_k) = \sum_k q(E_k)\sum_j p(H_j|E_k) = 1$. Therefore, to produce $q(H_j)$ satisfying the normalization condition $\sum_j q(H_j) = 1$, the adjustment parameters $\lambda_{j;k,m}$ have to satisfy the following constraint:

$$\sum_j \sum_{k<m} \lambda_{j;k,m}\sqrt{p(H_j|E_k)q(E_k)p(H_j|E_m)q(E_m)} = 0. \tag{5.4}$$

This constraint is not redundant, i.e. the PU adjustment parameters cannot be arbitrary. (Here, it is important that we proceed with a very special class of observables, of the Lüders–von Neumann type. By considering observables that do not belong to this class, we can relax this constraint, but not completely.) We also remark that updated probabilities $q(H_j)$ should be non-negative. This induces an additional constraint on the PU-adjustment parameters:

$$\sum_k p(H_j|E_k)q(E_k) + 2\sum_{k<m} \lambda_{j;k,m}\sqrt{p(H_j|E_k)q(E_k)p(H_j|E_m)q(E_m)} \geq 0. \tag{5.5}$$

This constraint is also non-trivial. Take, for example, $n = 3$ and $q(E_k) = \frac{1}{3}$ and also suppose that $p(H_j|E_k) = c_j$, where c_j is a constant, $c_j \in [0, 1]$. Then, we have that $q(H_j) = c[1 - 2] < 0$.

Thus by using the Växjö scheme for Jeffrey-like conditioning, Alice has to use adjustment parameters $\lambda_{j;k,m} \in [0, 1]$ satisfying two constraints, (5.4) and (5.5). Such selection of these parameters implies the consistent procedure of Jeffrey-like conditioning and learning about probabilities for the hypotheses $H = (H_j)$ on the basis of information about the data $E = (E_i)$.

However, a selection the PU-adjustment parameters $\lambda_{j;k,m} \in [0, 1]$ satisfying (5.4) and (5.5) does not guarantee that it is possible to construct a quantum state (in this paper, we use the Växjö approach to PU and learning only for pure states, but in principle, the approach can be generalized to arbitrary quantum states) and two Hermitian operators H and E generating the vectors of probabilities $p(H_j|E_k), q(E_k), k = 1, \ldots, n$, and the phases $\theta_{j;k,m}$, where $\lambda_{j;k,m} = \cos\theta_{j;k,m}$. This is a complex problem.[10]

It can be easily solved in the case of dichotomous observables H and E, see [10]. Here, the sufficient and necessary condition for existence of such quantum representation of probabilistic data is *double stochasticity* of the matrix of conditional probabilities, $P = (p(H_j|E_k))_{j,k=1,2}$, i.e.

$$p(H_j|E_1) + p(H_j|E_2) = 1,$$

for any j. We remark that the matrix P is also always stochastic, i.e. for each k,

$$p(H_1|E_k) + p(H_2|E_k) = 1.$$

However, already the case of triple-valued observables is very difficult from the mathematical viewpoint, see Nyman & Basieva [62] for some partial results. This inverse Born problem can be formulated in a more general setting: to construct a density operator and POVM observables from probabilities. The problem has not yet been solved, i.e. the situation is similar to the QBism.

6. Concluding remarks

In this paper, we have analysed the possibility to treat QBism as a general subjective probability scheme for DM. Thus, we try to extend the domain of its applicability outside of quantum physics—to cognition, psychology, economics. We stress again that this is not an attempt of quantum mechanical explanation of cognition and consciousness. Following Appleby [63], we can say that our study supports the project of creation of unified psychophysical model.

Then, we made a step towards merging the Växjö interpretation of QM (originally—statistical contextual realist interpretation) with QBism by switching from the statistical interpretation of probability to subjective one. We were motivated by evident advantages of the use of subjective probability in modelling the process of DM (by an individual agent). Our main aim was to explore the Växjö version of the FTP as the basis of the generalized scheme of PU based on the quantum representation in complex Hilbert space. This formula differs from QBist FTP of Fuch and Schack. We analysed advantages and problems of both types of quantum FTPs.

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Appendix A. Växjö interpretation

This interpretation was born in attempt to combine consistently the views of Einstein and Bohr [7]: realism and contextuality. This is the (ensemble) statistical interpretation. It was born from the observation that, in fact, Bohr's contextuality does not imply non-realism. Thus, a theory can (but need not) be both contextual and realistic. Contextuality has to be treated statistically as contextuality of probabilities, their dependence on experimental contexts.

Such contextuality means that each (experimental) context C determines its own Kolmogorov probability space,[11] $\mathcal{P}_C = (\Omega_C, \mathcal{F}_C, p_C)$. Compatibility of a family of contexts $\mathcal{C} = (C_\alpha)$ means that

[10]We call it the 'inverse Born problem'. Born solved the problem of the probabilistic interpretation of the quantum measurement scheme. He presented the explicit formula transferring quantum entities into probabilities. Now, we want to construct quantum entities from probabilities.

[11]As was pointed out in Introduction, this viewpoint on contextuality is more general than the one used in the quantum community in discussions related to violation of Bell's inequality. The latter contextuality is defined as dependence of the results of measurement of a quantum observable A on results of measurements of another observable B which is measurable jointly with A. This is a very special case of 'Växjö contextuality' which definitely matches Bohr's views better. We remark that recently the notion of contextuality which is close (but not identical) to 'Växjö contextuality' was invented by Dzhafarov [64,65], who proposed to assign context dependence to observables and not to probabilities.

they can be represented within a single probability space

$$\mathcal{P}_{\mathcal{C}} = (\Omega_{\mathcal{C}}, \mathcal{F}_{\mathcal{C}}, p_{\mathcal{C}})$$

in such a way that the concrete context probability $p_C, C \in \mathcal{C}$, is given as (classical) conditional probability in $\mathcal{P}_{\mathcal{C}}$ given by the Bayes formula. In the opposite case, a family of contexts $\mathcal{C}$ is treated as *incompatible*.[12]

By the Växjö interpretation, QM is a special mathematical formalism for working with contextual probabilities for families of contexts which are, in general, incompatible. Of course, QP is not the only possible formalism to operate with contextual probabilities. A general theory of contextual probability was presented in monograph [10].

The main distinguishing feature of QP is its complex Hilbert space representation. All quantum contexts can be unified with the aid of a quantum state ψ. Of course, ψ represents only a part of context, another part is given by an observable. From the Växjö viewpoint *this contexts unifying function of a quantum state ψ is the key-element of QP*. It is not about just a collection of Kolmogorov probability spaces corresponding to different experimental arrangements. All these arrangements are coupled with the aid of quantum states. The formal mathematical structure of this coupling is simple and clear. However, we are far from understanding its mechanism. From the viewpoint of contextual probability, *this context (probability space) unifying function of a quantum state is one of the mysteries of QM*.

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[12]We also remark that in many applications, even if contexts are incompatible, it is possible to select the sample space (space of elementary events) Ω and the σ-algebra of events $\mathcal{F}$ independently of contexts belonging to $\mathcal{C}$; the non-trivial issue is the dependence of probability measures on contexts, $C \in \mathcal{C} \to p_C$.

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