Published in final edited form as:
Linear Algebra Appl. 2016 August 1; 502: 104–125. doi:10.1016/j.laa.2015.09.052.

On the construction of general cubature formula by flat extensions

Marta Abril Bucero[a], **Chandrajit Bajaj**[b], and **Bernard Mourrain**[a]

Marta Abril Bucero: marta.abril_bucero@inria.fr; Chandrajit Bajaj: bajaj@cs.utexas.edu; Bernard Mourrain: bernard.mourrain@inria.fr

[a]Inria Sophia Antipolis Méditerranée, 2004 route des Lucioles, BP 93, 06902 Sophia Antipolis, France

[b]Department of Computer Science, The Institute of Computational Engineering and Sciences, The University of Texas at Austin, 201 East 24th Street, POB 2.324A, 1 University Station, C0200, Austin, TX 78712-0027, USA

Abstract

We describe a new method to compute general cubature formulae. The problem is initially transformed into the computation of truncated Hankel operators with flat extensions. We then analyze the algebraic properties associated to flat extensions and show how to recover the cubature points and weights from the truncated Hankel operator. We next present an algorithm to test the flat extension property and to additionally compute the decomposition. To generate cubature formulae with a minimal number of points, we propose a new relaxation hierarchy of convex optimization problems minimizing the nuclear norm of the Hankel operators. For a suitably high order of convex relaxation, the minimizer of the optimization problem corresponds to a cubature formula. Furthermore cubature formulae with a minimal number of points are associated to faces of the convex sets. We illustrate our method on some examples, and for each we obtain a new minimal cubature formula.

Keywords

Cubature formula; Hankel matrix; Flat extension; Orthogonal polynomials; Border basis; Semidefinite programming

1. Cubature formula

1.1. Statement of the problem

Consider the integral for a continuous function *f*,

$$I[f]=\int_{\Omega} w(\mathbf{x})f(\mathbf{x})d\mathbf{x}$$

Correspondence to: Bernard Mourrain, bernard.mourrain@inria.fr.

where $\Omega \subset \mathbb{R}^n$ and w is a positive function on Ω.

We are looking for a cubature formula which has the form

$$\langle\sigma|f\rangle=\sum_{j=1}^{r} w_j f(\zeta_j) \quad (1)$$

where the points $\zeta_j \in \mathbb{C}^n$ and the weights $w_j \in \mathbb{R}$ are independent of the function f. They are chosen so that

$$\langle\sigma|f\rangle=I[f], \forall f \in V,$$

where V is a finite dimensional vector space of functions. Usually, the vector space V is the vector space of polynomials of degree ≤ d, because a well-behaved function f can be approximated by a polynomial, so that $Q[f]$ approximates the integral $I[f]$.

Given a cubature formula (1) for I, its algebraic degree is the largest degree d for which $I[f] = \langle\sigma|f\rangle$ for all f of degree ≤ d.

1.2. Related works

Prior approaches to the solution of cubature problem can be grouped into roughly two classes. One, where the goal is to estimate the fewest weighted, aka cubature points possible for satisfying a prescribed cubature rule of fixed degree [9,24,26,29,30,33]. The other class focusses on the determination and construction of cubature rules which would yield the fewest cubature points possible [7,34,38–41,44,45]. In [34], for example, Radon introduced a fundamental technique for constructing minimal cubature rules where the cubature points are common zeros of multivariate orthogonal polynomials. This fundamental technique has since been extended by many, including e.g. [33,41,45] where notably, the paper [45] uses multivariate ideal theory, while [33] uses operator dilation theory. In this paper, we propose another approach to the second class of cubature solutions, namely, constructing a suitable finite dimensional Hankel matrix and extracting the cubature points using sub-operators of the Hankel matrix [18]. This approach is related to [21–23], which in turn are based on the methods of multivariate truncated moment matrices, their positivity and extension properties [11–13].

Applications of such algorithms determining cubature rules and cubature points over general domains occur in isogeometric modeling and finite element analysis using generalized Barycentric finite elements [17,1,35,36]. Additional applications abound in numerical integration for low dimensional (6–100 dimensions) convolution integrals that appear naturally in computational molecular biology [3,2], as well in truly high dimensional (tens of thousands of dimensions) integrals that occur in finance [32,8].

1.3. Reformulation

Let $R = \mathbb{R}[\mathbf{x}]$ be the ring of polynomials in the variables $\mathbf{x} = (x_1,\dots, x_n)$ with coefficients in $\mathbb{R}$. Let R_d be the set of polynomials of degree $\le d$. The set of linear forms on R, that is, the set of linear maps from R to $\mathbb{R}$ is denoted by R^*. The value of a linear form $\Lambda \in R^*$ on a polynomial $p \in R$ is denoted by $\langle \Lambda|p\rangle$. The set R^* can be identified with the ring of formal power series in new variables $\mathbf{y} = (y_1,\dots, y_n)$:

$$\begin{array}{lll} R^* & & \to \mathbb{R}[[\mathbf{y}]] \\ \Lambda & \mapsto \Lambda(\mathbf{y}) = & \sum_{\alpha\in\mathbb{N}^n} \langle \Lambda|\mathbf{x}^\alpha\rangle \mathbf{y}^\alpha. \end{array}$$

The coefficients $\langle \Lambda|\mathbf{x}^\alpha\rangle$ of these series are called the *moments* of Λ. The evaluation at a point $\zeta \in \mathbb{R}^n$ is an element of R, denoted by $\mathbf{e}_\zeta$, and defined by $\mathbf{e}_\zeta : f \in R \mapsto f(\zeta) \in \mathbb{R}$. For any $p \in R$ and any $\Lambda \in R^*$, let $p \star \Lambda : q \in R \mapsto \Lambda(pq)$.

Cubature problem—Let $V \subset R$ be a vector space of polynomials and consider the linear form $\bar{I} \in V^*$ defined by

$$\begin{array}{rl} \bar{I}:V & \to \mathbb{R} \\ \mathbf{v} & \mapsto I[\mathbf{v}] \end{array}$$

Computing a cubature formula for I on V then consists in finding a linear form

$$\sigma = \sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i} : f \mapsto \sum_{j=1}^{r} w_j f(\zeta_j),$$

which coincides on V with $\bar{I}$. In other words, given the linear form $\bar{I}$ on R_d, we wish to find a linear form $\sigma = \sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i}$ which extends $\bar{I}$.

2. Cubature formulae and Hankel operators

To find such a linear form $\sigma \in R^*$, we exploit the properties of its associated bilinear form $H_\sigma : (p, q) \in R \times R \to \langle \sigma|pq\rangle$, or equivalently, the associated Hankel operator:

$$\begin{array}{rl} H_\sigma : R & \to R^* \\ p & \mapsto p \star \sigma \end{array}$$

The kernel of H_σ is $\ker H_\sigma = \{p \in R \mid \forall q \in R, \langle \sigma|pq\rangle = 0\}$. It is an ideal of R. Let $\mathcal{A}_\sigma = R/\ker H_\sigma$ be the associated quotient ring.

The matrix of the bilinear form or the Hankel operator H_σ associated to σ in the monomial basis, and its dual are $(\langle \Lambda|\mathbf{x}^{\alpha+\beta}\rangle)_{\alpha,\beta\in\mathbb{N}^n}$. If we restrict them to a space V spanned by the monomial basis $(\mathbf{x}^\alpha)_{\alpha\in A}$ for some finite set $A \subset \mathbb{N}^n$, we obtain a finite dimensional matrix

$[H_\sigma^{A,A}]=(\langle\Lambda|\mathbf{x}^{\alpha+\beta}\rangle)_{\alpha,\beta\in A}$, and which is a Hankel matrix. More generally, for any vector spaces $V, V' \subset R$, we define the truncated bilinear form and Hankel operators: $H_\sigma^{V,V'}:(v,v') \in V \times V' \mapsto \langle\sigma|pq\rangle \in \mathbb{R}$ and $H_\sigma^{V,V'}:v \in V \mapsto v\star\sigma \in V'^*$. If V (resp. V') is spanned by a monomial set $\mathbf{x}^A$ for $A \subset \mathbb{N}^n$ (resp. $\mathbf{x}^B$ for $B \subset \mathbb{N}^n$), the truncated bilinear form and truncated Hankel operator are also denoted by $H_\sigma^{A,B}$. The associated Hankel matrix in the monomial basis is then $[H_\sigma^{A,B}]=(\langle\Lambda|\mathbf{x}^{\alpha+\beta}\rangle)_{\alpha\in A,\beta\in B}$.

The main property that we will use to characterize a cubature formula is the following (see [22,20]):

Proposition 2.1— *A linear form $\sigma \in R^*$ can be decomposed as $\sigma=\sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i}$ with $w_i \in \mathbb{C} \setminus \{0\}$, $\zeta_i \in \mathbb{C}^n$ iff*

- *$H_\sigma : p \mapsto p \star \sigma$ is of rank r,*
- *ker H_σ is the ideal of polynomials vanishing at the points $\{\zeta_1,\ldots,\zeta_r\}$.*

This shows that in order to find the points ζ_i of a cubature formula, it is sufficient to compute the polynomials $p \in R$ such that $\forall q \in R$, $\langle\sigma|pq\rangle = 0$, and to determine their common zeroes. In Section 4 we describe a direct way to recover the points ζ_i, and the weights ω_i from suboperators of H_σ.

In the case of cubature formulae with real points and positive weights, we already have the following stronger result (see [22,20]):

Proposition 2.2— *Let $\sigma \in R^*$.*

$$\sigma=\sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i}$$

with $w_i > 0$, $\zeta_i \in \mathbb{R}^n$ iff rank$H_\sigma = r$ *and* $H_\sigma \succcurlyeq 0$.

A linear form $\sigma=\sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i}$ with $w_i > 0$, $\zeta_i \in \mathbb{R}^n$ is called a r-atomic measure since it coincides with the weighted sum of the r Dirac measures at the points ζ_i.

Therefore, the problem of constructing a cubature formula σ for I exact on $V \subset R$ can be reformulated as follows: Construct a linear form $\sigma \in R^*$ such that

- rank $H_\sigma = r < \infty$ and $H_\sigma \succcurlyeq 0$.
- $v \in V$, $I[v] = \langle\sigma|v\rangle$.

The rank r of H_σ is given by the number of points of the cubature formula, which is expected to be small or even minimal.

The following result states that a cubature formula with dim(V) points, always exists.

Theorem 2.3. *(See* [42,4].)—*If a sequence* $(\sigma_\alpha)_{\alpha\in\mathbb{N}^n,|\alpha|\le t}$ *is the truncated moment sequence of a measure* μ *(i.e.* $\sigma_\alpha = \int \mathbf{x}^\alpha d\mu$ *for* $|\alpha| \le t$*), then it can also be represented by an r-atomic measure: for* $|\alpha| \le t$, $\sigma_\alpha=\sum_{i=1}^r w_i\xi_i^\alpha$ *where* $r \le s_t$, $w_i > 0$, $\zeta_i \in supp(\mu)$.

This result can be generalized to any set of linearly independent polynomials $v_1,\ldots, v_r \in R$ (see the proof in [4] or Theorem 5.9 in [22]). We deduce that the cubature problem always has a solution with dim(V) or less points.

Definition 2.4—Let $r_c(I)$ be the maximum rank of the bilinear form $H_I^{W,W'}:(w,w') \in W \times W' \mapsto I[ww']$ where $W, W' \subset V$ are such that $\forall w \in W, \forall w' \in W'$, $ww' \in V$. It is called the Catalecticant rank of I.

Proposition 2.5—*Any cubature formula for I exact on V involves at least* $r_c(I)$ *points*.

Proof: Suppose that σ is a cubature formula for I exact on V with r points. Let $W, W' \subset V$ be vector spaces such that $\forall w \in W, \forall w' \in W'$, $ww' \in V$. Since $H_I^{W,W'}$ coincides with $H_\sigma^{W,W'}$, which is the restriction of the bilinear form H_σ to $W \times W'$, we deduce that $r=\text{rank}\, H_\sigma \ge \text{rank}\, H_\sigma^{W,W'}=\text{rank}(H_I^{W,W'})$. Thus $r \ge r_c(I)$.

Corollary 2.6—*Let* $W \subset V$ *such that* $\forall w, w' \in W$, $ww' \in V$. *Then any cubature formula of I exact on V involves at least* dim(W) *points*.

Proof: As we have $\forall p \in W$, $p^2 \in V$ so that $I(p^2) = 0$ implies $p = 0$. Therefore the quadratic form $H_I^{W,W}:(p,q) \in W \times W \to I[pq]$ is positive definite of rank dim(W). By Proposition 2.5, a cubature formula of I exact on V involves at least $r_c(I) \ge$ dim(W) points.

In particular, if $V = R_d$ any cubature formula of I exact on V involves at least $\dim R_{\lfloor \frac{d}{2} \rfloor} = \binom{\lfloor \frac{d}{2} \rfloor + n}{n}$ points.

In [25], this lower bound is improved for cubature problems in two variables.

3. Flat extensions

In order to reduce the extension problem to a finite-dimensional problem, we consider hereafter only truncated Hankel operators. Given two subspaces W, W' of R and a linear form σ defined on $W \cdot W'$ (i.e. $\sigma \in \langle W \cdot W' \rangle^*$), we define

$$\begin{aligned} H_\sigma^{W,W'}: W \times W' & \to \mathbb{R} \\ (w,w') & \mapsto \langle \sigma | ww' \rangle. \end{aligned}$$

If $\mathbf{w}$ (resp. $\mathbf{w}'$) is a basis of W (resp. W'), then we will also denote $H_\sigma^{\mathbf{w},\mathbf{w}'} := H_\sigma^{W,W'}$. The matrix of $H_\sigma^{\mathbf{w},\mathbf{w}'}$ in the basis $\mathbf{w} = \{w_1,\dots, w_s\}$, $\mathbf{w}' = \{w'_1, \dots, w'_{s'}\}$ is $[\langle\sigma|w_i w_j\rangle]_{1\le i\le s, 1\le j\le s'}$.

Definition 3.1—Let $W \subset V$, $W' \subset V'$ be subvector spaces of R and $\sigma \in \langle V \cdot V'\rangle^*$. We say that $H_\sigma^{V,V'}$ is a *flat extension* of $H_\sigma^{W,W'}$ if $\operatorname{rank} H_\sigma^{V,V'} = \operatorname{rank} H_\sigma^{W,W'}$.

A set B of monomials of R is *connected to* 1 if it contains 1 and if for any $m \neq 1 \in B$, there exist $1 \le i \le n$ and $m' \in B$ such that $m = x_i m'$.

As a quotient $R/\ker H_\sigma$ has always a monomial basis connected to 1, so in the first step we take for $\mathbf{w}$, $\mathbf{w}'$, monomial sets that are connected to 1.

For a set B of monomials in R, let us define $B^+ = B \cup x_1 B \cup \cdots \cup x_n B$ and $\partial B = B^+ \setminus B$.

The next theorem gives a characterization of flat extensions for Hankel operators defined on monomial sets connected to 1. It is a generalized form of the Curto–Fialkow theorem [13].

Theorem 3.2. (See [23,6,5].)—*Let $B \subset C$, $B' \subset C'$ be sets of monomials connected to* 1 *such that $|B| = |B'| = r$ and $C \cdot C'$ contains $B^+ \cdot B'^+$. If $\sigma \in \langle C \cdot C'\rangle^*$ is such that $H_\sigma^{B,B'} = \operatorname{rank} H_\sigma^{C,C'} = r$, then $H_\sigma^{C,C'}$ has a unique flat extension $H_{\tilde\sigma}$ for some $\tilde\sigma \in R^*$. Moreover, we have $\ker H_{\tilde\sigma} = (\ker H_\sigma^{C,C'})$ and $R = \langle B\rangle \oplus \ker H_{\tilde\sigma} = \langle B'\rangle \oplus \ker H_{\tilde\sigma}$. In the case where $B' = B$, if $H_\sigma^{B,B} \succcurlyeq 0$, then $H_{\tilde\sigma} \succcurlyeq 0$.*

Based on this theorem, in order to find a flat extension of $H_\sigma^{B,B'}$, it suffices to construct an extension $H_\sigma^{B^+,B'^+}$ of the same rank r.

Corollary 3.3—*Let $V \subset R$ be a finite dimensional vector space. If there exists a set B of monomials connected to* 1 *such that $V \subset \langle B^+ \cdot B^+\rangle$ and $\sigma \in \langle B^+ \cdot B^+\rangle^*$ such that $\forall v \in V$, $\langle\sigma|v\rangle = I[v]$ and $\operatorname{rank} H_\sigma^{B,B} = \operatorname{rank} H_\sigma^{B^+,B^+} = |B| = r$, then there exist $w_i > 0$, $\zeta_i \in \mathbb{R}^n$, $i = 1,\dots, r$ such that $\forall v \in V$,*

$$I[v] = \sum_{i=1}^{r} w_i\, v(\xi_i).$$

This characterization leads to equations which are at most of degree 2 in a set of variables related to unknown moments and relation coefficients as described by the following proposition:

Proposition 3.4—*Let B and B' be two sets of monomials of R of size r, connected to* 1 *and σ be a linear form on $\langle B'^+ \cdot B^+\rangle$. Then, $H_\sigma^{B^+,B'^+}$ admits a flat extension $H_{\tilde\sigma}$ such that $H_{\tilde\sigma}$ is of rank r and B (resp. B') a basis of $R/\ker H_{\tilde\sigma}$ iff*

$$[H_\sigma^{B^+,B'^+}]=\begin{pmatrix}\mathbb{Q} & \mathbb{M}' \\ \mathbb{M}^t & \mathbb{N}\end{pmatrix}, \quad (2)$$

with $\mathbb{Q}=[H_\sigma^{B,B'}]$, $\mathbb{M}'=[H_\sigma^{B,\partial B'}]$, $\mathbb{M}^t=[H_\sigma^{\partial B,B'}]$, $\mathbb{N}=[H_\sigma^{\partial B,\partial B'}]$ *such that* $\mathbb{Q}$ *is invertible and*

$$\mathbb{M}=\mathbb{Q}^t\mathbb{P},\ \mathbb{M}'=\mathbb{Q}\mathbb{P}',\ \mathbb{N}=\mathbb{P}^t\mathbb{Q}\mathbb{P}', \quad (3)$$

for some matrices $\mathbb{P}\in\mathbb{C}^{B\times\partial B'}$, $\mathbb{P}'\in\mathbb{C}^{B'\times\partial B}$.

Proof: If we have $\mathbb{M}=\mathbb{Q}^t\mathbb{P}$, $\mathbb{M}=\mathbb{Q}\mathbb{P}'$, $\mathbb{N}=\mathbb{P}^t\mathbb{Q}\mathbb{P}'$, then

$$[H_\sigma^{B^+,B'^+}]=\begin{pmatrix}\mathbb{Q} & \mathbb{Q}\mathbb{P}' \\ \mathbb{P}^t\mathbb{Q} & \mathbb{P}^t\mathbb{Q}\mathbb{P}'\end{pmatrix}$$

has clearly the same rank as $\mathbb{Q}=[H_\sigma^{B,B'}]$. According to Theorem 3.2, $H_\sigma^{B^+,B'^+}$ admits a flat extension $H_{\tilde\sigma}$ with $\tilde\sigma\in R^*$ such that B and B' are bases of $\mathscr{A}_{\tilde\sigma}=R/\ker H_{\tilde\sigma}$.

Conversely, if $H_{\tilde\sigma}$ is a flat extension of $H_\sigma^{B^+,B'^+}$ with B and B' bases of $\mathscr{A}_{\tilde\sigma}=R/\ker H_{\tilde\sigma}$, then $[H_{\tilde\sigma}^{B,B'}]=[H_\sigma^{B,B'}]=\mathbb{Q}$ is invertible and of size $r=|B|=|B'|$. As $H_{\sigma\sigma}$ is of rank r, $H_\sigma^{B^+,B'^+}$ is also of rank r. Thus, there exists $\mathbb{P}'\in\mathbb{C}^{B'\times\partial B}$ ($\mathbb{P}'=\mathbb{Q}^{-1}\mathbb{M}$) such that $\mathbb{M}=\mathbb{Q}\mathbb{P}'$. Similarly, there exists $\mathbb{P}\in\mathbb{C}^{B\times\partial B'}$ such that $\mathbb{M}=\mathbb{Q}^t\mathbb{P}$. Thus, the kernel of $[H_\sigma^{B^+,B'^+}]$ (resp. $[H_\sigma^{B'^+,B^+}]=[H_\sigma^{B^+,B'^+}]^t$) is the image of $\begin{pmatrix}-\mathbb{P}' \\ \mathbb{I}\end{pmatrix}$ (resp. $\begin{pmatrix}-\mathbb{P} \\ \mathbb{I}\end{pmatrix}$). We deduce that $\mathbb{N}=\mathbb{M}^t\mathbb{P}'=\mathbb{P}^t\mathbb{Q}\mathbb{P}'$.

Remark 3.5—A basis of the kernel of $H_\sigma^{B^+,B'^+}$ is given by the columns of $\begin{pmatrix}-\mathbb{P}' \\ \mathbb{I}\end{pmatrix}$, which represent polynomials of the form

$$p_\alpha=\mathbf{x}^\alpha-\sum_{\beta\in B}p_{\alpha,\beta}\mathbf{x}^\beta$$

for $\alpha\in\partial B$. These polynomials are border relations which project the monomials $\mathbf{x}^\alpha$ of ∂B on the vector space spanned by the monomials B, modulo $\ker H_\sigma^{B^+,B'^+}$. It is proved in [6] that they form a border basis of the ideal ker $H_{\tilde{\sigma}}$ when $H_\sigma^{B^+,B'^+}$ is a flat extension and $H_\sigma^{B,B'}$ is invertible.

Remark 3.6—Let $A \subset \mathbb{N}^n$ be a set of monomials such that $\langle \sigma | \mathbf{x}^\alpha \rangle = I[\mathbf{x}^\alpha]$. Considering the entries of $\mathbb{P}$, $\mathbb{P}'$ and the entries σ_α of $\mathbb{Q}$ with $\alpha \notin A$ as variables, the constraints (3) are multilinear equations in these variables of total degree at most 3 if $\mathbb{Q}$ contains unknown entries and 2 otherwise.

Example 3.7—We consider here $V = R_{2k}$ for $k > 0$. By Proposition 3.4, any cubature formula for I exact on V has at least $r_k := \dim R_k$ points. Let us take B to be all the monomials of degree $\leq k$ so that B^+ is the set of monomials of degree $\leq k + 1$. If a cubature formula for I is exact on R_{2k} and has r_k points, then $H_\sigma^{B^+,B^+}$ is a flat extension of $H_\sigma^{B,B}$ of rank r_k. Consider a decomposition of $H_\sigma^{B^+,B^+}$ as in (2). By Proposition 3.4, we have the relations

$$\mathbb{M} = \mathbb{Q}\mathbb{P}, \ \mathbb{N} = \mathbb{P}^t \mathbb{Q}\mathbb{P}, \qquad (4)$$

where

- $\mathbb{Q} = (I[\mathbf{x}^{\beta+\beta'}])_{\beta,\beta' \in B}$,
- $\mathbb{M} = (\langle \sigma | \mathbf{x}^{\beta+\beta'} \rangle)_{\beta \in B, \beta' \in \partial B}$ with $\langle \sigma | \mathbf{x}^{\beta+\beta'} \rangle = I[\mathbf{x}^{\beta+\beta'}]$ when $|\beta + \beta'| \leq 2k$,
- $\mathbb{N} = (\langle \sigma | \mathbf{x}^{\beta+\beta'} \rangle)_{\beta,\beta' \in \partial B}$,
- $\mathbb{P} = (p_{\beta,\alpha})_{\beta \in B, \alpha \in \partial B}$.

The equations (4) are quadratic in the variables $\mathbb{P}$ and linear in the variables in $\mathbb{M}$ Solving these equations yields a flat extension $H_\sigma^{B^+,B^+}$ of $H_\sigma^{B,B}$. As $H_\sigma^{B,B} \succcurlyeq 0$, any real solution of this system of equations corresponds to a cubature for I on exact R_{2k} of the form $\sigma = \sum_{i=1}^{r_k} w_i \mathbf{e}_{\zeta_i}$ with $w_i > 0$, $\zeta_i \in \mathbb{R}^n$.

We illustrate the approach with $R = \mathbb{R}[x_1, x_2]$, $V = R_4$, $B = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, $B^+ = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$. Let

$$\sigma = 8 + 17y_2 - 4y_1 + 15y_2^2 + 14y_1y_2 - 16y_1^2 + 47y_2^3 - 6y_1y_2^2 + 34y_1^2y_2 - 52y_1^3 + 51y_2^4 + 38y_1y_2^3 - 18y_1^2y_2^2 + 86y_1^3y_2 - 160y_1^4$$

be the series truncated in degree 4, corresponding to the first moments (not necessarily given by an integral).

$$H_\sigma^{B^+,B^+}=\begin{bmatrix} 8 & -4 & 17 & -16 & 14 & 15 & -52 & 34 & -6 & 47 \\ -4 & -16 & 14 & -52 & 34 & -6 & -160 & 86 & -18 & 38 \\ 17 & 14 & 15 & 34 & -6 & 47 & 86 & -18 & 38 & 51 \\ -16 & -52 & 34 & -160 & 86 & -18 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 14 & 34 & -6 & 86 & -18 & 38 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \\ 15 & -6 & 47 & -18 & 38 & 51 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ -52 & -160 & 86 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} \\ 34 & 86 & -18 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} \\ -6 & -18 & 38 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} \\ 47 & 38 & 51 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \end{bmatrix}$$

where $\sigma_1 = \sigma_{5,0}$, $\sigma_2 = \sigma_{4,1}$, $\sigma_2 = \sigma_{3,2}$, $\sigma_4 = \sigma_{2,3}$, $\sigma_5 = \sigma_{1,4}$, $\sigma_6 = \sigma_{0,5}$, $\sigma_7 = \sigma_{6,0}$, $\sigma_8 = \sigma_{5,1}$, $\sigma_9 = \sigma_{4,2}$, $\sigma_{10} = \sigma_{3,3}$, $\sigma_{11} = \sigma_{2,4}$, $\sigma_{12} = \sigma_{1,5}$, $\sigma_{13} = \sigma_{0,6}$.

The first 6 × 6 diagonal block $H_\sigma^{B,B}$ is invertible. To have a flat extension $H_\sigma^{B^+,B^+}$, we impose the condition that the sixteen 7 × 7 minors of $H_\sigma^{B^+,B^+}$, which contains the first 6 rows and columns, must vanish. This yields the following system of quadratic equations:

$$\begin{cases} -814\,592\sigma_1^2-1\,351\,680\sigma_1\sigma_2-476\,864\sigma_1\sigma_3-599\,040\sigma_2^2-301\,440\sigma_2\sigma_3-35\,072\sigma_3^2 \\ -520\,892\,032\sigma_1-396\,821\,760\sigma_2-164\,529\,152\sigma_3+1\,693\,440\sigma_7-86\,394\,672\,128=0 \\ -814\,592\sigma_2^2-1\,351\,680\sigma_2\sigma_3-476\,864\sigma_2\sigma_4-599\,040\sigma_3^2-301\,440\sigma_3\sigma_4-35\,072\sigma_4^2 \\ +335\,275\,392\sigma_2+257\,276\,160\sigma_3+96\,277\,632\sigma_4+1\,693\,440\sigma_9-34\,904\,464\,128=0 \\ -814\,592\sigma_3^2-1\,351\,680\sigma_3\sigma_4-476\,864\sigma_3\sigma_5-599\,040\sigma_4^2-301\,440\sigma_4\sigma_5-35\,072\sigma_5^2 \\ +13\,226\,880\sigma_3+13\,282\,560\sigma_4-8\,227\,200\sigma_5+1\,693\,440\sigma_{11}+31\,714\,560=0 \\ -814\,592\sigma_4^2-1\,351\,680\sigma_4\sigma_5-476\,864\sigma_4\sigma_6-599\,040\sigma_5^2-301\,440\sigma_5\sigma_6-35\,072\sigma_6^2 \\ +212\,860\,736\sigma_4+162\,698\,880\sigma_5+51\,427\,456\sigma_6+1\,693\,440\sigma_{13}-13\,356\,881\,792=0 \\ -814\,592\sigma_1\sigma_2-675\,840\sigma_1\sigma_3-238\,432\sigma_1\sigma_4-675\,840\sigma_2^2-837\,472\sigma_2\sigma_3 \\ -150\,720\sigma_2\sigma_4-150\,720\sigma_3^2-35\,072\sigma_3\sigma_4+167\,637\,696\sigma_1-131\,807\,936\sigma_2 \\ -150\,272\,064\sigma_3-82\,264\,576\sigma_4+1\,693\,440\sigma_8+54\,990\,043\,008=0 \\ -814\,592\sigma_2\sigma_3-675\,840\sigma_2\sigma_4-238\,432\sigma_2\sigma_5-675\,840\sigma_3^2-837\,472\sigma_3\sigma_4-150\,720\sigma_3\sigma_5 \\ -150\,720\sigma_4^2-35\,072\sigma_4\sigma_5+6\,613\,440\sigma_2+174\,278\,976\sigma_3+124\,524\,480\sigma_4 \\ +48\,138\,816\sigma_5+1\,693\,440\sigma_{10}-746\,438\,400=0 \\ -814\,592\sigma_3\sigma_4-675\,840\sigma_3\sigma_5-238\,432\sigma_3\sigma_6-675\,840\sigma_4^2-837\,472\sigma_4\sigma_5-150\,720\sigma_4\sigma_6 \\ -150\,720\sigma_5^2-35\,072\sigma_5\sigma_6+106\,430\,368\sigma_3+87\,962\,880\sigma_4+32\,355\,008\sigma_5 \\ -4\,113\,600\sigma_6+1\,693\,440\sigma_{12}-183\,201\,600=0 \\ -814\,592\sigma_2\sigma_4-675\,840\sigma_2\sigma_5-238\,432\sigma_2\sigma_6-675\,840\sigma_3\sigma_4-599\,040\sigma_3\sigma_5 \\ -150\,720\sigma_3\sigma_6-238\,432\sigma_4^2-150\,720\sigma_4\sigma_5-35\,072\sigma_4\sigma_6+106\,430\,368\sigma_2 \\ +81\,349\,440\sigma_3+193\,351\,424\sigma_4+128\,638\,080\sigma_5+48\,138\,816\sigma_6 \\ +1\,693\,440\sigma_{11}-21\,721\,986\,624=0 \\ -814\,592\sigma_1\sigma_3-675\,840\sigma_1\sigma_4-238\,432\sigma_1\sigma_5-675\,840\sigma_2\sigma_3-599\,040\sigma_2\sigma_4 \\ -150\,720\sigma_2\sigma_5-238\,432\sigma_3^2-150\,720\sigma_3\sigma_4-35\,072\sigma_3\sigma_5+6\,613\,440\sigma_1+6\,641\,280\sigma_2 \\ -264\,559\,616\sigma_3-198\,410\,880\sigma_4-82\,264\,576\sigma_5+1\,693\,440\sigma_9+1\,312\,368\,000=0 \\ -814\,592\sigma_1\sigma_4-675\,840\sigma_1\sigma_5-238\,432\sigma_1\sigma_6-675\,840\sigma_2\sigma_4-599\,040\sigma_2\sigma_5 \\ -150\,720\sigma_2\sigma_6-238\,432\sigma_3\sigma_4-150\,720\sigma_3\sigma_5-35\,072\sigma_3\sigma_6+106\,430\,368\sigma_1 \\ +81\,349\,440\sigma_2+25\,713\,728\sigma_3-260\,446\,016\sigma_4-198\,410\,880\sigma_5-82\,264\,576\sigma_6 \\ +1\,693\,440\sigma_{10}+34\,550\,702\,464=0 \end{cases}$$

The set of solutions of this system is an algebraic variety of dimension 3 and degree 52. A solution is $\sigma_1 = -484$, $\sigma_2 = 226$, $\sigma_3 = -54$, $\sigma_4 = 82$, $\sigma_5 = -6$, $\sigma_6 = 167$, $\sigma_7 = -1456$, $\sigma_8 = 614$, $\sigma_9 = -162$, $\sigma_{10} = 182$, $\sigma_{11} = -18$, $\sigma_{12} = 134$, $\sigma_{13} = 195$.

3.1. Computing an orthogonal basis of $\mathscr{A}_\sigma$

In this section, we describe a new method to construct a basis B of $\mathscr{A}_\sigma$ and to detect flat extensions, from the knowledge of the moments σ_α of $\sigma(\mathbf{y})$. We are going to inductively construct a family P of polynomials, orthogonal for the inner product

$$(p,q)\mapsto \langle p,q\rangle_\sigma := \langle \sigma|pq\rangle,$$

and a monomial set B connected to 1 such that $\langle B\rangle = \langle P\rangle$.

We start with $B = \{1\}$, $P = \{1\} \subset R$. As $\langle 1, 1\rangle_\sigma = \langle \sigma \mid 1\rangle \neq 0$, the family P is orthogonal for σ and $\langle B\rangle = \langle P\rangle$.

We now describe the induction step. Assume that we have a set $B = \{m_1,\dots, m_s\}$ and $P = \{p_1,\dots, p_s\}$ such that

- $\langle B\rangle = \langle P\rangle$;
- $\langle p_i, p_j\rangle_\sigma \neq 0$ if $i = j$ and 0 otherwise.

To construct the next orthogonal polynomials, we consider the monomials in $\partial B = \{m_1^{'},\dots,m_l^{'}\}$ and project them on $\langle P\rangle$:

$$p_i^{'} = m_i^{'} - \sum_{j=1}^{s} \frac{\langle m_i^{'}, p_j\rangle_\sigma}{\langle p_j, p_j\rangle_\sigma} p_j,\ i=1,\dots,l.$$

By construction, $\langle p_i^{'}, p_j\rangle_\sigma = 0$ and $\langle p_1,\dots,p_s,p_i^{'}\rangle = \langle m_1,\dots,m_s,m_i^{'}\rangle$. We extend B by choosing a subset of monomials $B^{'} = \{m_{i_1}^{'},\dots,m_{i_k}^{'}\}$ such that the matrix

$$[\langle p_{i_j}^{'}, p_{i_{j'}}^{'}\rangle_\sigma]_{1\le j,j'\le k}$$

is invertible. The family P is then extended by adding an orthogonal family of polynomials $\{p_{s+1},\dots, p_{s+k}\}$ constructed from $\{p_{i_1}^{'},\dots,p_{i_k}^{'}\}$. If all the polynomials $p_i^{'}$ are such that $\langle p_i^{'}, p_j^{'}\rangle_\sigma = 0$, the process stops.

This leads to the following algorithm:

Algorithm 1—*Input:* *the coefficients* σ_α *of a series* $\sigma \in \mathbb{C}[[\mathbf{y}]]$ *for* $\alpha \in A \subset \mathbb{N}^n$ *connected to* 1 *with* $\sigma_0 \neq 0$.

- *Let* $B := \{1\}$; $P = \{1\}$; $r := 1$; $E = \langle \mathbf{y}^{\alpha} \rangle_{\alpha \in A}$;
- *While* $s > 0$ *and* $B^+ \cdot B^+ \subset E$ *do*
  - *Compute* $\partial B=\{m_1^{'},\ldots,m_l^{'}\}$ *and* $p_i^{'}=m_i^{'}-\sum_{j=1}^{s}\frac{\langle m_i^{'},p_j\rangle_\sigma}{\langle p_j,p_j\rangle_\sigma}p_j$;
  - *Compute a (maximal) subset* $B^{'}=\{m_{i_1}^{'},\ldots,m_{i_k}^{'}\}$ *of* ∂B *such that* $\left[\langle p_{i_j}^{'},p_{i_{j'}}^{'}\rangle_\sigma\right]_{1\le j,j'\le k}$ *is invertible;*
  - *Compute an orthogonal family of polynomials* $\{p_{s+1},\ldots,p_{s+k}\}$ *from* $\{p_{i_1}^{'},\ldots,p_{i_k}^{'}\}$;
  - $B := B \cup B'$, $P := P \cup \{p_{s+1},\ldots,p_{s+k}\}$; $r += k$;
- *If* $B^+ \cdot B^+ \not\subset \subset E$ *then return failed.*

Output: *failed or success with*

- *a set of monomials* $B = \{m_1,\ldots,m_r\}$ *connected to* 1*, and non-degenerate for* $\langle \cdot, \cdot \rangle_\sigma$;
- *a set of polynomials* $P = \{p_1,\ldots,p_r\}$ *orthogonal for* σ *and such that* $\langle B \rangle = \langle P \rangle$;
- *the relations* $\rho_i:=m_i^{'}-\sum_{j=1}^{s}\frac{\langle m_i^{'},p_j\rangle_\sigma}{\langle p_j,p_j\rangle_\sigma}p_j$ *for the monomials* $m_i^{'}$ *in* $\partial B=\{m_1^{'},\ldots,m_l^{'}\}$.

The above algorithm is a Gramm–Schmidt-type orthogonalization method, where, at each step, new monomials are taken in ∂B and projected onto the space spanned by the previous monomial set B. Notice that if the polynomials p_i are of degree at most $d' < d$, then only the moments of σ of degree $\le 2d' + 1$ are involved in this computation.

Proposition 3.8—*If Algorithm 1 outputs with success a set* $B = \{m_1,\ldots,m_r\}$ *and the relations* $\rho_i:=m_i^{'}-\sum_{j=1}^{s}\frac{\langle m_i^{'},p_j\rangle_\sigma}{\langle p_j,p_j\rangle_\sigma}p_j$, *for* $m_i^{'}$ *in* $\partial B=\{m_1^{'},\ldots,m_l^{'}\}$, *then* σ *coincides on* $\langle B^+ \cdot B^+ \rangle$ *with the series* $\tilde{\sigma}$ *such that*

- rank $H_{\tilde{\sigma}} = r$;
- *B and P are bases of* $\mathscr{A}_{\tilde{\sigma}}$ *for the inner product* $\langle \cdot, \cdot \rangle_{\tilde{\sigma}}$;
- *The ideal* $I_{\tilde{\sigma}} = \ker H_{\tilde{\sigma}}$ *is generated by* $(\rho_i)_{i=1,\ldots,l}$;
- *The matrix of multiplication by* x_k *in the basis P of* $\mathscr{A}_{\tilde{\sigma}}$ *is*

$$M_k=\left(\frac{\langle\sigma|x_k p_i p_j\rangle}{\langle\sigma|p_j^2\rangle}\right)_{1\le i,j\le r}.$$

Proof: By construction, B is connected to 1. A basis B' of $\langle B^+\rangle$ is formed by the elements of B and the polynomials ρ_i, $i = 1,\ldots, l$. Since Algorithm 1 stops with success, we have $\forall i, j \in [1, l]$, $\forall b \in \langle B\rangle$, $\langle \rho_i, b\rangle_\sigma = \langle \rho_i, \rho_j\rangle_\sigma = 0$ and $\rho_1, \ldots, \rho_l \in \ker H_\sigma^{B^+,B^+}$. As $\langle B^+\rangle = \langle B\rangle \oplus \langle \rho_1,\ldots, \rho_l\rangle$, $\mathrm{rank}\, H_\sigma^{B^+,B^+} = \mathrm{rank}\, H_\sigma^{B,B}$ and $H_\sigma^{B^+,B^+}$ is a flat extension of $H_\sigma^{B,B}$. By construction, P is an orthogonal basis of $\langle B\rangle$ and the matrix of $H_\sigma^{B,B}$ in this basis is diagonal with non-zero entries on the diagonal. Thus $H_\sigma^{B,B}$ is of rank r.

By Theorem 3.2, σ coincides on $\langle B^+ \cdot B^+\rangle$ with a series $\tilde{\sigma} \in R^*$ such that B is a basis of $\mathscr{A}_{\tilde{\sigma}} = R/I_{\tilde{\sigma}}$ and $I_{\tilde{\sigma}} = (\ker H_{\tilde{\sigma}}^{B^+,B^+}) = (\rho_1, \ldots, \rho_l)$.

As $\langle B^+\rangle = \langle B\rangle \oplus \langle \rho_1,\ldots, \rho_l\rangle = \langle P\rangle \oplus \langle \rho_1,\ldots, \rho_l\rangle$ and P is an orthogonal basis of $\mathscr{A}_{\tilde{\sigma}}$, which is orthogonal to $\langle \rho_1,\ldots, \rho_l\rangle$, we have

$$x_k p_i = \sum_{j=1}^{r} \frac{\langle \sigma | x_k p_i p_j\rangle}{\langle \sigma | p_j^2\rangle} p_j + \rho$$

with $\rho \in \langle \rho_1,\ldots, \rho_l\rangle$. This shows that the matrix of the multiplication by x_k modulo $I_{\tilde{\sigma}} = (\rho_1, \ldots, \rho_l)$, in the basis $P = \{p_1,\ldots, p_r\}$ is $M_k = \left(\frac{\langle \sigma | x_k p_i p_j\rangle}{\langle \sigma | p_j^2\rangle}\right)_{1 \le i,j \le r}$.

Remark 3.9—It can be shown that the polynomials $(\rho_i)_{i=1,\cdot,l}$ are a border basis of $I_{\tilde{\sigma}}$ for the basis B [23,6,27,28].

Remark 3.10—If $H_\sigma^{B,B} \succcurlyeq 0$, then by Proposition 2.2, the common roots $\zeta_1,\ldots, \zeta_r$ of the polynomials $\rho_1,\ldots, \rho_l$ are simple and real $\in \mathbb{R}^n$. They are the cubature points:

$$\overline{\sigma} = \sum_{i=1}^{r} w_j \mathbf{e}_{\zeta_j},$$

with $w_j > 0$.

4. The cubature formula from the moment matrix

We now describe how to recover the cubature formula, from the moment matrix $H_\sigma^{B^+,B'^+}$. We assume that the flat extension condition is satisfied:

$$\mathrm{rank}\, H_\sigma^{B^+,B'^+} = H_\sigma^{B,B'} = |B| = |B'|. \quad (5)$$

Theorem 4.1— *Let B and B' be monomial subsets of R of size r connected to 1 and $\sigma \in \langle B^+ \cdot B'^+ \rangle^*$. Suppose that $H_\sigma^{B^+,B'^+} = \text{rank} H_\sigma^{B,B'} = |B| = |B'|$. Let $M_i = [H_\sigma^{B,B'}]^{-1}[H_\sigma^{x_i,B,B'}]$ and $(M_i')^t = [H_\sigma^{B,x_iB'}][H_\sigma^{B,B'}]^{-1}$. Then,*

1. *B and B' are bases of $\mathscr{A}_{\tilde{\sigma}} = R/\ker H_{\tilde{\sigma}}$,*
2. *M_i (resp. M_i') is the matrix of multiplication by x_i in the basis B (resp. B') of $\mathscr{A}_{\tilde{\sigma}}$,*

Proof: By the flat extension Theorem 3.2, there exists $\tilde{\sigma} \in R^*$ such that $H_{\tilde{\sigma}}$ is a flat extension of $H_\sigma^{B^+,B'^+}$ of rank $r = |B| = |B'|$ and $\ker H_{\tilde{\sigma}} = (\ker H_\sigma^{B^+,B'^+})$. As $R = \langle B \rangle \oplus \ker H_{\tilde{\sigma}}$ and rank $H_{\tilde{\sigma}} = r$, $\mathscr{A}_{\tilde{\sigma}} = R/\ker H_{\tilde{\sigma}}$ is of dimension r and generated by B. Thus B is a basis of $\mathscr{A}_{\tilde{\sigma}}$. A similar argument shows that B' is also a basis of $\mathscr{A}_{\tilde{\sigma}}$. We denote by $\pi : \mathscr{A}_{\tilde{\sigma}} \to \langle B \rangle$ and $\pi' : \mathscr{A}_{\tilde{\sigma}} \to \langle B' \rangle$ the isomorphisms associated to these bases representations.

The matrix $[H_\sigma^{B,B'}]$ is the matrix of the Hankel operator

$$\begin{aligned} \overline{H}_{\tilde{\sigma}} : \mathscr{A}_{\tilde{\sigma}} &\to \mathscr{A}_{\tilde{\sigma}}^* \\ a &\mapsto a \star \overline{\sigma} \end{aligned}$$

in the basis B and the dual basis of B'. Similarly, $[H_\sigma^{x_iB,B'}]$ is the matrix of

$$\begin{aligned} \overline{H}_{x_i \star \tilde{\sigma}} : \mathscr{A}_{\tilde{\sigma}} &\to \mathscr{A}_{\tilde{\sigma}}^* \\ a &\mapsto a \star x_i \star \overline{\sigma} \end{aligned}$$

in the same bases. As $x_i \star \bar{\sigma} = \bar{\sigma} \circ M_i$ where $M_i : \mathscr{A}_{\tilde{\sigma}} \to \mathscr{A}_{\tilde{\sigma}}$ is the multiplication by x_i in $\mathscr{A}_{\tilde{\sigma}}$ we deduce that $\bar{H}_{x_i \star \tilde{\sigma}} = \bar{H}_{\tilde{\sigma}} \circ M_i$ and $[H_\sigma^{B,B'}]^{-1}[H_\sigma^{x_iB,B'}]$ is the matrix of multiplication by x_i in the basis B of $\mathscr{A}_{\tilde{\sigma}}$. By exchanging the role of B and B' and by transposition ($[H_\sigma^{B,B'}]^t = [H_\sigma^{B',B}]$), we obtain that $[H_\sigma^{B,x_iB'}][H_\sigma^{B,B'}]^{-1}$ is the transpose of the matrix of multiplication by x_i in the basis B' of $\mathscr{A}_{\tilde{\sigma}}$.

Theorem 4.2— *Let B be a monomial subset of R of size r connected to 1 and $\sigma \in \langle B^+ \cdot B^+ \rangle^*$. Suppose that $\text{rank}\, H_\sigma^{B^+,B^+} = \text{rank}\, H_\sigma^{B,B} = |B| = r$ and that $H_\sigma^{B,B} \succcurlyeq 0$. Let $M_i = [H_\sigma^{B,B'}]^{-1}[H_\sigma^{x_i,B,B'}]$. Then σ can be decomposed as*

$$\sigma = \sum_{j=1}^{r} w_j \mathbf{e}_{\zeta_j}$$

with $w_j > 0$ *and* $\zeta_j \in \mathbb{R}^n$ *such that* M_i *have r common linearly independent eigenvectors* $\mathbf{u}_j$, $j = 1, \dots, r$ *and*

- $\zeta_{j,i}=\frac{\langle\sigma|x_i\mathbf{u}_j\rangle}{\langle\sigma|\mathbf{u}_j\rangle}$ *for* $1 \le i \le n$, $1 \le j \le r$;
- $w_j=\frac{\langle\sigma|\mathbf{u}_j\rangle}{\mathbf{u}_j(\zeta_{j,1},\dots,\zeta_{j,n})}$.

Proof: By Theorem 4.1, the matrix M_i is the matrix of multiplication by x_i in the basis B of $\mathscr{A}_{\tilde{\sigma}}$. As $H_{\sigma}^{B,B} \succcurlyeq 0$, the flat extension Theorem 3.2 implies that $H_{\bar{\sigma}} \succcurlyeq 0$ and that

$$\overline{\sigma}=\sum_{j=1}^{r} w_j \mathbf{e}_{\zeta_j}$$

where $w_j > 0$ and $\zeta_j \in \mathbb{R}^n$ are the simple roots of the ideal ker $H_{\bar{\sigma}}$. Thus the commuting operators M_i are diagonalizable in a common basis of eigenvectors $\mathbf{u}_i$, $i = 1, \dots, r$, which are scalar multiples of the interpolation polynomials at the roots $\zeta_1, \dots, \zeta_r$: $\mathbf{u}_i(\zeta_i) = \lambda_i \neq 0$ and $\mathbf{u}_i(\zeta_j) = 0$ if $j \neq i$ (see [15, Chap. 4] or [10]). We deduce that

$$\langle\overline{\sigma}|\mathbf{u}_j\rangle=\sum_{k=1}^{r} w_k\mathbf{u}_j(\zeta_k)=w_j\lambda_j \text{ and } \langle\overline{\sigma}|x_i\mathbf{u}_j\rangle=\zeta_{j,i}w_j\lambda_j,$$

so that $\zeta_{j,i}=\frac{\langle\sigma|x_i\mathbf{u}_j\rangle}{\langle\sigma|\mathbf{u}_j\rangle}$. As $\mathbf{u}_i(\zeta_i) = \lambda_i$, we have $w_j=\frac{\langle\sigma|\mathbf{u}_j\rangle}{\mathbf{u}_j(\zeta_{j,1},\dots,\zeta_{j,n})}$.

Algorithm 2—*Input:* *B is a set of monomials connected to* 1, $\sigma \in \langle B^+ \cdot B^+\rangle^*$ *such that* $H_{\sigma}^{B^+,B^+}$ *is a flat extension of* $H_{\sigma}^{B,B}$ *of rank* $|B|$.

- *Compute an orthogonal basis* $\{p_1, \dots, p_r\}$ *of B for* σ;
- *Compute the matrices* $M_k=\left(\frac{\langle\sigma|x_k p_i p_j\rangle}{\langle\sigma|p_j^2\rangle}\right)_{1\le i,j\le r}$;
- *Compute their common eigenvectors* $\mathbf{u}_1, \dots, \mathbf{u}_r$.

Output*: For* $j = 1, \dots, r$,

- $\zeta_j=\left(\frac{\langle\sigma|x_1\mathbf{u}_j\rangle}{\langle\sigma|\mathbf{u}_j\rangle},\dots,\frac{\langle\sigma|x_{in}\mathbf{u}_j\rangle}{\langle\sigma|\mathbf{u}_j\rangle}\right)$;
- $w_j=\frac{\langle\sigma|\mathbf{u}_j\rangle}{\mathbf{u}_j(\zeta_{j,1},\dots,\zeta_{j,n})}$.

Remark 4.3—Since the matrices M_k commute and are diagonalizable with the same basis, their common eigenvectors can be obtained by computing the eigenvectors of a generic linear combination $l_1M_1 + \dots + l_nM_n$, $l_i \in \mathbb{R}$.

5. Cubature formula by convex optimization

As described in the previous section, the computation of cubature formulae reduces to a low rank Hankel matrix completion problem, using the flat extension property. In this section, we describe a new approach which relaxes this problem into a convex optimization problem.

Let $V \subset R$ be a vector space spanned by monomials $\mathbf{x}^{\alpha}$ for $\alpha \in A \subset \mathbb{N}^n$. Our aim is to construct a cubature formula for an integral function I exact on V. Let $\mathbf{i} = (I[\mathbf{x}^{\alpha}])_{\alpha \in A}$ be the sequence of moments given by the integral I. We also denote $\mathbf{i} \in V^*$ the associated linear form such that $\forall v \in V$, $\langle \mathbf{i} \mid v \rangle = I[v]$.

For $k \in \mathbb{N}$, we denote by

$$\mathscr{H}^k(\mathbf{i}) = \{H_\sigma | \sigma \in R_{2k}^*, \sigma_\alpha = \mathbf{i}_\alpha \text{ for } \alpha \in A, H_\sigma \succcurlyeq 0\},$$

the set of semi-definite Hankel operators on R_t is associated to moment sequences which extend $\mathbf{i}$. We can easily check that $\mathscr{H}^k(\mathbf{i})$ is a convex set. We denote by $\mathscr{H}_r^k(\mathbf{i})$ the set of elements of $\mathscr{H}^k(\mathbf{i})$ of rank $\leq r$.

A subset of $\mathscr{H}_r^k(\mathbf{i})$ is the set of Hankel operators associated to cubature formulae of r points:

$$\mathscr{E}_r^k(\mathbf{i}) = \left\{ H_\sigma \in \mathscr{H}^k(\mathbf{i}) | \sigma = \sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i}, \omega_i > 0, \zeta_i \in \mathbb{R}^n \right\}.$$

We can check that $\mathscr{E}^k(\mathbf{i}) = \cup_{r \in \mathbb{N}} \mathscr{E}_r^k(\mathbf{i})$ is also a convex set.

To impose the cubature points to be in a semialgebraic set $\mathscr{S}$ defined by equality and inequalities $\mathscr{S} = \{\mathbf{x} \in \mathbb{R}^n | g_1^0(\mathbf{x}) = 0, \ldots, g_{n_1}^0(\mathbf{x}) = 0, g_1^+(\mathbf{x}) \geq 0, \ldots, g_{n_2}^+(\mathbf{x}) \geq 0\}$, one can refine the space of $\mathscr{H}^k(\mathbf{i})$ by imposing that σ is positive on the quadratic module (resp. preordering) associated to the constraints $\mathbf{g} = \{g_1^0, \ldots, g_n^0; g_1^+, \ldots, g_n^+\}$ [19]. For the sake of simplicity, we don't analyze this case here, which can be done in a similar way.

The Hankel operator $H_\sigma \in \mathscr{E}_r^k(\mathbf{i})$ associated to a cubature formula of r points is an element of $\mathscr{H}_r^k(\mathbf{i})$. In order to find a cubature formula of minimal rank, we would like to compute a minimizer solution of the following optimization problem:

$$\min_{H \in \mathscr{H}^k(\mathbf{i})} \operatorname{rank}(H)$$

However this problem is NP-hard [16]. We therefore relax it into the minimization of the nuclear norm of the Hankel operators, i.e. the minimization of the sum of the singular values of the Hankel matrix [37]. More precisely, for a generic matrix $P \in \mathbb{R}^{s_t \times s_t}$, we consider the following minimization problem:

$$\min_{H \in \mathscr{H}^k(\mathbf{i})} \operatorname{trace}(P^t H P) \qquad (6)$$

Let $(A, B) \in \mathbb{R}^{s_k \times s_k} \times \mathbb{R}^{s_k \times s_k} \to \langle A, B \rangle = \operatorname{trace}(AB)$ denote the inner product induced by the trace on the space of $s_k \times s_k$ matrices. The optimization problem (6) requires minimizing the linear form $H \to \operatorname{trace}(HPP^t) = \langle H, PP^t \rangle$ on the convex set $\mathscr{H}^k(\mathbf{i})$. As the trace of P^tHP is bounded by below by 0 when $H \succcurlyeq 0$, our optimization problem (6) has a non-negative minimum ≥ 0.

Problem (6) is a Semi-Definite Program (SDP), which can be solved efficiently by interior point methods. See [31]. SDP is an important ingredient of relaxation techniques in polynomial optimization. See [19,22].

Let $\sum^k = \left\{ p = \sum_{i=1}^{l} p_i^2 | p_i \in R_k \right\}$ be the set of polynomials of degree ≤ $2k$ which are sums of squares, let $\mathbf{x}^{(k)}$ be the vector of all monomials in $\mathbf{x}$ of degree ≤ k and let $q(\mathbf{x}) = (\mathbf{x}^{(k)})^t PP^t \mathbf{x}^{(k)} \in \Sigma^k$. Let $p_i(\mathbf{x})$ $(1 \le i \le s_k)$ denote the polynomial $\langle P_i, \mathbf{x}^{(k)} \rangle$ associated to the column P_i of P. We have $q(\mathbf{x}) = \sum_{i=1}^{s_k} p_i(\mathbf{x})^2$ and for any $\sigma \in R_{2k}$,

$$\operatorname{trace}(P^t H_\sigma P) = \langle H_\sigma | PP^t \rangle = \langle \sigma | q(\mathbf{x}) \rangle.$$

For any $l \in \mathbb{N}$, we denote by π_l: $R_l \to R_l$ the linear map which associates to a polynomial $p \in R_l$ its homogeneous component of degree l. We say that P is a proper matrix if $\pi_{2k}(q(\mathbf{x})) \neq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

We are thus looking for cubature formulae with a small number of points, which correspond to Hankel operators with small rank. The next result describes the structure of truncated Hankel operators, when the degree of truncation is high enough, compared to the rank.

Theorem 5.1—*Let $\sigma \in R_{2k}^*$ and let H_σ be its truncated Hankel operator on R_k. If $H_\sigma \succcurlyeq 0$ and H_σ is of rank $r \le k$, then*

$$\sigma \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i} + \sum_{i=r'+1}^{r} w_i \mathbf{e}_{\zeta_i} \circ \pi_{2k}$$

with $\omega_i > 0$ and $\zeta_i \in \mathbb{R}^n$ distinct for $i = 1, \ldots, r$.

Proof: The substitution τ_0: $S_{[2k]} \to R_{2k}$ which replaces x_0 by 1 is an isomorphism of $\mathbb{K}$ vector spaces. Let $\tau_0^*: R_{2k}^* \to S_{[2k]}^*$ be the pull-back map on the dual $(\tau_0^*(\sigma) = \sigma \circ \tau_0)$. Let $\overline{\sigma} = \sigma \circ \tau_0 = \tau_0^*(\sigma) \in S_{[2k]}^*$ be the linear form induced by σ on $S_{[2k]}$ and let $H_{\overline{\sigma}}: S_{[k]} \to S_{[k]}^*$ be the corresponding truncated operator on $S_{[k]}$. The kernel $\bar{K}$ of $H_{\bar{\sigma}}$ is the vector space spanned by the homogenization in x_0 of the elements of the kernel K of H_σ.

Let $\geqslant$ be the lexicographic ordering such that $x_0 \geqslant \cdots \geqslant x_n$. By [14, Theorem 15.20, p. 351], after a generic change of coordinates, the initial J of the homogeneous ideal $(\bar{K}) \subset S$ is Borel fixed. That is, if $x_i\mathbf{x}^\alpha \in J$, then $x_j\mathbf{x}^\alpha \in J$ for $j > i$. Let $\bar{B}$ be the set of monomials of degree k, which are not in J. As J is Borel fixed and different from $S_{[2k]}$, $x_0^k \in \overline{B}$. Similarly we check that if $x_0^{\alpha_0}\cdots x_n^{\alpha_n} \in J$ with $\alpha_1 = \cdots = \alpha_{l-1} = 0$, then $x_0^{\alpha_0+1}x_l^{\alpha_l-1}\cdots x_n^{\alpha_n} \in J$. This shows that $B = \tau_0(\bar{B})$ is connected to 1.

As $\langle \bar{B} \rangle \oplus \langle J \rangle = \langle \bar{B} \rangle \oplus \bar{K} = S_{[k]}$ where $\bar{K} = \ker H_{\bar{\sigma}}$, we have $|B| = r$. As B is connected to 1, $\deg(B) < r \leq k$ and $B^+ \subset R^k$.

By the substitution $x_0 = 1$, we have $R_k = \langle B \rangle \oplus K$ with $K = \ker H_\sigma$. Therefore, H_σ is a flat extension of $H_\sigma^{B,B}$. By the flat extension Theorem 3.2, there exist $\lambda_i > 0$, $\bar{\zeta}_i = (\zeta_{i,0}, \zeta_{i,1}, \ldots, \zeta_{i,n}) \in \mathbb{R}^{n+1}$ distinct for $i = 1, \ldots, r$ such that

$$\overline{\sigma} \equiv \sum_{i=1}^{r} \lambda_i \mathbf{e}_{\overline{\zeta}_i} \text{ on } S_{[2k]}. \quad (7)$$

Notice that for any $\lambda \neq 0$, $\mathbf{e}_{\bar{\zeta}_i} = \lambda^{-k}\mathbf{e}_{\lambda\bar{\zeta}_i}$ on $S_{[2k]}$.

By an inverse change of coordinates, the points $\bar{\zeta}_i$ of (7) are transformed into some points $\bar{\zeta}_i = (\zeta_{i,0}, \zeta_{i,1}, \ldots, \zeta_{i,n}) \in \mathbb{K}^{n+1}$ such that $\zeta_{i,0} \neq 0$ (say for $i = 1, \ldots, r'$) and the remaining $r - r'$ points with $\zeta_{i,0} = 0$. The image by τ_0^* of $\mathbf{e}_{\overline{\zeta}_i} \in S_{[2k]}^*$ with $\zeta_{i,0} \neq 0$ is

$$\tau_0^*(\mathbf{e}_{\zeta_i}) \equiv \zeta_{i,0}^{2k}\mathbf{e}_{\zeta_i}$$

where $\zeta_i = \frac{1}{\zeta_{i,0}}(\zeta_{i,1}, \ldots, \zeta_{i,n})$. The image by τ_0^* of $\mathbf{e}_{\overline{\zeta}_i} \in S_{[2k]}^*$ with $\zeta_{i,0} = 0$ is a linear form $\in R_{2k}^*$, which vanishes on all the monomials $\mathbf{x}^\alpha$ with $|\alpha| < 2k$, since their homogenization in degree $2k$ is $x_0^{2k-|\alpha|}\mathbf{x}^\alpha$ and their evaluation at $\bar{\zeta}_i = (0, \zeta_{i,1}, \ldots, \zeta_{i,n})$ gives 0. The value of $\tau_0^*(\mathbf{e}_{\overline{\zeta}_i})$ at $\mathbf{x}^\alpha$ with $|\alpha| = 2k$, $\zeta_{i,1}^{\alpha_1}\cdots\zeta_{i,n}^{\alpha_n} = \mathbf{e}_{\zeta_i}(\mathbf{x}^\alpha)$ where $\zeta_i = (\zeta_{i,1}, \ldots, \zeta_{i,n})$. We deduce that

$$\tau_0^*(\mathbf{e}_{\overline{\zeta}_i}) \equiv \mathbf{e}_{\zeta_i} \circ \pi_{2k}.$$

By dehomogenization, we have $\omega_i = \lambda_i \zeta_{i,0}^{2k} > 0$, $\zeta_i = \frac{1}{\zeta_{i,0}}(\zeta_{i,1}, \ldots, \zeta_{i,n}) \in \mathbb{R}^n$ for $i = 1, \ldots, r'$ and $\zeta_i = (\zeta_{i,1}, \ldots, \zeta_{i,n}) \in \mathbb{R}^n$ for $i = r' + 1, \ldots, n$.

We exploit this structure theorem to show that if the truncation order is sufficiently high, a minimizer of (6) corresponds to a cubature formula.

Theorem 5.2—*Let P be a proper operator and* $k \geq \frac{\deg(V)+1}{2}$. *Assume that there exists* $\sigma^* \in R_{2k}^*$ *such that* H_{σ^*} *is a minimizer of* (6) *of rank r with* $r \leq k$. *Then* $H_{\sigma^*} \in \mathscr{E}_r^k(\mathbf{i})$ *i.e. there exist* $\omega_i > 0$ *and* $\zeta_i \in \mathbb{R}^n$ *such that*

$$\sigma^* \equiv \sum_{i=1}^{r} \omega_i \mathbf{e}_{\zeta_i}.$$

Proof: By Theorem 5.1,

$$\sigma^* \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i} + \sum_{i=r'+1}^{r} w_i \mathbf{e}_{\zeta_i} \circ \pi_{2k},$$

with $\omega_i > 0$ and $\zeta_i \in \mathbb{R}^n$ for $i = 1, \ldots, r$.

Let us suppose that $r \neq r'$. As $k \geq \frac{\deg(V)+1}{2}$, the elements of V are of degree $< 2k$, therefore σ^* and $\sigma' \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i}$ coincide on V and $H_{\sigma'} \in \mathscr{H}^k(\mathbf{i})$. We have the decomposition

$$\mathrm{trace}(PH_{\sigma^*}P) = \langle \sigma^* | q \rangle = \langle \sigma' | q \rangle + \sum_{i=1}^{r'} \omega_i \pi_{2k}(q)(\zeta_i)$$

The homogeneous component of highest degree $\pi_{2k}(q)$ of $q(\mathbf{x}) = \sum_{i=1}^{s_k} p_i(\mathbf{x})^2$ is the sum of the squares of the degree-k components of the p_i:

$$\pi_{2k}(q) = \sum_{i=1}^{s_k} (\pi_k(p_i))^2,$$

so that $\sum_{i=1}^{r'} \omega_i \pi_{2k}(q)(\zeta_i) \geq 0$. As trace($PH_{\sigma^*}P$) is minimal, we must have $\sum_{i=r'+1}^{r} \omega_i \pi_{2k}(q)(\zeta_i) = 0$, which implies that $\pi_{2k}(q)(\zeta_i)$ for $i = r' + 1, \ldots, r$. However, this is impossible, since P is proper. We thus deduce that $r' = r$, which concludes the proof of the theorem.

This theorem shows that an optimal solution of the minimization problem (6) of small rank ($r \leq k$) yields a cubature formula, which is exact on V. Among such minimizers, we have those of minimal rank as shown in the next proposition.

Proposition 5.3—*Let* $k \geq \frac{\deg(V)+1}{2}$ *and H be an element of* $\mathcal{H}^k(\mathbf{i})$ *with minimal rank r. If k* ≥ *r, then* $H \in \mathcal{E}_r^k(\mathbf{i})$ *and it is either an extremal point of* $\mathcal{H}^k(\mathbf{i})$ *or on a face of* $\mathcal{H}^k(\mathbf{i})$, *which is included in* $\mathcal{E}_r^k(\mathbf{i})$.

Proof: Let $H_\sigma \in \mathcal{E}_r^k(\mathbf{i})$ be of minimal rank *r*.

By Theorem 5.1, $\sigma \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i} + \sum_{i=r'+1}^{r} w_i \mathbf{e}_{\zeta_i} \circ \pi_{2k}$ with $\omega_i > 0$ and $\zeta_i \in \mathbb{R}^n$ for $i = 1, \ldots, r$. The elements of *V* are of degree < 2*k*, therefore σ and $\sigma' \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i}$ coincide on *V*. We deduce that $H_{\sigma'} \in \mathcal{H}^k(\mathbf{i})$.

As rank $H_{\sigma'} = r' \leq r$ and $H_\sigma \in \mathcal{H}^k(\mathbf{i})$ is of minimal rank *r*, $r = r'$ and $H_\sigma \in \mathcal{E}_r^k(\mathbf{i})$.

Let us assume that H_σ is not an extremal point of $\mathcal{H}^k(\mathbf{i})$. Then it is in the relative interior of a face *F* of $\mathcal{H}^k(\mathbf{i})$. For any H_{σ_1} in a sufficiently small ball of *F* around H_σ, there exist $t \in]0, 1[$ and $H_{\sigma_2} \in F$ such that

$$H_\sigma = tH_{\sigma_1} + (1-t)H_{\sigma_2}.$$

The kernel of H_σ is the set of polynomials $p \in R_k$ such that

$$0 = H_\sigma(p,p) = tH_{\sigma_1}(p,p) + (1-t)H_{\sigma_2}(p,p).$$

As $H_{\sigma_i} \succcurlyeq 0$, we have $H_{\sigma_i}(p, p) = 0$ for $i = 1, 2$. This implies that $\ker H_\sigma \subset \ker H_{\sigma_i}$, for $i = 1, 2$. From the inclusion $\ker H_{\sigma_1} \cap \ker H_{\sigma_2} \subset \ker H_\sigma$, we deduce that

$$\ker H_\sigma = \ker H_{\sigma_1} \cap \ker H_{\sigma_2}.$$

As H_σ is of minimal rank *r*, we have $\dim \ker H_\sigma \geq \dim \ker H_{\sigma_i}$. This implies that $\ker H_\sigma = \ker H_{\sigma_1} = \ker H_{\sigma_2}$.

As $r \leq k$, $\mathcal{A}_\sigma$ has a monomial basis *B* (connected to 1) in degree < *k* and $R_k = \langle B \rangle \oplus \ker H_\sigma$. Consequently, H_σ (resp. H_{σ_i}) is a flat extension of $H_\sigma^{B,B}$ (resp. $H_{\sigma_i}^{B,B}$) and we have the decomposition

$$\sigma_1 \equiv \sum_{i=1}^{r} \omega_{i,1} \mathbf{e}_{\zeta_i}, \qquad \sigma_2 \equiv \sum_{i=1}^{r} \omega_{i,2} \mathbf{e}_{\zeta_i},$$

with $\omega_{i,j} > 0$, $i = 1, \ldots, r$, $j = 1, 2$. We deduce that $H_{\sigma_1} \in \mathcal{E}_r^k(\mathbf{i})$ and all the elements of the line (H_σ, H_{σ_1}) which are in *F* are also in $\mathcal{E}_r^k(\mathbf{i})$. Since *F* is convex, we deduce that $F \subset \mathcal{E}_r^k(\mathbf{i})$.

Remark 5.4—A cubature formula is *interpolatory* when the weights are uniquely determined from the points. From the previous theorem and proposition, we see that if a cubature formula is of minimal rank and interpolatory, then it is an extremal point of $\mathcal{H}^k(\mathbf{i})$.

According to the previous proposition, by minimizing the nuclear norm of a random matrix, we expect to find an element of minimal rank in one of the faces of $\mathcal{H}^k(\mathbf{i})$, provided k is big enough. This yields the following simple algorithm, which solves the SDP problem, and checks the flat extension property using Algorithm 1. Furthermore it computes the decomposition using Algorithm 2 or increases the degree if there is no flat extension:

Algorithm 3

- $k := \lceil \frac{\deg(V)}{2} \rceil$*; notflat := true; P:= random* $s_k \times s_k$ *matrix;*
- While (notflat) do
  - *Let* σ *be a solution of the SDP problem:* $\min_{H \in \mathcal{H}^k(\mathbf{i})}$ trace($P^t H P$);
  - *If* H_σ^k *is not a flat extension, then* $k := k + 1$*; else notflat := false;*
- *Compute the decomposition of* $\sigma = \sum_{i=1}^{r} w_i \mathbf{e}_{\zeta_i}$, $\omega_i > 0$, $\zeta_i \in \mathbb{R}^n$.

6. Examples

We now illustrate our cubature method on a few explicit examples.

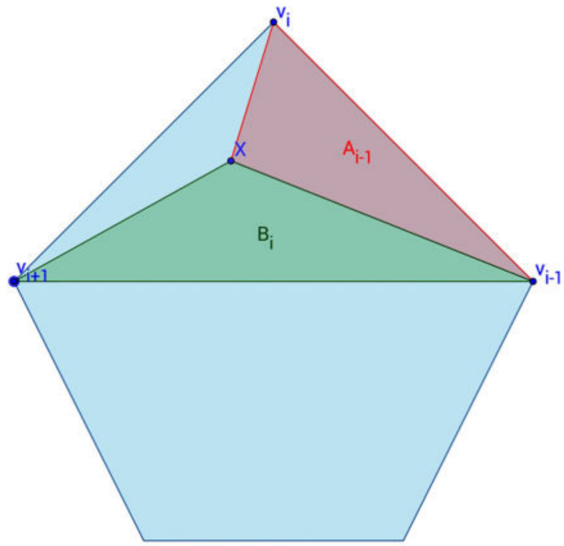
Example 6.1 (*Cubature on a square*)—Our first application is a well known case, namely, the square domain $\Omega = [-1, 1] \times [-1, 1]$. We solve the SDP problem (6), with a random matrix P and with no constraint on the support of the points. In the following table, we give the degree of the cubature formula (i.e. the degree of the polynomials for which the cubature formula is exact), the number N of cubature points, the coordinates of the cubature points and the associated weights.

Degree	N	Points	Weights
3	4	±(0.46503, 0.464462)	1.545
		±(0.855875, −0.855943)	0.454996
5	7	±(0.673625, 0.692362)	0.595115
		±(0.40546, −0.878538)	0.43343
		±(−0.901706, 0.340618)	0.3993
		(0, 0)	1.14305
7	12	±(0.757951, 0.778815)	0.304141
		±(0.902107, 0.0795967)	0.203806
		±(0.04182, 0.9432)	0.194607
		±(0.36885, 0.19394)	0.756312

Degree	N	Points	Weights
		±(0.875533, −0.873448)	0.0363
		±(0.589325, −0.54688)	0.50478

The cubature points are symmetric with respect to the origin (0, 0). The computed cubature formula involves the minimal number of points, which all lie in the domain Ω.

Example 6.2 (*Barycentric Wachpress coordinates on a pentagon*)—Here we consider the pentagon C of vertices $v_1 = (0, 1)$, $v_2 = (1, 0)$, $v_3 = (-1, 0)$, $v_4 = (-0.5, -1)$, $v_5 = (0.5, -1)$.



To this pentagon, we associate (Wachpress) barycentric coordinates [43], which are defined as follows. The weighted function associated to the vertex v_i is defined as:

$$w_i(\mathbf{x}) = \frac{A_{i-1} - B_i + A_i}{A_{i-1} \cdot A_i}$$

where A_i is the signed area of the triangle $(\mathbf{x}, v_{i-1}, v_i)$ and B_i is the area of $(\mathbf{x}, v_{i+1}, v_{i-1})$. The coordinate function associated to v_i is:

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{i=1}^{5} w_i(\mathbf{x})}.$$

These coordinate functions satisfy:

- $\lambda_i(\mathbf{x}) \geq 0$ for $\mathbf{x} \in C$
- $\sum_{i=1}^{5} \lambda_i = 1,$

- $\sum_{i=1}^{5} v_i \cdot \lambda_i(\mathbf{x}) = \mathbf{x}$

For all polynomials $p \in R = \mathbb{R}[u_0, u_1, u_2, u_3, u_4]$, we consider

$$I[p] = \int_{\mathbf{x}\in\Omega} p \circ \lambda(\mathbf{x}) d\mathbf{x}.$$

We look for a cubature formula $\sigma \in R^*$ of the form:

$$\langle \sigma | p \rangle = \sum_{j=1}^{r} w_j p(\zeta_j) \quad (8)$$

with $w_i > 0$, $\zeta_i \in \mathbb{R}^5$, such that $I[p] = \langle \sigma \mid p \rangle$ for all polynomials p of degree ≤ 2.

The moment matrix $H_\sigma^{B,B}$ associated to $B = \{1, u_0, u_1, u_2, u_3, u_4\}$ involves moments of degree ≤ 2:

$$H_\sigma^{B,B} = \begin{pmatrix} 2.5000 & 0.5167 & 0.5666 & 0.5167 & 0.4500 & 0.4500 \\ 0.5167 & 0.2108 & 0.1165 & 0.0461 & 0.0440 & 0.0992 \\ 0.5666 & 0.1165 & 0.2427 & 0.1165 & 0.0454 & 0.0454 \\ 0.5167 & 0.0461 & 0.1165 & 0.2108 & 0.0992 & 0.0440 \\ 0.4500 & 0.0440 & 0.0454 & 0.0992 & 0.1701 & 0.0911 \\ 0.4500 & 0.0992 & 0.0454 & 0.0440 & 0.0911 & 0.1701 \end{pmatrix}$$

Its rank is $\operatorname{rank}(H_\sigma^{B,B}) = 5$.

We compute $H_\sigma^{B^+,B^+}$. In this matrix, there are 105 unknown parameters. We solve the following SDP problem

$$\begin{array}{ll} \min & \operatorname{trace}(H_\sigma^{B^+,B^+}) \\ \text{s. t.} & H_\sigma^{B^+,B^+} \succcurlyeq 0 \end{array} \quad (9)$$

which yields a solution with minimal rank 5. Since the rank of the solution matrix is the rank of $H_\sigma^{B,B}$, we do have a flat extension. Applying Algorithm 1, we find the orthogonal polynomials ρ_i, the matrices of the operators of multiplication by a variable, their common eigenvectors, which gives the following cubature points and weights:

Points	Weights
(0.249888, −0.20028, 0.249993, 0.350146, 0.350193)	0.485759
(0.376647, 0.277438, −0.186609, 0.20327, 0.329016)	0.498813

Points	Weights
(0.348358, 0.379898, 0.244967, −0.174627, 0.201363)	0.509684
(−0.18472, 0.277593, 0.376188, 0.329316, 0.201622)	0.490663
(0.242468, 0.379314, 0.348244, 0.200593, −0.170579)	0.51508

Acknowledgments

The research of C. Bajaj was supported in part by grants from NSF (OCI-1216701737), NIH (R01-GM117594) and a contract (BD-4485) from Sandia National Labs.

References

1. Gillette AA, Rand A, Bajaj C. Error estimates for generalized barycentric coordinates. Adv Comput Math. 2012; 37(3):417–439. [PubMed: 23338826]
2. Abrams C, Bussi G. Enhanced sampling in molecular dynamics using metadynamics, replicaexchange and temperature acceleration. Entropy. 2014; 16:163–199.
3. Bajaj C, Bauer B, Bettadapura R, Vollrath A. Non-uniform Fourier transforms for multi-dimensional rotational correlations. SIAM J Sci Comput. 2013; 35(4):821–845.
4. Bayer C, Teichmann J. The proof of Tchakaloff's Theorem. Proc Amer Math Soc. 2006; 134:3035–3040.
5. Bernardi A, Brachat J, Comon P, Mourrain B. General tensor decomposition, moment matrices and applications. J Symbolic Comput. 2013; 52:51–71.
6. Brachat J, Comon P, Mourrain B, Tsigaridas E. Symmetric tensor decomposition. Linear Algebra Appl. 2010; 433:1851–1872.
7. Patterson TNL, Morrow CR. Construction of algebraic cubature rules using polynomial ideal theory. SIAM J Numer Anal. 1978; 15:953–976.
8. Caflisch R, Morokoff W, Owen A. Valuation of mortgage backed securities using Brownian bridges to reduce effective dimension. J Comput Finance. 1997; 1:27–46.
9. Cools R. Constructing cubature formulae: the science behind the art. Acta Numer. 1997; 6:1–54.
10. Cox, D. Solving equations via algebra. In: Dickenstein, A.; Emiris, IZ., editors. Algorithms Comput Math. Vol. 14. Springer; 2005. p. 63-123.Solving Polynomial Equations: Foundations, Algorithms, and Applications
11. Curto RE, Fialkow LA. Recursiveness, positivity, and truncated moment problems. Houston J Math. 1991; 17(4):603–635.
12. Curto RE, Fialkow L. Flat extensions of positive moment matrices: recursively generated relations. Mem Amer Math Soc. 1998; 136(648):1–64.
13. Curto RE, Fialkow L. The truncated complex k-moment problem. Trans Amer Math Soc. 2000; 352:2825–2855.
14. Eisenbud, D. Texts in Math. Vol. 150. Springer-Verlag; Berlin: 1994. Commutative Algebra with a View Toward Algebraic Geometry, Grad.
15. Elkadi, M.; Mourrain, B. Math Appl. Vol. 59. Springer-Verlag; 2007. Introduction à la résolution des systèmes d'équations algébriques.
16. Fazel, M. PhD thesis. Stanford University; 2002. Matrix rank minimization with applications.
17. Gillette A, Bajaj C. Dual formulations of mixed finite element methods with applications. Comput Aided Design. 2011; 43(10):1213–1221.
18. Iohvidov, IS. Hankel and Toeplitz Matrices and Forms: Algebraic Theory. Birkhäuser; Boston, MA: 1982. Translated from the Russian by G. Philip A. Thijsse. With an introduction by I. Gohberg
19. Lasserre, JB. Moments, Positive Polynomials and Their Applications. Imperial College Press; 2009.

20. Lasserre JB, Laurent M, Mourrain B, Rostalski P, Trébuchet P. Moment matrices, border bases and real radical computation. J Symbolic Comput. 2012:63–85.
21. Laurent M. Revisiting two theorems of Curto and Fialkow on moment matrices. Proc Amer Math Soc. 2005; 133:2965–2976.
22. Laurent, M. IMA Vol Math Appl. Vol. 149. Springer; 2009. Sums of Squares, Moment Matrices and Optimization over Polynomials; p. 157-270.
23. Laurent M, Mourrain B. A sparse flat extension theorem for moment matrices. Arch Math. 2009; 93:87–98.
24. Möller HM. Kubaturformeln mit minimaler knotenzahl. Numer Math. 1975/1976; 25(2):185–200.
25. Möller, HM. Lower bounds for the number of nodes in cubature formulae. In: Hämmerlin, G., editor. Numerische Integration. Vol. 45. Birkhäuser; Basel: 1979. p. 221-230.International Series of Numerical Mathematics/Internationale Schriftenreihe zur Numerischen Mathematik/Série Internationale D'Analyse Numérique
26. Möller HM. Lower bounds for the number of nodes in cubature formulae. Numerische Integration. 1979; 45:221–230. Internat. Ser. Numer. Math.
27. Mourrain B. A new criterion for normal form algorithms. AAECC. 1999:430–443.
28. Mourrain, B.; Trébuchet, P. Generalized normal forms and polynomials system solving. In: Kauers, M., editor. ISSAC: Proceedings of the ACM SIGSAM International Symposium on Symbolic and Algebraic Computation; 2005; p. 253-260.
29. Mysovskikh IP. A proof of minimality of the number of nodes of a cubature formula for a hypersphere. Z Vycisl Mat i Mat Fiz. 1966; 6:621–630. (in Russian).
30. Mysovskikh, IP. Interpolational Cubature Formulas. Nauka, Moscow: 1981. p. 1-336.(in Russian)
31. Nesterov, Y.; Nemirovski, A. Interior-Point Polynomial Algorithms in Convex Programming. SIAM; Philadelphia: 1994.
32. Ninomiya S, Tezuka S. Toward real time pricing of complex financial derivatives. Appl Math Finance. 1996; 3:1–20.
33. Putinar M. A dilation theory approach to cubature formulas. Expo Math. 1997; 15:183–192.
34. Radon J. Zur mechanischen kubatur. Monatsh Math. 1948; 52:286–300. (in German).
35. Rand A, Gillette A, Bajaj C. Interpolation error estimates for mean value coordinates. Adv Comput Math. 2013; 39:327–347. [PubMed: 24027379]
36. Rand A, Gillette A, Bajaj C. Quadratic serendipity finite elements on polygons using generalized barycentric coordinates. Math Comp. 2014; 83:2691–2716.
37. Recht B, Fazel M, Parrilo P. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. SIAM Rev. 2010; 52(3):471–501.
38. Schmid HJ. Two-dimensional minimal cubature formulas and matrix equations. SIAM J Matrix Anal Appl. 1995; 16:898–921.
39. Schmid HJ, Xu Y. On bivariate Gaussian cubature formulae. Proc Amer Math Soc. 1994; 122:833–841.
40. Stroud AH. Quadrature methods for functions of more than one variable. Ann NY Acad Sci. 1960; 86:776–791.
41. Stroud AH. Integration formulas and orthogonal polynomials. SIAM J Numer Anal. 1970; 7:271–276.
42. Tchakaloff V. Formules de cubatures mécaniques à coefficients non négatifs. Bull Sci Math. 1957; 81(2):123–134.
43. Wachpress, E. A Rational Finite Element Basis. Academic Press; 1975.
44. Xu Y. Common zeros of polynomials in several variables and higher-dimensional quadrature. Pitman Res Notes Math Ser. 1994; 312(312):119.
45. Xu Y. Cubature formulae and polynomial ideals. Adv in Appl Math. 1999; 23:211–233.