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Interval oscillation criteria for second-order forced impulsive delay differential equations with damping term

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Abstract

In this paper, we present some sufficient conditions for the oscillation of all solutions of a second order forced impulsive delay differential equation with damping term. Three factors-impulse, delay and damping that affect the interval qualitative properties of solutions of equations are taken into account together. The results obtained in this paper extend and generalize some of the the known results for forced impulsive differential equations. An example is provided to illustrate the main result.

Keywords: Oscillation, Second-order, Impulse, Damping term, Differential equation

Mathematics Subject Classification: 34A37, 34C10

Background

In this paper, we consider the second-order impulsive differential equation with mixed nonlinearities of the form

$$\begin{cases} (r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + q(t)x^{\gamma}(t-\delta) \\ \qquad\qquad + \sum_{i=1}^{n} q_i(t)|x(t-\delta)|^{\alpha_i - 1}x(t-\delta) = e(t), \quad t \neq \tau_k; \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k) \end{cases} \tag{1}$$

where $t \geq t_0$, $k \in \mathbb{N}$, $\{\tau_k\}$ is the impulse moments sequence with

$$0 \leq t_0 = \tau_0 < \tau_1 \ldots, \quad \lim_{t\to\infty} \tau_k = \infty,$$

and

$$x(\tau_k) = x(\tau_k^-) = \lim_{t\to\tau_k^-} x(t), \; x(\tau_k^+) = \lim_{t\to\tau_k^+} x(t)$$

$$x'(\tau_k) = x'(\tau_k^-) = \lim_{h\to 0^-} \frac{x(\tau_k + h) - x(\tau_k)}{h},$$

$$x'(\tau_k^+) = \lim_{h\to 0^+} \frac{x(\tau_k + h) - x(\tau_k^+)}{h}.$$

Let $J \subset \mathbb{R}$ be an interval and define $PLC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous on each interval (τ_k, τ_{k+1}), $x(\tau_k^{\pm})$ exist, and $x(\tau_k) = x(\tau_k^-)$ for all $k \in \mathbb{N}\}$.

For given t_0 and $\phi \in PLC([t_0 - \delta, t_0], \mathbb{R})$, we say $x \in PLC([t_0 - \delta, \infty], \mathbb{R})$ is a solution of Eq. (1) with initial value ϕ if $x(t)$ satisfies (1) for $t \geq t_0$ and $x(t) = \phi(t)$ for all $t \in [t_0 - \delta, t_0]$. A non-trivial solution is called oscillatory if it has infinitely many zeros;otherwise it is called non-oscillatory.

In recent years the theory of impulsive differential equations emerge as an important area of research, since such equations have applications in the control theory, physics, biology, population dynamics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equation, see Bainov and Simenov (1993), Lakshmikantham et al. (1989). In the last decades, interval oscillation of impulsive differential equations was arousing the interest of many researchers, see Li and Cheung (2013), Liu and Xu (2007, 2009), Muthulakshmi and Thandapani (2011) and Özbekler and Zafer (2009, 2011) considered the following equations

$$\begin{cases} (r(t)(\Phi_\alpha(x'))' + p(t)\Phi_\alpha(x') + q(t)\Phi_\beta(x) = e(t), & t \neq \tau_k; \\ \Delta(r(t)\Phi_\alpha(x')) + q_i\Phi_\beta(x) = e_i, & t = \tau_k,\ k \in \mathbb{N}. \end{cases} \tag{2}$$

As far as we know, it is the first article focusing on the interval oscillation for the impulsive differential equation with damping term. Their results well improved and extended the earlier one for the equations without impulse or damping. Recently Guo et al. (2014) considered a class of second order nonlinear impulsive delay differential equations with damping term and established some interval oscillation criteria for that equation.

However, for the impulsive equations, almost all of interval oscillation results in the existing literature were established only for the case of "without delay", in other words, for the case of "with delay" the study on the interval oscillation is very scarce. To the best of our knowledge, Huang and Feng (2010) gave the first research in this direction recently. They considered second order delay differential equations with impulses

$$\begin{cases} x''(t) + p(t)f(x(t-\tau)) = e(t), & t \geq t_0, \quad t \neq t_k; \\ x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), & k = 1, 2, \ldots \end{cases} \tag{3}$$

and established some interval oscillation criteria which developed some known results for the equations without delay or impulses (Liu and Xu 2007; El Sayed 1993). It is natural to ask if it is possible to research the interval oscillation of the impulsive delay equations with damping term. In this paper, motivated mainly by Huang and Feng (2010) and Özbekler and Zafer (2009), we study the interval oscillation of second order nonlinear impulsive delay differential equations with damping term (1). We establish some interval oscillation criteria which generalize or improve some known results of Guo et al. (2012a, b, 2014), Liu and Xu (2007, 2009), Muthulakshmi and Thandapani (2011), Pandian and Purushothaman (2012), Özbekler and Zafer (2009, 2011) and Li and Cheung (2013). Finally we give an example to illustrate our main result.

Main results

Throughout this paper, we assume that the following conditions hold:

(A1) $r(t) \in C^1([t_0,\infty),(0,\infty))$ and $p(t)$, $q(t)$, $q_i(t)$, $e(t) \in PLC([t_0,\infty),\mathbb{R})$, $i = 1, 2\ldots,n$, with $r'(t) + p(t) \geq 0$ for all $t \geq t_o$;

(A2) $\delta \geq 0$, $\tau_{k+1} - \tau_k > \delta$, $k \in \mathbb{N}$, $\alpha_1 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0$ are constants;

(A3) a_k, b_k are real constants satisfying $b_k \geq a_k > 0$, $k = 1, 2, \ldots$.

We begin with the following notations: $I(s) = \max\{i : t_0 < \tau_i < s\}$, $r_j = \max\{r(t) : t \in [c_j, d_j]\}$, $j = 1, 2$ and

$$E_{c_j,d_j} = \{u \in C^1([c_j,d_j],\mathbb{R}) : \ u(t) \not\equiv 0,\, u(c_j) = u(d_j) = 0\}.$$

For two constants $c, d \notin \{\tau_k\}$ with $c < d$ and a function $\varphi \in C([c,d],\mathbb{R})$, we define an operator $\Omega : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\Omega_c^d[\varphi] = \begin{cases} 0 & \text{for} \quad I(c) = I(d), \\ \varphi(\tau_{I(c)+1})\theta(c) + \sum_{i=I(c)+2}^{I(d)} \varphi(\tau_i)\varepsilon(\tau_i) & \text{for} \quad I(c) < I(d), \end{cases}$$

where

$$\theta(c) = \frac{(a_{I(c)+1})^\gamma - (b_{I(c)+1})^\gamma}{(a_{I(c)+1})^\gamma(\tau_{I(c)+1} - c)^\gamma} \quad \text{and} \quad \varepsilon(\tau_i) = \frac{a_i^\gamma - b_i^\gamma}{a_i^\gamma(\tau_i - \tau_{i-1})^\gamma}.$$

To prove our main results, we need the following lemmas.

Lemma 1 *Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0$. Then there exists an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ satisfying*

$$\sum_{i=1}^n \alpha_i\eta_i = \gamma \tag{4}$$

and also either

$$\sum_{i=1}^n \eta_i < 1, \quad 0 < \eta_i < 1 \tag{5}$$

or

$$\sum_{i=1}^n \eta_i = 1, \quad 0 < \eta_i < 1. \tag{6}$$

The proof of Lemma 1 can be found in Hassan et al. (2011) and Özbekler and Zafer (2011) which is the extension of (Lemma 1, Sun and Wong 2007).

Remark 1 For given constants $\alpha_1 > \alpha_2 > \ldots \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0$, Lemma 1 ensures the existence of n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ such that either (4) and (5) or (4) and (6) hold. Particularly when $n = 2$, and $\alpha_1 > \gamma > \alpha_2 > 0$ in the first case we have

$$\eta_1 = \frac{\gamma - \alpha_2(1-\eta_0)}{\alpha_1 - \alpha_2}, \quad \eta_2 = \frac{\alpha_1(1-\eta_0) - \gamma}{\alpha_1 - \alpha_2}$$

where η_0 be any positive number satisfying $0 < \eta_0 < \frac{\alpha_1 - \gamma}{\alpha_1}$. This will ensure that $0 < \eta_1, \eta_2 < 1$ and conditions (4) and (5) are satisfied. In the second case, we can solve (4) and (6) and obtain

$$\eta_1 = \frac{\gamma - \alpha_2}{\alpha_1 - \alpha_2}, \quad \eta_2 = \frac{\alpha_1 - \gamma}{\alpha_1 - \alpha_2}.$$

The Lemma below can be found in Hardy et al. (1934).

Lemma 2 *Let X and Y be non-negative real numbers. Then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1) Y^{\lambda}, \quad \lambda > 1$$

where equality holds if and only if $X = Y$.

Let $\gamma > 0$, $A \ge 0$, $B > 0$ and $y > 0$. Put $\lambda = 1 + \frac{1}{\gamma}$, $X = B^{\frac{\gamma}{\gamma+1}} y$, $Y = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma} A^{\gamma} B^{\frac{-\gamma^2}{\gamma+1}}$ in Lemma 2, we have

$$A - B \le \left(\frac{A}{\gamma+1}\right)^{\gamma+1} \left(\frac{\gamma}{B}\right)^{\gamma}. \tag{7}$$

Theorem 1 *Suppose that for any $T > 0$, there exist $c_j, d_j \notin \{\tau_k\}$, $j = 1, 2$ such that $c_1 < d_1 \le d_1 + \delta \le c_2 < d_2$ and $q(t), q_i(t) \ge 0$, $t \in [c_1 - \delta, d_1] \cup [c_2 - \delta, d_2]$, $i = 1, 2, \ldots, n$ and*

$$e(t) = \begin{cases} \le 0 & \text{if } t \in [c_1 - \delta, d_1], \\ \ge 0 & \text{if } t \in [c_2 - \delta, d_2], \end{cases} \tag{8}$$

and $u_j \in E_{c_j, d_j}$ such that

$$\begin{aligned} &\int_{c_j}^{d_j} \left[\frac{r(t)}{(\gamma+1)^{\gamma+1}} \left| (\gamma+1) u'(t) - \frac{p(t)u(t)}{r(t)} \right|^{\gamma+1} \right] dt - \int_{c_j}^{\tau_{I(c_j)+1}} Q(t) Q^j_{I(c_j)}(t) |u(t)|^{\gamma+1} dt \\ &\quad - \sum_{k=I(c_j)+2}^{I(d_j)} \int_{\tau_{k-1}}^{\tau_k} Q(t) Q^j_k(t) |u(t)|^{\gamma+1} dt - \int_{\tau_{I(d_j)}}^{d_j} Q(t) Q^j_{I(d_j)}(t) |u(t)|^{\gamma+1} dt \\ &\quad < r_j \Omega_{c_j}^{d_j} [|u(t)|^{\gamma+1}], \quad j = 1, 2 \end{aligned} \tag{9}$$

where

$$Q(t) = q(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} (\eta_i^{-1} q_i(t))^{\eta_i} |e(t)|^{\eta_0}, \quad \eta_0 = 1 - \sum_{i=1}^{n} \eta_i$$

where $\eta_i > 0$ are chosen according to given $\alpha_1, \alpha_2, \ldots \alpha_n$ as in Lemma 1 satisfying (4) and (5), and

$$Q_k^j(t) = \begin{cases} \frac{(t-\tau_k)^\gamma}{(a_k\delta + b_k(\tau-\tau_k))^\gamma}, & t \in (\tau_k, \tau_k+\delta), \\ \frac{(\tau-\tau_k-\delta)^\gamma}{(\tau-\tau_k)^\gamma}, & t \in [\tau_k+\delta, \tau_{k+1}], \end{cases} \quad k = I(c_j), I(c_j)+1, \ldots, I(d_j),$$

then every solution of Eq. (1) *is oscillatory.*

Proof Let $x(t)$ be a non-oscillatory solution of Eq. (1). Without loss of generality, we may assume that $x(t) > 0$ and $x(t-\delta) > 0$ for all $t \geq t_0 > 0$. Define

$$\omega(t) = \frac{r(t)(x'(t))^\gamma}{x^\gamma(t)}, \quad t \in [c_1 - \delta, d_1]. \tag{10}$$

Then for all $t \neq \tau_k$, $t \geq t_0$, we have

$$\omega'(t) = -q(t)\frac{x^\gamma(t-\delta)}{x^\gamma(t)} - \sum_{i=1}^n q_i(t)|x(t-\delta)|^{\alpha_i-\gamma}\frac{x^\gamma(t-\delta)}{x^\gamma(t)} + \frac{e(t)}{x^\gamma(t)} - \frac{p(t)\omega(t)}{r(t)} - \frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}. \tag{11}$$

By taking $\eta_0 := 1 - \sum_{i=1}^n \eta_i$,

$$\zeta_0 = \eta_0^{-1}\left|\frac{e(t)x^\gamma(t-\delta)}{x^\gamma(t)}\right|x^{-\gamma}(t-\delta)$$
$$\zeta_i = \eta_i^{-1}q_i(t)\frac{x^\gamma(t-\delta)}{x^\gamma(t)}x^{\alpha_i-\gamma}(t-\delta), \quad i = 1, 2, \ldots, n$$

and using the the arithmetic–geometric mean inequality,

$$\sum_{i=0}^n \eta_i\zeta_i \geq \prod_{i=0}^n \zeta_i^{\eta_i}, \ \zeta_i \geq 0$$

we have

$$\sum_{i=1}^n q_i(t)\frac{x^{\alpha_i-\gamma}(t-\delta)}{x^\gamma(t)}x^\gamma(t-\delta) + \frac{|e(t)|}{x^\gamma(t)} \geq \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=0}^n \eta_i^{-\eta_i}q_i^{\eta_i}(t)\frac{x^{\eta_i(\alpha_i-\gamma)}(t-\delta)}{x^{\eta_i\gamma}(t)}x^{\eta_i\gamma}(t-\delta)$$
$$\times \frac{x^{\eta_0\gamma}(t-\delta)}{x^{\eta_0\gamma}(t)}x^{-\eta_0\gamma}(t-\delta). \tag{12}$$

Since

$$\prod_{i=0}^n \frac{x^{\eta_i\gamma}(t-\delta)}{x^{\eta_i\gamma}(t)} = \frac{x^{(\eta_0+\eta_1+\cdots+\eta_n)\gamma}(t-\delta)}{x^{(\eta_0+\eta_1+\cdots+\eta_n)\gamma}(t)} = \frac{x^\gamma(t-\delta)}{x^\gamma(t)}$$

and

$$\prod_{i=1}^n x^{(\alpha_i-\gamma)\eta_i}(t-\delta)x^{-\eta_0\gamma}(t-\delta) = 1,$$

from (12), (11) becomes

$$\begin{aligned}\omega'(t) \le &- q(t)\frac{x^{\gamma}(t-\delta)}{x^{\gamma}(t)} - \eta_0^{-\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}q_i^{\eta_i}(t)|e(t)|^{\eta_0} - \frac{p(t)\omega(t)}{r(t)} - \frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}\\ =&- Q(t)\frac{x^{\gamma}(t-\delta)}{x^{\gamma}(t)} - \frac{p(t)\omega(t)}{r(t)} - \frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}, \quad t \ne \tau_k.\end{aligned} \tag{13}$$

For $t = \tau_k$, $k = 1, 2, \ldots,$ we have

$$\omega(\tau_k^{+}) = \frac{b_k^{\gamma}}{a_k^{\gamma}}\omega(\tau_k). \tag{14}$$

Multiply both sides of (13) by $|u(t)|^{\gamma+1}$ where $u(t) \in E_{c_1,d_1}$ and integrating from c_1 to d_1, then using integration by parts on the left side, we have

$$\begin{aligned}&\sum_{k=I(c_1)+1}^{I(d_1)} |u(\tau_k)|^{\gamma+1}[\omega(\tau_k) - \omega(\tau_k^{+})]\\ &\le \int_{c_1}^{d_1}(\gamma+1)u^{\gamma}(t)u'(t)\omega(t)dt - \int_{c_1}^{d_1} Q(t)|u(t)|^{\gamma+1}\frac{x^{\gamma}(t-\delta)}{x^{\gamma}(t)}dt\\ &\quad - \int_{c_1}^{d_1}\frac{p(t)\omega(t)}{r(t)}|u(t)|^{\gamma+1}dt - \int_{c_1}^{d_1}\frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}|u(t)|^{\gamma+1}dt\\ &\le -\int_{c_1}^{\tau_{I(c_1)+1}} Q(t)|u(t)|^{\gamma+1}\frac{x^{\gamma}(t-\delta)}{x^{\gamma}(t)}dt - \sum_{k=I(c_1)+1}^{I(d_1-1)}\int_{\tau_k}^{\tau_{k+1}} Q(t)|u(t)|^{\gamma+1}\frac{x^{\gamma}(t-\delta)}{x^{\gamma}(t)}dt\\ &\quad - \int_{\tau_{I(d_1)}}^{d_1} Q(t)|u(t)|^{\gamma+1}\frac{x^{\gamma}(t-\delta)}{x^{\gamma}(t)}dt + \int_{c_1}^{d_1}\left[\left(\left|(\gamma+1)u'(t) - \frac{p(t)u(t)}{r(t)}\right|\right)|\omega(t)||u(t)|^{\gamma}\right.\\ &\quad \left.- \frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}|u(t)|^{\gamma+1}\right]dt.\end{aligned} \tag{15}$$

Using (7) with

$$A = \left(\left|(\gamma+1)u'(t) - \frac{p(t)u(t)}{r(t)}\right|\right), \quad B = \frac{\gamma}{(r(t))^{\frac{1}{\gamma}}}, \quad \text{and } y = |\omega(t)||u(t)|^{\gamma}$$

we have

$$\begin{aligned}&\left((\gamma+1)|u'(t)| - \frac{p(t)|u(t)|}{r(t)}\right)|\omega(t)||u(t)|^{\gamma} - \frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}|u(t)|^{\gamma+1}\\ &\quad \le \frac{r(t)}{(\gamma+1)^{\gamma+1}}\left((\gamma+1)|u'(t)| - \frac{p(t)|u(t)|}{r(t)}\right)^{\gamma+1}.\end{aligned} \tag{16}$$

Now for $t \in [c_1, d_1] \setminus \tau_k$, $k \in \mathbb{N}$ from (1) it is clear that

$$(r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} = e(t) - q(t)x^{\gamma}(t-\delta) - \sum_{i=1}^{n} q_i(t)|x(t-\delta)|^{\alpha_i-1}x(t-\delta) \le 0.$$

That is

$$((x'(t))^{\gamma})' + \Big(\frac{r'(t) + p(t)}{r(t)}\Big)(x'(t))^{\gamma} \le 0$$

which implies that

$$(x'(t))^{\gamma} \exp \int_{c_1}^{t} \frac{r'(s) + p(s)}{r(s)} ds$$

is non-increasing on $[c_1, d_1] \setminus \tau_k$.

Because there are different integration intervals in (15), we will estimate $x(t-\delta)/x(t)$ in each interval of t as follows. We first consider the situation where $I(c_1) \le I(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $\tau_{I(c_1)+1}, \tau_{I(c_2)+1}, \ldots \tau_{I(d_1)}$.

Case 1 For $t \in (\tau_k, \tau_{k+1}] \subset [c_1, d_1]$ we have the following two sub cases:

(a) If $\tau_k + \delta \le t \le \tau_{k+1}$, then $(t-\delta, t) \subset (\tau_k, \tau_{k+1}]$. Thus there is no impulse moment in $(t-\delta, t)$. For any $s \in (t-\delta, t)$, we have $x(s) > x(s) - x(\tau_k^+) = x'(\xi)(s - \tau_k)$, $\xi \in (\tau_k, s)$. Then

$$(x(s))^{\gamma} \ge (x'(\xi))^{\gamma}(s - \tau_k)^{\gamma}. \tag{17}$$

Since $(x'(s))^{\gamma} \exp \int_{c_1}^{s} \frac{r'(v)+p(v)}{r(v)} dv$ is non-increasing in $[c_1, t]$, we have

$$(x'(\xi))^{\gamma} \exp \int_{c_1}^{\xi} \frac{r'(v) + p(v)}{r(v)} dv \ge (x'(s))^{\gamma} \exp \int_{c_1}^{s} \frac{r'(v) + p(v)}{r(v)} dv. \tag{18}$$

From (17) and (18) we have

$$\begin{aligned}(x(s))^{\gamma} &\ge \frac{(x'(s))^{\gamma} \exp \int_{c_1}^{s} \frac{r'(v)+p(v)}{r(v)} dv}{\exp \int_{c_1}^{\xi} \frac{r'(v)+p(v)}{r(v)} dv}(s - \tau_k)^{\gamma} \\ &\ge (x'(s))^{\gamma}(s - \tau_k)^{\gamma}.\end{aligned} \tag{19}$$

Therefore $\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_k}$. Integrating both sides of the above inequality from $t-\delta$ to t, we obtain

$$\frac{x(t-\delta)}{x(t)} > \frac{t - \tau_k - \delta}{t - \tau_k} > 0.$$

(b) If $\tau_k < t < \tau_k + \delta$, then $\tau_k - \delta < t - \delta < \tau_k < t < \tau_k + \delta$. There is an impulsive moment τ_k in $(t-\delta, t)$. Similar to (a), we have $\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_k+\delta}$ for any $s \in (\tau_k - \delta, \tau_k]$. Upon integrating from $t-\delta$ to τ_k, we obtain

$$\frac{x(t-\delta)}{x(\tau_k)} > \frac{t - \tau_k}{\delta} \ge 0. \tag{20}$$

For any $t \in (\tau_k, \tau_k + \delta)$, we have

$$x(t) - x(\tau_k^+) < x'(t_k^+)(t - \tau_k), \ \xi \in (\tau_k, t).$$

Using the impulsive conditions in Eq. (1) we get

$$x(t) - a_k x(\tau_k) < b_k x'(\tau_k)(t-\tau_k)$$

$$\frac{x(t)}{x(\tau_k)} \le \frac{b_k x'(\tau_k)}{x(\tau_k)}(t-\tau_k) + a_k.$$

Using $\frac{x'(\tau_k)}{x(\tau_k)} < \frac{1}{\delta}$, we obtain

$$\frac{x(t)}{x(\tau_k)} < a_k + \frac{b_k}{\delta}(t-\tau_k).$$

That is

$$\frac{x(\tau_k)}{x(t)} > \frac{\delta}{a_k\delta + b_k(t-\tau_k)}. \tag{21}$$

From (20) and (21), we have

$$\frac{x(t-\delta)}{x(t)} > \frac{t-\tau_k}{a_k\delta + b_k(t-\tau_k)} \ge 0.$$

Case 2 For $t \in [c_1, \tau_{I(c_1)+1})$ we have the following three sub-cases:

(a) If $c_1 < t < \tau_{I(c_1)} + \delta$ and $\tau_{I(c_1)} > c_1 - \delta$, then $t - \delta \in [c_1 - \delta, \tau_{I(c_1)})$ and there is an impulsive moment $\tau_{I(c_1)}$ in $(t-\delta, t)$. Similar to Case 1(b), we have

$$\frac{x(t-\delta)}{x(t)} > \frac{t-\tau_{I(c_1)}}{a_{I(c_1)}\delta + b_{I(c_1)}(t-\tau_{I(c_1)})} \ge 0.$$

(b) If $\tau_{I(c_1)} + \tau < t < \tau_{I(c_1)+1}$ and $\tau_{I(c_1)} > c_1 - \delta$, then there are no impulsive moments in $(t-\delta, t)$. Making a similar analysis of Case 1(a), we obtain $\frac{x(t-\delta)}{x(t)} > \frac{t-\delta-\tau_{I(c_1)}}{t-\tau_{I(c_1)}} \ge 0$.

(c) If $\tau_{I(c_1)} > c_1 - \delta$, then there are no impulsive moments in $(t-\delta, t)$. So

$$\frac{x(t-\delta)}{x(t)} > \frac{t-\delta-\tau_{I(c_1)}}{t-\tau_{I(c_1)}} \ge 0.$$

Case 3 For $t \in (\tau_{I(d_1)}, d_1]$, there are three sub-cases:

(a) If $\tau_{I(d_1)} + \delta < d_1,\ t \in [\tau_{I(d_1)}, \tau_{I(d_1)} + \delta)$, then there is an impulsive moment $\tau_{I(d_1)}$. Similar to Case 2(a), we have

$$\frac{x(t-\delta)}{x(t)} > \frac{t-\tau_{I(d_1)}}{a_{I(d_1)}\delta + b_{I(d_1)}(t-\tau_{I(d_1)})} \ge 0.$$

(b) If $\tau_{I(d_1)} + \delta < t < d_1$ then there are no impulsive moments in $(t-\delta, t)$. Making a similar analysis of Case 2(b), we obtain

$$\frac{x(t-\delta)}{x(t)} > \frac{t-\delta-\tau_{I(d_1)}}{t-\tau_{I(d_1)}} \ge 0.$$

(c) If $\tau_{I(d_1)} + \delta \geq d_1$, then there is an impulsive moment $\tau_{I(d_1)}$ in $(t-\delta, t)$.

Similar to Case 3(a), we obtain

$$\frac{x(t-\delta)}{x(t)} > \frac{t-\tau_{I(d_1)}}{a_{I(d_1)}\delta + b_{I(d_1)}(t-\tau_{I(d_1)})} \geq 0.$$

Combining all these cases, we have

$$\frac{x^\gamma(t-\delta)}{x^\gamma(t)} > \begin{cases} Q^1_{I(c_1)}(t) & \text{for} \quad t \in [c_1, \tau_{I(c_1)+1}], \\ Q^1_k(t) & \text{for} \quad t \in (\tau_k, \tau_{k+1}],\ k = I(c_1)+1, \ldots, I(d_1)-1, \\ Q^1_{I(d_1)}(t) & \text{for} \quad t \in (\tau_{I(d_1)+1}, d_1] \end{cases} \tag{22}$$

Using (16) and (22) in (15) we get

$$\begin{aligned} &\sum_{k=I(c_1)+1}^{I(d_1)} |u(\tau_k)|^{\gamma+1}[\omega(\tau_k) - \omega(\tau_k^+)] \\ &\leq \int_{c_1}^{d_1} \frac{r(t)}{(\gamma+1)^{\gamma+1}}\left((\gamma+1)|u'(t)| - \frac{p(t)|u(t)|}{r(t)}\right)^{\gamma+1} dt - \int_{c_1}^{\tau_{I(c_1)+1}} Q(t)|u(t)|^{\gamma+1} Q^1_{I(c_1)}(t)dt \\ &\quad - \sum_{k=I(c_1)+1}^{I(d_1-1)} \int_{\tau_k}^{\tau_{k+1}} Q(t)|u(t)|^{\gamma+1} Q^1_k(t)dt - \int_{\tau_{I(d_1)}}^{d_1} Q(t)|u(t)|^{\gamma+1} Q^1_{I(d_1)}(t)dt. \end{aligned} \tag{23}$$

For any $t \in (c_1, \tau_{I(c_1)+1}]$, we have $x(t) - x(c_1) = x'(\xi)(t-c_1)$, $\xi \in (c_1, t)$. Since $x(c_1) > 0$, we have $x(t) > x'(\xi)(t-c_1)$. Then

$$(x(t))^\gamma > (x'(\xi))^\gamma (t-c_1)^\gamma. \tag{24}$$

Using the monotonicity of $(x'(t))^\gamma \exp\left(\int_{c_1}^t \frac{r'(s)+p(s)}{r(s)} ds\right)$, and (24) we have

$$\begin{aligned} (x(t))^\gamma &\geq \frac{(x'(t))^\gamma \exp\left(\int_{c_1}^t \frac{r'(s)+p(s)}{r(s)} ds\right)}{\exp\left(\int_{c_1}^\xi \frac{r'(s)+p(s)}{r(s)} ds\right)}(t-c_1)^\gamma \\ &\geq (x'(t))^\gamma (t-c_1)^\gamma \end{aligned}$$

for some $\xi \in (c_1, t)$. It follows

$$\frac{(x'(t))^\gamma}{(x(t))^\gamma} \leq \frac{1}{(t-c_1)^\gamma}.$$

Letting $t \to \tau_{I(c_1)+1}$, from (9), we have

$$\omega(\tau_{I(c_1)+1}) \leq \frac{r_1}{(\tau_{I(c_1)+1} - c_1)^\gamma}. \tag{25}$$

Making a similar analysis on $(\tau_{k-1}, \tau_k]$, $k = I(c_1)+2, \ldots, I(d_1)$, we can prove that

$$\omega(\tau_k) \leq \frac{r_1}{(\tau_k - \tau_{k-1})^\gamma}. \tag{26}$$

From (24), (25) and (A3), we obtain

$$\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} \frac{a_k^\gamma - b_k^\gamma}{a_k^\gamma} |u(\tau_k)|^{\gamma+1} \omega(\tau_k) \\
&\geq \frac{a_{I(c_1)+1}^\gamma - b_{I(c_1)+1}^\gamma}{a_{I(c_1)+1}^\gamma (\tau_{I(c_1)+1} - c_1)^\gamma} |u(\tau_{I(c_1)+1})|^{\gamma+1} r_1 + \sum_{k=I(c_1)+2}^{I(d_1)} \frac{a_k^\gamma - b_k^\gamma}{a_k^\gamma (\tau_k - \tau_{k-1})^\gamma} |u(\tau_k)|^{\gamma+1} r_1 \\
&= r_1 \Omega_{c_1}^{d_1} [|u(t)|^{\gamma+1}].
\end{aligned} \tag{27}$$

Since

$$\sum_{k=I(c_1)+2}^{I(d_1)} |u(\tau_k)|^{\gamma+1} [\omega(\tau_k) - \omega(\tau_k^+)] = \sum_{k=I(c_1)+1}^{I(d_1)} \frac{a_k^\gamma - b_k^\gamma}{a_k^\gamma} |u(\tau_k)|^{\gamma+1} \omega(\tau_k),$$

from (23) we have

$$\begin{aligned}
&\int_{c_1}^{d_1} \frac{r(t)}{(\gamma+1)^{\gamma+1}} \left((\gamma+1)|u'(t)| - \frac{p(t)|u(t)|}{r(t)} \right)^{\gamma+1} dt - \int_{c_1}^{\tau_{I(c_1)+1}} Q(t)|u(t)|^{\gamma+1} Q_{I(c_1)}^1(t) dt \\
&- \sum_{k=I(c_1)+2}^{I(d_1)-1} \int_{\tau_{k-1}}^{\tau_k} Q(t)|u(t)|^{\gamma+1} Q_k^1(t) dt - \int_{\tau_{I(d_1)}}^{d_1} Q(t)|u(t)|^{\gamma+1} Q_{I(d_1)}^1(t) dt > r_1 \Omega_{c_1}^{d_1} [|u(t)|^{\gamma+1}]
\end{aligned}$$

which contradicts (9).

If $I(c_1) = I(d_1)$, then $\Omega_{c_1}^{d_1}[|u(t)|^{\gamma+1}] = 0$ and there is no impulsive moments in $[c_1, d_1]$. Similar to the proof of (22), we obtain

$$\int_{c_1}^{d_1} \frac{r(t)}{(\gamma+1)^{\gamma+1}} \left((\gamma+1)|u'(t)| - \frac{p(t)|u(t)|}{r(t)} \right)^{\gamma+1} dt - \int_{c_1}^{\tau_{I(c_1)+1}} Q(t)|u(t)|^{\gamma+1} Q_{I(c_1)}^1(t) dt > 0.$$

It is again a contraction with our assumption. The proof when $x(t)$ is eventually negative is analogous by repeating a similar argument on the interval $[c_2, d_2]$. □

Following Kong (1999) and Philos (1989), we introduce a class of functions: $D = \{(t,s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(D, \mathbb{R})$. A pair of functions (H_1, H_2) is said to belong to a function class $\mathcal{H}$, if $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0, H_2(t,s) > 0$ for $t > s$ and there exist $h_1, h_2 \in L_{loc}(D, \mathbb{R})$ such that

$$\frac{\partial H_1(t,s)}{\partial t} = h_1(t,s) H_1(t,s), \qquad \frac{\partial H_2(t,s)}{\partial s} = -h_2(t,s) H_2(t,s). \tag{28}$$

For $\lambda \in (c_j, d_j)$, $j = 1, 2$,

$$\begin{aligned}
\Gamma_{1,j} = &\int_{c_j}^{\tau_{I(c_j)+1}} H_1(t,c_j) Q(t) Q_{I(c_j)}^1(t) dt + \sum_{k=I(c_j)+1}^{I(d_j)-1} \int_{\tau_k}^{\tau_{k+1}} H_1(t,c_j) Q(t) Q_k^1(t) dt \\
&+ \int_{\tau_{I(d_j)}}^{d_j} H_1(t,c_j) Q(t) Q_{I(d_j)}^1(t) dt \\
&- \frac{1}{(\gamma+1)^{\gamma+1}} \int_{c_j}^{\lambda_j} r(t) H_1(t,c_j) \left| h_1(t,c_j) - \frac{p(t)}{r(t)} \right|^{\gamma+1} dt,
\end{aligned}$$

and

$$\Gamma_{2,j} = \int_{\lambda_j}^{\tau_{I(\lambda_j)+1}} H_2(d_j,t)Q(t)Q^1_{I(\lambda_j)}(t)dt + \sum_{k=I(\lambda_j)+1}^{I(d_j)-1} \int_{\tau_k}^{\tau_{k+1}} H_2(d_j,t)Q(t)Q^1_k(t)dt$$
$$+ \int_{\tau_{I(d_j)}}^{d_j} H_2(d_j,t)Q(t)Q^1_{I(d_j)}(t)dt$$
$$- \frac{1}{(\gamma+1)^{\gamma+1}} \int_{\lambda_j}^{d_j} r(t)H_2(t,c_j)\left| h_1(d_j,t) - \frac{p(t)}{r(t)} \right|^{\gamma+1} dt.$$

Theorem 2 *Suppose that for any $T > 0$, there exist $c_j, d_j, j = 1,2, \lambda \notin \{\tau_k\}$ such that $c_1 < \lambda_1 < d_1 \leq c_2 < \lambda_2 < d_2$ and (8) holds. If there exists $(H_1, H_2) \in \mathcal{H}$ such that*

$$\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1} > \Lambda(H_1,H_2;c_j,d_j) \tag{29}$$

where

$$\Lambda(H_1,H_2;c_j,d_j) = -\left[\frac{r_j}{H_1(\lambda_j,c_j)} \Omega_{c_j}^{\lambda_j}[H_1(.,c_j)] + \frac{r_j}{H_2(d_j,\lambda_j)} \Omega_{\lambda_j}^{d_j}[H_2(d_j,.)] \right], \tag{30}$$

then every solution of Eq. (1) is oscillatory.

Proof Let $x(t)$ be a non-oscillatory solution of Eq. (1). Proceeding as in proof of Theorem 1, we get (13) and (14). Noticing whether or not there are impulsive moments in $[c_1,\lambda_1]$ and $[\lambda_1,d_1]$, we should consider the following four cases, namely: $I(c_1) < I(\lambda_1) < I(d_1)$; $I(c_1) = I(\lambda_1) < I(d_1)$; $I(c_1) < I(\lambda_1) = I(d_1)$ and $I(c_1) = I(\lambda_1) = I(d_1)$. Moreover, in the discussion of the impulse moments of $x(t-\delta)$, it is necessary to consider the following two cases: $\tau_{I(\lambda_j)+\delta} > \lambda_j$ and $\tau_{I(\lambda_j)+\delta} \leq \lambda_j$. Here we only consider the case $I(c_1) < I(\lambda_1) < I(d_1)$; with $\tau_{I(\lambda_j)+\delta} > \lambda_j$. For the other cases, similar conclusions can be obtained.

For this case there are impulsive moments $\tau_{I(c_1)}+1, \tau_{I(c_1)}+2, \ldots, \tau_{I(\lambda_1)}$ in $[c_1,d_1]$ and $\tau_{I(\lambda_1)+1}, \tau_{I(\lambda_1)+2}, \ldots, \tau_{I(d_1)}$ in $[\lambda_1,d_1]$. Multiplying both sides of (13) by $H_1(t,c_1)$ and integrating it from c_1 to λ_1, we have

$$\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x^\gamma(t-\delta)}{x^\gamma(t)}dt \leq -\int_{c_1}^{\lambda_1} H_1(t,c_1)\omega'(t)dt$$
$$- \int_{c_1}^{\lambda_1} \frac{p(t)\omega(t)}{r(t)} H_1(t,c_1)dt$$
$$- \int_{c_1}^{\lambda_1} \frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}} H_1(t,c_1)dt.$$

Applying integration by parts on first integral of R.H.S of last inequality, we get

$$
\begin{aligned}
&\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x^\gamma(t-\delta)}{x^\gamma(t)}dt \\
&\leq -\sum_{k=I(c_1)+1}^{I(d_1)} H_1(\tau_k,c_1)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)-H_1(\lambda_1,c_1)\omega(\lambda_1) \\
&\quad+\left(\int_{c_1}^{\tau_{I(c_1)+1}}+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{\tau_k}^{\tau_{k+1}}+\int_{\tau_{I(d_1)}}^{\lambda_1}\right)\left[h_1(t,c_1)\omega(t)\right. \\
&\quad\left.-\frac{p(t)}{r(t)}\omega(t)-\frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}\right]H_1(t,c_1)dt \\
&\leq -\sum_{k=I(c_1)+1}^{I(d_1)} H_1(\tau_k,c_1)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)-H_1(\lambda_1,c_1)\omega(\lambda_1) \\
&\quad+\left(\int_{c_1}^{\tau_{I(c_1)+1}}+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{\tau_k}^{\tau_{k+1}}+\int_{\tau_{I(d_1)}}^{\lambda_1}\right)\left[\left|h_1(t,c_1)\omega(t)\right.\right. \\
&\quad\left.\left.-\frac{p(t)}{r(t)}\right||\omega(t)|-\frac{\gamma|\omega(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}\right]H_1(t,c_1)dt.
\end{aligned}
$$

Using (7) with $A=\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|$, $B=\frac{\gamma}{r(t)^{\frac{1}{\gamma}}}$, $y=|\omega(t)|$ in the last inequality, we have

$$
\begin{aligned}
&\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x^\gamma(t-\delta)}{x^\gamma(t)}dt \\
&\leq -\sum_{k=I(c_1)+1}^{I(d_1)} H_1(\tau_k,c_1)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)-H_1(\lambda_1,c_1)\omega(\lambda_1) \\
&\quad+\frac{1}{(\gamma+1)^{\gamma+1}}\int_{c_1}^{\lambda_1} r(t)H_1(t,c_1)\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\gamma+1}dt.
\end{aligned}
\tag{31}
$$

Similar to the proof of Theorem 1, we need to divide the integration interval $[c_1,\lambda_1]$ into several subintervals for estimating the function $\frac{x(t-\delta)}{x(t)}$. Now,

$$
\begin{aligned}
&\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x^\gamma(t-\delta)}{x^\gamma(t)}dt \geq \int_{c_1}^{\tau_{I(c_1)+1}} H_1(t,c_1)Q(t)Q^1_{I(c_1)}(t)dt \\
&+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{\tau_k}^{\tau_{k+1}} H_1(t,c_1)Q(t)Q_k^1(t)dt+\int_{\tau_{I(d_1)}}^{d_1} H_1(t,c_1)Q(t)Q^1_{I(d_1)}(t)dt.
\end{aligned}
\tag{32}
$$

From (31) and (32),we obtain

$$\begin{aligned}&\int_{c_1}^{\tau_{I(c_1)+1}} H_1(t,c_1)Q(t)Q^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{\tau_k}^{\tau_{k+1}} H_1(t,c_1)Q(t)Q^1_k(t)dt\\&\quad+\int_{\tau_{I(d_1)}}^{d_1} H_1(t,c_1)Q(t)Q^1_{I(d_1)}(t)dt - \frac{1}{(\gamma+1)^{\gamma+1}}\int_{c_1}^{\lambda_1} r(t)H_1(t,c_1)\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\gamma+1}dt\\&\leq \sum_{k=I(c_1)+1}^{I_{(\lambda_1)}} H_1(\tau_k,c_1)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)-H_1(\lambda_1,c_1)\omega(\lambda_1).\end{aligned}\tag{33}$$

On the other hand multiplying both sides of (13) by $H_2(d_1,t)$ and then integrating from λ_1 to d_1 and using similar analysis to the above, we can obtain

$$\begin{aligned}&\int_{\lambda_1}^{\tau_{I(\lambda_1)+1}} H_2(d_1,t)Q(t)Q^1_{I(\lambda_1)}(t)dt + \sum_{k=I(\lambda_1)+1}^{I(d_1)-1}\int_{\tau_k}^{\tau_{k+1}} H_2(d_1,t)Q(t)Q^1_k(t)dt\\&\quad+\int_{\tau_{I(d_1)}}^{d_1} H_2(d_1,t)Q(t)Q^1_{I(d_1)}(t)dt - \frac{1}{(\gamma+1)^{\gamma+1}}\int_{\lambda_1}^{d_1} r(t)H_2(t,c_1)\left|h_1(d_1,t)-\frac{p(t)}{r(t)}\right|^{\gamma+1}dt\\&\leq -\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,\tau_k)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)-H_2(d_1,\lambda_1)\omega(\lambda_1).\end{aligned}\tag{34}$$

Dividing (33) and (34) by $H_1(\lambda_1,c_1)$ and $H_2(d_1,\lambda_1)$ respectively and adding them, we get

$$\begin{aligned}&\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1}\\&\quad\leq -\left(\frac{1}{H_1(\lambda_1,c_1)}\sum_{k=I(c_1)+1}^{I(d_1)} H_1(\tau_k,c_1)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)\right.\\&\qquad\left.+\frac{1}{H_2(d_1,\lambda_1)}\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,\tau_k)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)\right).\end{aligned}\tag{35}$$

Using the same method as in (27), we obtain

$$\begin{aligned}&-\sum_{k=I(c_1)+1}^{I(d_1)} H_1(\tau_k,c_1)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)\leq -r_1\Omega_{c_1}^{\lambda_1}[H_1(.,c_1)]\\&-\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,\tau_k)\left(\frac{a_k^\gamma-b_k^\gamma}{a_k^\gamma}\right)\omega(\tau_k)\leq -r_2\Omega_{\lambda_1}^{d_1}[H_2(d_1,.)].\end{aligned}\tag{36}$$

From (33) and (36), we obtain

$$\begin{aligned}\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1}&\leq -\left(r_1\Omega_{c_1}^{\lambda_1}[H_1(.,c_1)]+r_2\Omega_{\lambda_1}^{d_1}[H_2(d_1,.)]\right)\\&\leq \Lambda(H_1,H_2;c_j,d_j)\end{aligned}$$

which is a contradiction to the condition (29). When $x(t)<0$, we choose interval $[c_2,d_2]$ to study Eq. (1). The proof is similar and hence omitted. Now the proof is complete. □

Remark 2 When $p(t)=0$, Eq. (1) reduces to the equation studied by Guo et. al (2012b). Therefore our Theorem 1 provides an extension of Theorem 2.3 with $\rho(t)=1$ to damped impulsive differential equation.

Remark 3 When $\delta = 0$, that is, the delay disappears and our results reduces to that of Theorem 2.1 and Theorem 1 with $\rho(t) = 1$ in Pandian and Purushothaman (2012).

Remark 4 When $p(t) = 0$ and $\gamma = 1$ our Theorem 1 is a generalization of Theorem 2.2 in Li and Cheung (2013).

Remark 5 When the impulse is disappear, i.e., $a_k = b_k = 1$ for all $k = 1, 2, \ldots$, the delay term $\delta = 0$ and $p(t) = 0$ Eq. (1) reduces to the situation studied in Hassan et al. (2011). Therefore our Theorem 1 extends Theorem 2.1 of Hassan et al. (2011).

Example 1 Consider the following impulsive differential equation

$$\begin{cases} (((2+cost)x'(t)^{\frac{9}{5}}))' + (1+\sin t)(x'(t))^{\frac{9}{5}} + m_1(\cos t)|x(t-\frac{\pi}{8})|^{\frac{3}{2}}x(t-\frac{\pi}{8}) \\ \qquad +m_2(\cos t)|x(t-\frac{\pi}{8})|^{\frac{1}{2}}x(t-\frac{\pi}{8}) = \sin 2t,\ t \neq 2k\pi \pm \frac{\pi}{4}; \\ x(\tau_k^+) = \frac{1}{3}x(\tau_k), x^{'}(\tau_k^+) = \frac{2}{3}x'(\tau_k),\ \tau_k = 2k\pi \pm \frac{\pi}{4},\ k = 1, 2, \ldots \end{cases} \tag{37}$$

Here $r(t) = 2 + cost$, $p(t) = 1 + \sin t$, $q_1(t) = m_1 \cos t$, $q_2(t) = m_2\, cost$, $e(t) = \sin 2t$, $\gamma = \frac{9}{5}$, $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{3}{2}$ and m_1, m_2 are positive constants. Also $\delta = \frac{\pi}{8}$, $\tau_{k+1} - \tau_k = \pi/2 > \pi/8$. For any $T > 0$, we can choose k large enough such that $T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi$ and $c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}$, $k = 1, 2 \ldots$. Then there is an impulsive moment $\tau_k = 4k\pi - \frac{\pi}{4}$ in $[c_1, d_1]$ and an impulsive moment $\tau_{k+1} = 4k\pi + \frac{\pi}{4}$ in $[c_2, d_2]$. Now choose $\eta_0 = 1/5$, $\eta_1 = 3/5$, $\eta_2 = 1/5$, therefore

$$Q(t) = 5\frac{2^{\frac{1}{5}}}{3^{\frac{3}{5}}}(m_1)^{\frac{3}{5}}(m_2)^{\frac{1}{5}}|\cos t||\sin t|^{\frac{1}{5}}$$

If we take $u1(t) = u_2(t) = \sin 4t$, $\tau_{I(c_1)} = 4k\pi - \frac{7}{4}\pi$, $\tau_{I(d_1)} = 4k\pi - \frac{\pi}{4}$, then by a simple calculation, the left side of Eq. (9) is the following:

$$\begin{aligned} &\int_{c_1}^{d_1} \frac{r(t)}{(\gamma+1)^{\gamma+1}}\left((\gamma+1)|u'(t)| - \frac{p(t)|u(t)|}{r(t)}\right)^{\gamma+1} dt - \int_{c_1}^{\tau_{I(c_1)+1}} Q(t)|u(t)|^{\gamma+1}Q^1_{I(c_1)}(t)dt \\ &\quad - \sum_{k=I(c_1)+1}^{I(d_1-1)} \int_{\tau_k}^{\tau_{k+1}} Q(t)|u(t)|^{\gamma+1}Q^1_k(t)dt - \int_{\tau_{I(d_1)}}^{d_1} Q(t)|u(t)|^{\gamma+1}Q^1_{I(d_1)}(t)dt \\ &\geq \frac{1}{(\frac{14}{5})^{\frac{14}{5}}}\int_{4k\pi-\frac{\pi}{2}}^{4k\pi}(2+\cos t)\left(\frac{56}{5}|\cos 4t| - \frac{(1+\sin t)|\sin 4t|}{(2+\cos t)}\right)^{\frac{14}{5}} dt \\ &\quad - \int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}} Q(t)|\sin 4t|^{\frac{14}{5}}\left(\frac{t-\frac{\pi}{8}-4k\pi+\frac{7\pi}{4}}{t-4k\pi+\frac{7\pi}{4}}\right)^{\frac{9}{5}} dt \\ &\quad - \int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}} Q(t)|\sin 4t|^{\frac{14}{5}}\left(\frac{t-4k\pi+\frac{\pi}{4}}{a_{I(c_1)+1}(t+\frac{\pi}{8}-4k\pi+\frac{\pi}{4})}\right)^{\frac{9}{5}} dt \\ &\quad - \int_{4k\pi-\frac{\pi}{8}}^{4k\pi} Q(t)|\sin 4t|^{\frac{14}{5}}\left(\frac{t-\frac{\pi}{8}-4k\pi+\frac{\pi}{4}}{t-4k\pi+\frac{\pi}{4}}\right)^{\frac{9}{5}} dt \\ &\approx (m_1)^{\frac{3}{5}}(m_2)^{\frac{1}{5}}(1.5196) - 0.6739. \end{aligned}$$

Since $I(c_1) = k - 1$, $I(d_1) = k, r_1 = 3$, we have

$$r_1 \Omega_{c_1}^{d_1}[|u(t)|^{\gamma+1}] = 3|\sin 4(\tau_k)|^{\frac{14}{5}} \left(\frac{a_k^{\frac{9}{5}} - b_k^{\frac{9}{5}}}{a_k^{\frac{9}{5}}} \right)$$
$$= 0.$$

The condition (9) is satisfied in $[c_1, d_1]$ if

$$(m_1)^{\frac{3}{5}}(m_2)^{\frac{1}{5}}(1.5196) < 0.6739 \tag{38}$$

Similarly, we can show that for $t \in [c_2, d_2]$, the condition (9) is satisfied if

$$(m_1)^{\frac{3}{5}}(m_2)^{\frac{1}{5}}(0.7553) < 0.5233 \tag{39}$$

Since the condition (38) is weaker than (39) we can choose the constants m_1, m_2 small enough such that (39) holds. Hence by Theorem 1 every solution of Eq. (37) is oscillatory. In fact for $m_1 = 1/5$, $m_2 = 1/6$, every solution of Eq. (37) is oscillatory.

Remark 6 The result obtained in Guo et al. (2012a, b, 2014) and Erbe et al. (2010) cannot be applied to Example 1, since the results in Guo et al. (2012a) can be applicable only to equations having only one nonlinear term and the results in Guo et al. (2012b), Guo et al. (2014), Erbe et al. (2010) can be applied to equations without damping term. Therefore our results extent and compliment to Guo et al. (2012a, b, 2014), Hassan et al. (2011), Li and Cheung (2013), Pandian and Purushothaman (2012) and Erbe et al. (2010).

Conclusion

In this paper we have obtained interval oscillation criteria for Eq. (1) which extend and generalize some known results in Guo et al. (2012a), Li and Cheung (2013), Hassan et al. (2011) and Özbekler and Zafer (2011), Pandian and Purushothaman (2012).

Authors' contributions

All authors contributed equally to this paper. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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