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Author for correspondence:
Patrizio Neff
e-mail: patrizio.neff@uni-due.de

Real wave propagation in the isotropic-relaxed micromorphic model

Patrizio Neff[1], Angela Madeo[2,4], Gabriele Barbagallo[2,3], Marco Valerio d'Agostino[2], Rafael Abreu[5] and Ionel-Dumitrel Ghiba[6,7,8]

[1]Nonlinear Analysis and Modelling, Fakultät für Mathematik, Universität Duisburg-Essen, Mathematik-Carrée, Thea-Leymann-Straße 9, 45127 Essen, Germany
[2]LGCIE SMS-ID and [3]LaMCoS-CNRS, INSA-Lyon, Université de Lyon, 20 avenue Albert Einstein, 69621 Villeurbanne Cedex, France
[4]IUF, Institut Universitaire de France, 1 rue Descartes, 75231 Paris Cedex 05, France
[5]Institut für Geophysik, Westfälische Wilhelms-Universität Münster, Corrensstraße 24, 48149 Münster, Germany
[6]Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Straße 9, 45127 Essen, Germany
[7]Department of Mathematics, Alexandru Ioan Cuza University of Iaşi, Boulevard, Carol I, No. 11, 700506 Iaşi, Romania
[8]Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi Branch, 700505 Iaşi, Romania

AM, 0000-0003-1940-9853

For the recently introduced isotropic-relaxed micromorphic generalized continuum model, we show that, under the assumption of positive-definite energy, planar harmonic waves have real velocity. We also obtain a necessary and sufficient condition for real wave velocity which is weaker than the positive definiteness of the energy. Connections to isotropic linear elasticity and micropolar elasticity are established. Notably, we show that strong ellipticity does not imply real wave velocity in micropolar elasticity, whereas it does in isotropic linear elasticity.

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1. Introduction

Investigations of real wave propagation and ellipticity, in principle, are not new. Indeed, it is textbook knowledge for linear elasticity that positive definiteness of the elastic energy implies real wave velocities (phase velocities) $v = \omega/k$, where ω [rad s^{-1}] is the angular frequency and k [rad m^{-1}] $\in \mathbb{R}$ is the wavenumber of planar propagating waves. In classical elasticity, having real wave velocities is equivalent to rank-one convexity (strong ellipticity or Legendre–Hadamard ellipticity). Moreover, ellipticity is equivalent to the positive definiteness of the acoustic tensor. For anisotropic linear elasticity, see [1], whereas for anisotropic nonlinear elasticity we refer the reader to [2–5].

The same question of ellipticity and real wave velocities in generalized continuum mechanics has been discussed for micropolar models, e.g. in [6] and for elastic materials with voids in [7]. For the isotropic micromorphic model, results can be found with respect to positive-definite energy and/or real wave velocity in Nowacki [8], Smith [9], Mindlin [10,11] and Eringen [12, pp. 277–280]. These latter results present conditions which are neither easily verifiable nor are truly transparent. This is due to the very high number of material coefficients of the Eringen–Mindlin theory that are strongly reduced in the relaxed micromorphic model [13]. Indeed, the implication that positive definiteness of the energy always implies real wave velocities is not directly established and demonstrated. In this paper, we investigate the relaxed micromorphic model in terms of conditions for real wave velocities for planar waves and establish a necessary and sufficient conditions for this to happen.

This paper is organized as follows. We shortly recall the basics of the relaxed micromorphic model and discuss the wave propagation problem for propagating planar waves. Because we deal with an isotropic model, we can, without loss of generality, assume wave propagation in one specific direction only. The dispersion relations are then obtained, and real wave velocities, under the assumption of uniform positiveness of the elastic energy, are established.

We next present a set of necessary and sufficient conditions for real wave velocities in the relaxed micromorphic model which is weaker than the positivity of the energy, as is the strong ellipticity condition with respect to positive definiteness of the energy in the case of linear elasticity. Then, for didactic purposes, we repeat the analysis for isotropic linear elasticity in order to see relations of our necessary and sufficient condition to the strong ellipticity condition in linear elasticity. Similarly, we discuss micropolar elasticity and establish the necessary and sufficient conditions for real wave propagation. We finally show that strong ellipticity in micropolar and micromorphic models is *not* sufficient for having real wave velocities, when dealing with plane waves.

2. The relaxed micromorphic model

The relaxed micromorphic model has been recently introduced into continuum mechanics in [14]. In subsequent works [15–18], the model has shown its wider applicability compared with the classical Mindlin–Eringen micromorphic model in diverse areas [10–12,19].

The dynamic relaxed micromorphic model counts only eight constitutive parameters in the (simplified) isotropic case, namely five elastic moduli μ_e, λ_e, μ_{micro}, λ_{micro}, μ_c [Pa], one characteristic length L_c [m], the average macroscopic inertia ρ [kg] and the microinertia η [kg m^{-1}]. The simplification consists of assuming one scalar microinertia parameter η and a uniconstant curvature expression. The characteristic length, L_c, is intrinsically related to non-local effects due to the fact that it weights a suitable combination of first-order space derivatives of the microdistortion tensor in the strain energy density (2.1). For a general presentation of the features of the relaxed micromorphic model in the anisotropic setting, we refer the reader to [20].

(a) Elastic energy density

The relaxed micromorphic model couples the macroscopic displacement $u \in \mathbb{R}^3$, and an affine substructure deformation attached at each macroscopic point is encoded by the

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microdistortion field $P \in \mathbb{R}^{3\times3}$. Our novel relaxed micromorphic model endows the Mindlin–Eringen representation of linear micromorphic models with the second-order *dislocation density tensor* $\alpha = -\text{Curl}\, P$ instead of the full gradient ∇P.[1] In the isotropic hyperelastic case, the elastic energy density reads

$$\begin{aligned} W &= \mu_e \| \operatorname{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2}(\operatorname{tr}(\nabla u - P))^2 + \mu_c \| \operatorname{skew}(\nabla u - P)\|^2 \\ &\quad + \mu_{\text{micro}} \| \operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2}(\operatorname{tr} P)^2 + \frac{\mu_e L_c^2}{2}\|\operatorname{Curl} P\|^2 \\ &= \underbrace{\mu_e \| \operatorname{dev}\operatorname{sym}(\nabla u - P)\|^2 + \frac{2\mu_e + 3\lambda_e}{3}(\operatorname{tr}(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \| \operatorname{skew}(\nabla u - P)\|^2}_{\text{rotational elastic coupling}} \\ &\quad + \underbrace{\mu_{\text{micro}} \| \operatorname{dev}\operatorname{sym} P\|^2 + \frac{2\mu_{\text{micro}} + 3\lambda_{\text{micro}}}{3}(\operatorname{tr} P)^2}_{\text{microself energy}} + \underbrace{\frac{\mu_e L_c^2}{2}\|\operatorname{Curl} P\|^2}_{\text{simplified isotropic curvature}}, \end{aligned} \tag{2.1}$$

where the parameters and the elastic stress are analogous to the standard Mindlin–Eringen micromorphic model. The model is well posed in the statical and dynamic case even for zero *Cosserat couple modulus* $\mu_c = 0$; see [21,22]. In that case, it is non-redundant in the sense of [23]. Well-posedness results for the statical and dynamic cases have been provided in [14], making decisive use of recently established new coercive inequalities, generalizing Korn's inequality to incompatible tensor fields [24–28].

Decisive for the relaxed micromorphic formulation is the definition of the elastic energy in terms of suitable strain tensors. Because ∇u is the macroscopic displacement gradient and P is the microdistortion, it appears possible to use the non-symmetric relative (elastic) strain tensor $\nabla u - P$ as the basic building block in the energy. Using the Cartan–Lie orthogonal decomposition, we may introduce

$$\mu_e \| \operatorname{dev}\operatorname{sym}(\nabla u - P)\|^2 + \frac{2\mu_e + 3\lambda_e}{3}(\operatorname{tr}(\nabla u - P))^2 + \mu_c \| \operatorname{skew}(\nabla u - P)\|^2. \tag{2.2}$$

The microstructure contribution based on P alone is restricted, by infinitesimal frame indifference, to

$$\mu_{\text{micro}} \| \operatorname{dev}\operatorname{sym} P\|^2 + \frac{2\mu_{\text{micro}} + 3\lambda_{\text{micro}}}{3}(\operatorname{tr} P)^2 + \frac{\mu_e L_c^2}{2}\|\operatorname{Curl} P\|^2. \tag{2.3}$$

Strict positive definiteness of the potential energy is equivalent to the following simple relations for the introduced parameters [14]:

$$\mu_e > 0, \quad \mu_c > 0, \quad 2\mu_e + 3\lambda_e > 0, \quad \mu_{\text{micro}} > 0, \quad 2\mu_{\text{micro}} + 3\lambda_{\text{micro}} > 0 \quad \text{and} \quad L_c > 0. \tag{2.4}$$

As for the kinetic energy density, we consider that it takes the following (simplified) form:

$$J = \frac{\rho}{2}\|u_{,t}\|^2 + \underbrace{\frac{\eta}{2}\|P_{,t}\|^2}_{\text{simplified microinertia}}, \tag{2.5}$$

where $\rho > 0$ is the value of the averaged macroscopic mass density of the considered material, whereas $\eta > 0$ is its microinertia density.

For very large sample sizes, a scaling argument shows easily that the relative characteristic length scale L_c of the micromorphic model must vanish. Therefore, we have a way of comparing a classical first-gradient formulation with the relaxed micromorphic model and to offer an *a priori* relation between the microscopic parameters $\lambda_e, \lambda_{\text{micro}}, \mu_e, \mu_{\text{micro}}$, on the one side, and

[1] The dislocation tensor is defined as $\alpha_{ij} = -(\text{Curl}\, P)_{ij} = -P_{ih,k}\epsilon_{jkh}$, where ϵ is the Levi–Civita tensor.

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the resulting macroscopic parameters $\lambda_{\mathrm{macro}}, \mu_{\mathrm{macro}}$, on the other side [20,29,30]. We have

$$(2\mu_{\mathrm{macro}}+3\lambda_{\mathrm{macro}})=\frac{(2\mu_{\mathrm{e}}+3\lambda_{\mathrm{e}})(2\mu_{\mathrm{micro}}+3\lambda_{\mathrm{micro}})}{(2\mu_{\mathrm{e}}+3\lambda_{\mathrm{e}})+(2\mu_{\mathrm{micro}}+3\lambda_{\mathrm{micro}})} \quad \text{and} \quad \mu_{\mathrm{macro}}=\frac{\mu_{\mathrm{e}}\,\mu_{\mathrm{micro}}}{\mu_{\mathrm{e}}+\mu_{\mathrm{micro}}}, \tag{2.6}$$

where $\mu_{\mathrm{macro}}, \lambda_{\mathrm{macro}}$ are the moduli obtained for $L_{\mathrm{c}} \to 0$.

For future use, we define the elastic bulk modulus κ_{e}, the microscopic bulk modulus κ_{micro} and the macroscopic bulk modulus κ_{macro}, respectively,

$$\kappa_{\mathrm{e}}=\frac{2\mu_{\mathrm{e}}+3\lambda_{\mathrm{e}}}{3}, \quad \kappa_{\mathrm{micro}}=\frac{2\mu_{\mathrm{micro}}+3\lambda_{\mathrm{micro}}}{3} \quad \text{and} \quad \kappa_{\mathrm{macro}}=\frac{2\mu_{\mathrm{macro}}+3\lambda_{\mathrm{macro}}}{3}. \tag{2.7}$$

In terms of these moduli, strict positive definiteness of the energy is equivalent to

$$\mu_{\mathrm{e}}>0, \quad \mu_{\mathrm{c}}>0, \quad \kappa_{\mathrm{e}}>0, \quad \mu_{\mathrm{micro}}>0, \quad \kappa_{\mathrm{micro}}>0, \quad L_{\mathrm{c}}>0. \tag{2.8}$$

If strict positive definiteness (2.8) holds, we can write the macroscopic consistency conditions as

$$\kappa_{\mathrm{macro}}=\frac{\kappa_{\mathrm{e}}\kappa_{\mathrm{micro}}}{\kappa_{\mathrm{e}}+\kappa_{\mathrm{micro}}} \quad \text{and} \quad \mu_{\mathrm{macro}}=\frac{\mu_{\mathrm{e}}\mu_{\mathrm{micro}}}{\mu_{\mathrm{e}}+\mu_{\mathrm{micro}}}, \tag{2.9}$$

and, again under condition (2.8),

$$\left.\begin{aligned}\kappa_{\mathrm{e}}&=\frac{\kappa_{\mathrm{micro}}\kappa_{\mathrm{macro}}}{\kappa_{\mathrm{micro}}-\kappa_{\mathrm{macro}}}, \quad \kappa_{\mathrm{micro}}=\frac{\kappa_{\mathrm{e}}\kappa_{\mathrm{macro}}}{\kappa_{\mathrm{e}}-\kappa_{\mathrm{macro}}}, \quad \mu_{\mathrm{e}}=\frac{\mu_{\mathrm{micro}}\mu_{\mathrm{macro}}}{\mu_{\mathrm{micro}}-\mu_{\mathrm{macro}}}\\ \text{and} \qquad \mu_{\mathrm{micro}}&=\frac{\mu_{\mathrm{e}}\mu_{\mathrm{macro}}}{\mu_{\mathrm{e}}-\mu_{\mathrm{macro}}}.\end{aligned}\right\} \tag{2.10}$$

Here, strict positivity (2.8) implies that

$$\left.\begin{aligned}&\kappa_{\mathrm{e}}+\kappa_{\mathrm{micro}}>0, \quad \mu_{\mathrm{e}}+\mu_{\mathrm{micro}}>0, \quad \kappa_{\mathrm{e}}>\kappa_{\mathrm{macro}}, \quad \kappa_{\mathrm{micro}}>\kappa_{\mathrm{macro}}\\ \text{and} \qquad &\mu_{\mathrm{e}}>\mu_{\mathrm{macro}}, \quad \mu_{\mathrm{micro}}>\mu_{\mathrm{macro}}.\end{aligned}\right\} \tag{2.11}$$

Because it is useful in what follows, we explicitly remark that

$$\left.\begin{aligned}2\mu_{\mathrm{e}}+\lambda_{\mathrm{e}}&=\frac{4}{3}\mu_{\mathrm{e}}+\frac{2\mu_{\mathrm{e}}+3\lambda_{\mathrm{e}}}{3}=\frac{4}{3}\mu_{\mathrm{e}}+\kappa_{\mathrm{e}}=\frac{4\mu_{\mathrm{e}}+3\kappa_{\mathrm{e}}}{3}\\ \text{and} \qquad 2\mu_{\mathrm{micro}}+\lambda_{\mathrm{micro}}&=\frac{4\mu_{\mathrm{micro}}+3\kappa_{\mathrm{micro}}}{3}.\end{aligned}\right\} \tag{2.12}$$

With these relations, it is easy to show how $\mu_{\mathrm{e}}>0$ and $\kappa_{\mathrm{e}}>0$ imply $2\mu_{\mathrm{e}}+\lambda_{\mathrm{e}}>0$. Moreover, as shown in appendix A (equations (A 2) and (A 3)), we note here that if only $\mu_{\mathrm{e}}+\mu_{\mathrm{micro}}>0$ and $\kappa_{\mathrm{e}}+\kappa_{\mathrm{micro}}>0$, then the macroscopic parameters are less than or equal to respective microscopic parameters, namely

$$\kappa_{\mathrm{e}}\geq\kappa_{\mathrm{macro}}, \quad \kappa_{\mathrm{micro}}\geq\kappa_{\mathrm{macro}}, \quad \mu_{\mathrm{e}}\geq\mu_{\mathrm{macro}} \quad \text{and} \quad \mu_{\mathrm{micro}}\geq\mu_{\mathrm{macro}}, \tag{2.13}$$

and, moreover, the following inequalities are satisfied:

$$\left.\begin{aligned}&2\mu_{\mathrm{e}}+\lambda_{\mathrm{e}}\geq2\mu_{\mathrm{macro}}+\lambda_{\mathrm{macro}}, \quad 2\mu_{\mathrm{micro}}+\lambda_{\mathrm{micro}}\geq2\mu_{\mathrm{macro}}+\lambda_{\mathrm{macro}}\\ \text{and} \qquad &\frac{4\mu_{\mathrm{macro}}+3\kappa_{\mathrm{e}}}{3}\geq2\mu_{\mathrm{macro}}+\lambda_{\mathrm{macro}}.\end{aligned}\right\} \tag{2.14}$$

Note that the Cosserat couple modulus μ_{c} [31] does not appear in the introduced scale between micro and macro.

(b) Dynamic formulation

The dynamic formulation is obtained by defining a joint Hamiltonian and assuming stationary action. The dynamic equilibrium equations are

$$\left.\begin{aligned} \rho u_{,tt} &= \mathrm{Div}[2\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\mu_c\,\mathrm{skew}(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\mathbb{1}] \\ \text{and} \qquad \eta P_{,tt} &= -\mu_e\,L_c^2\mathrm{Curl}\,\mathrm{Curl}\,P + 2\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\mu_c\,\mathrm{skew}(\nabla u - P) \\ &\quad + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\mathbb{1} - [2\mu_{\mathrm{micro}}\,\mathrm{sym}\,P + \lambda_{\mathrm{micro}}\,\mathrm{tr}\,(P)\mathbb{1}]. \end{aligned}\right\} \tag{2.15}$$

We note here that the presence of the Curl P in the energy generates a non-local term Curl Curl P in the equation of motion, whereas the possibility of band gaps is still present; see [15]. The presence of the Curl P term is essential to simultaneously allow us to describe the non-localities and band gap in an enriched continuum mechanics framework.

Sufficiently far from a source, dynamic wave solutions may be treated as planar waves. Therefore, we now want to study harmonic solutions travelling in an infinite domain for the differential system (2.15). To do so, we define

$$\left.\begin{aligned} P^S &:= \tfrac{1}{3}\,\mathrm{tr}\,(P), \qquad P_{[ij]} := (\mathrm{skew}P)_{ij} = \tfrac{1}{2}(P_{ij} - P_{ji}), \\ P^D &:= P_{11} - P^S, \quad P_{(ij)} := (\mathrm{sym}\,P)_{ij} = \tfrac{1}{2}(P_{ij} + P_{ji}) \\ \text{and} \qquad P^V &:= P_{22} - P_{33} \end{aligned}\right\} \tag{2.16}$$

and we introduce the unknown vectors

$$\underbrace{v_1 = (u_1, P^D, P^S)}_{\text{longitudinal}}, \quad \underbrace{v_\tau = (u_\tau, P_{(1\tau)}, P_{[1\tau]})}_{\text{transversal}}, \quad \tau = 2,3 \quad \text{and} \quad \underbrace{v_4 = (P_{(23)}, P_{[23]}, P^V)}_{\text{uncoupled}}. \tag{2.17}$$

The definition of the unknown vectors was made considering the coupling of the variables in the equations of motion; see [15–18,32–36]. More particularly, it has been shown in these previous works that three sets of equations can be isolated: one involving only longitudinal quantities, one involving only transverse quantities and one of three completely uncoupled equations. We suppose that the space dependences of all introduced kinematic fields are limited to a direction defined by a unit vector $\tilde{\xi} \in \mathbb{R}^3$, which is the direction of propagation of the wave and which is assumed given. Hence, we look for solutions of (2.15) in the form

$$v_1 = \beta^1\,e^{\mathrm{i}(k\langle\tilde{\xi},x\rangle_{\mathbb{R}^3} - \omega t)}, \quad v_\tau = \beta^\tau\,e^{\mathrm{i}(k\langle\tilde{\xi},x\rangle_{\mathbb{R}^3} - \omega t)}, \quad \tau = 2,3 \quad \text{and} \quad v_4 = \beta^4\,e^{\mathrm{i}(k\langle\tilde{\xi},x\rangle_{\mathbb{R}^3} - \omega t)}, \tag{2.18}$$

where $\beta^1 = (\beta_1^1, \beta_2^1, \beta_3^1)^{\mathrm{T}} \in \mathbb{C}^3$, $\beta^\tau = (\beta_1^\tau, \beta_2^\tau, \beta_3^\tau)^{\mathrm{T}} \in \mathbb{C}^3$ and $\beta^4 = (\beta_1^4, \beta_2^4, \beta_3^4)^{\mathrm{T}} \in \mathbb{C}^3$ are the unknown amplitudes of the considered waves, $\mathbb{C}^3$ is the space of complex constant three-dimensional vectors,[2] k is the wavenumber and ω is the wave frequency. Because our formulation is isotropic, we can, without loss of generality, specify the propagation direction $\tilde{\xi} = e_1$. Then, $X = \langle e_1, x\rangle_{\mathbb{R}^3} = x_1$, and we obtain that the space dependences of all introduced kinematic fields are limited to the component X, which is now the direction of propagation of the wave.[3] This means that we look for solutions in the form

$$v_1 = \beta^1\,e^{\mathrm{i}(kX - \omega t)}, \quad v_\tau = \beta^\tau\,e^{\mathrm{i}(kX - \omega t)}, \quad \tau = 2,3 \quad \text{and} \quad v_4 = \beta^4\,e^{\mathrm{i}(kX - \omega t)}. \tag{2.19}$$

Replacing these expressions in equations (2.15), it is possible to express the system (see [15,16]) as

$$A_1 \cdot \beta^1 = 0, \quad A_\tau \cdot \beta^\tau = 0, \quad \tau = 2,3 \quad \text{and} \quad A_4 \cdot \beta^4 = 0, \tag{2.20}$$

[2]Here, we understand that, having found the (in general, complex) solutions of (2.19), only the real or imaginary parts separately constitute actual wave solutions which can be observed in reality.

[3]In an isotropic model, it is clear that there is no direction dependence. More specifically, let us consider an arbitrary direction $\tilde{\xi} \in \mathbb{R}^3$. Now we consider an orthogonal spatial coordinate change $Q \cdot e_1 = \tilde{\xi}$ with $Q \in \mathrm{SO}(3)$. In the rotated variables, the ensuing system of PDEs (2.15) is form invariant; see [37].

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with

$$A_1(\omega,k)=\begin{pmatrix} -\omega^2+c_p^2k^2 & \dfrac{ik2\mu_e}{\rho} & \dfrac{ik(2\mu_e+3\lambda_e)}{\rho} \\ -ik\dfrac{4}{3}\dfrac{\mu_e}{\eta} & -\omega^2+\dfrac{1}{3}k^2c_m^2+\omega_s^2 & -\dfrac{2}{3}k^2c_m^2 \\ -\dfrac{1}{3}\dfrac{ik(2\mu_e+3\lambda_e)}{\eta} & -\dfrac{1}{3}k^2c_m^2 & -\omega^2+\dfrac{2}{3}k^2c_m^2+\omega_p^2 \end{pmatrix}, \tag{2.21}$$

$$A_2(\omega,k)=A_3(\omega,k)=\begin{pmatrix} -\omega^2+k^2c_s^2 & \dfrac{ik2\mu_e}{\rho} & -ik\dfrac{\eta}{\rho}\omega_r^2 \\ -\dfrac{ik\mu_e}{\eta} & -\omega^2+\dfrac{c_m^2}{2}k^2+\omega_s^2 & \dfrac{c_m^2}{2}k^2 \\ \dfrac{i}{2}\omega_r^2k & \dfrac{c_m^2}{2}k^2 & -\omega^2+\dfrac{c_m^2}{2}k^2+\omega_r^2 \end{pmatrix} \tag{2.22}$$

and

$$A_4(\omega,k)=\begin{pmatrix} -\omega^2+c_m^2k^2+\omega_s^2 & 0 & 0 \\ 0 & -\omega^2+c_m^2k^2+\omega_r^2 & 0 \\ 0 & 0 & -\omega^2+c_m^2k^2+\omega_s^2 \end{pmatrix}. \tag{2.23}$$

Here, we have defined

$$c_m=\sqrt{\frac{\mu_e L_c^2}{\eta}},\qquad c_s=\sqrt{\frac{\mu_e+\mu_c}{\rho}},\qquad c_p=\sqrt{\frac{2\mu_e+\lambda_e}{\rho}},$$

$$\omega_s=\sqrt{\frac{2(\mu_e+\mu_{micro})}{\eta}},\qquad \omega_p=\sqrt{\frac{(2\mu_e+3\lambda_e)+(2\mu_{micro}+3\lambda_{micro})}{\eta}},\qquad \omega_r=\sqrt{\frac{2\mu_c}{\eta}},$$

$$\omega_l=\sqrt{\frac{2\mu_{micro}+\lambda_{micro}}{\eta}},\qquad \omega_t=\sqrt{\frac{\mu_{micro}}{\eta}}.$$

Let us next define the diagonal matrix

$$\mathrm{diag}_1=\begin{pmatrix} \sqrt{\rho} & 0 & 0 \\ 0 & i\dfrac{\sqrt{6\eta}}{2} & 0 \\ 0 & 0 & i\sqrt{3\eta} \end{pmatrix}. \tag{2.24}$$

Considering $\gamma=\mathrm{diag}_1\cdot\beta$ and the matrix $\bar{A}_1(\omega,k)=\mathrm{diag}_1\cdot A_1(\omega,k)\cdot\mathrm{diag}_1^{-1}$, it is possible to formulate the problem (2.20) equivalently as[4]

$$\bar{A}_1\cdot\gamma=\begin{pmatrix} -\omega^2+c_p^2k^2 & \dfrac{2\sqrt{6}}{3}\dfrac{k\mu_e}{\sqrt{\rho\eta}} & \dfrac{\sqrt{3}}{3}\dfrac{k(2\mu_e+3\lambda_e)}{\sqrt{\rho\eta}} \\ \dfrac{2\sqrt{6}}{3}\dfrac{k\mu_e}{\sqrt{\rho\eta}} & -\omega^2+\dfrac{1}{3}k^2c_m^2+\omega_s^2 & -\dfrac{\sqrt{2}}{3}k^2c_m^2 \\ \dfrac{\sqrt{3}}{3}\dfrac{k(2\mu_e+3\lambda_e)}{\sqrt{\rho\eta}} & -\dfrac{\sqrt{2}}{3}k^2c_m^2 & -\omega^2+\dfrac{2}{3}k^2c_m^2+\omega_p^2 \end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}=0. \tag{2.25}$$

[4]It is possible to face the problem in two more equivalent ways. The first one is to consider from the start that the amplitudes of the microdistortion field are multiplied by the imaginary unit i, i.e. $\beta=(\beta_1, i\beta_2, i\beta_3)^T\in\mathbb{C}^3$, as done in [10, p. 24, equation 8.6]. Doing so, we obtain a real matrix that can be symmetrized with $\mathrm{diag}_1=\begin{pmatrix}\sqrt{\rho}&0&0\\0&\frac{\sqrt{6\eta}}{2}&0\\0&0&\sqrt{3\eta}\end{pmatrix}$. On the other hand, it is also possible to consider from the beginning $\beta=(\sqrt{\rho}\beta_1, i(\sqrt{6\eta}/2)\beta_2, i\sqrt{3\eta}\beta_3)^T\in\mathbb{C}^3$, obtaining directly a real symmetric matrix.

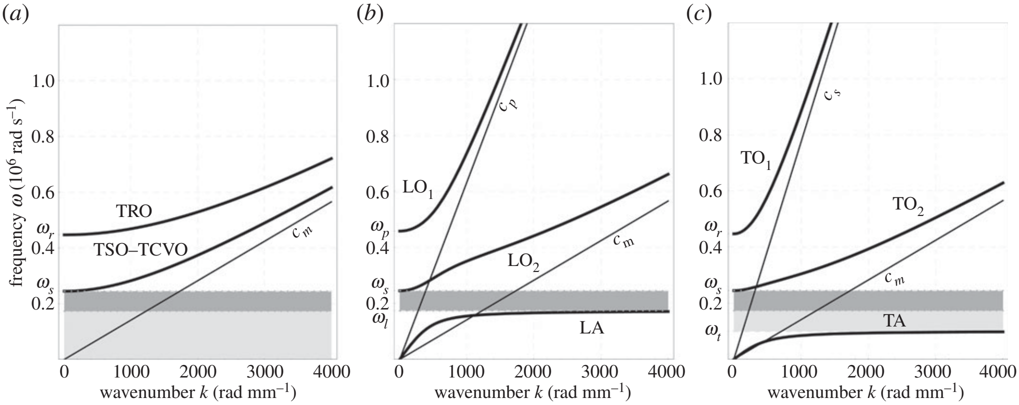

Figure 1. Dispersion relations $\omega = \omega(k)$ [17] for the *relaxed micromorphic model* with non-vanishing Cosserat couple modulus $\mu_c > 0$. Uncoupled waves (*a*), longitudinal waves (*b*) and transverse waves (*c*). TRO, transverse rotational optic; TSO, transverse shear optic; TCVO, transverse constant-volume optic; LA, longitudinal acoustic; LO_1–LO_2, first and second longitudinal optic; TA, transverse acoustic; TO_1–TO_2, first and second transverse optic. (*a*) $\det A_4(\omega, k) = 0$; (*b*) $\det \bar{A}_1(\omega, k) = 0$; (*c*) $\det \bar{A}_2(\omega, k) = 0$.

Analogously considering

$$\mathrm{diag}_2 = \begin{pmatrix} \sqrt{\rho} & 0 & 0 \\ 0 & \mathrm{i}\sqrt{2\eta} & 0 \\ 0 & 0 & \mathrm{i}\sqrt{2\eta} \end{pmatrix}, \tag{2.26}$$

it is possible to obtain $\bar{A}_2(\omega, k) = \bar{A}_3(\omega, k) = \mathrm{diag}_2 \cdot A_2(\omega, k) \cdot \mathrm{diag}_2^{-1}$

$$\bar{A}_2(\omega, k) = \bar{A}_3(\omega, k) = \begin{pmatrix} -\omega^2 + k^2 c_s^2 & \dfrac{k\sqrt{2}\mu_e}{\sqrt{\rho\eta}} & -\dfrac{k\sqrt{2}\mu_c}{\sqrt{\rho\eta}} \\ \dfrac{k\sqrt{2}\mu_e}{\sqrt{\rho\eta}} & -\omega^2 + \dfrac{c_m^2}{2}k^2 + \omega_s^2 & \dfrac{c_m^2}{2}k^2 \\ -\dfrac{k\sqrt{2}\mu_c}{\sqrt{\rho\eta}} & \dfrac{c_m^2}{2}k^2 & -\omega^2 + \dfrac{c_m^2}{2}k^2 + \omega_r^2 \end{pmatrix}. \tag{2.27}$$

In order to have non-trivial solutions of the algebraic systems (2.20), one must impose that

$$\det \bar{A}_1(\omega, k) = 0, \quad \det \bar{A}_2(\omega, k) = \det \bar{A}_3(\omega, k) = 0 \quad \text{and} \quad \det A_4(\omega, k) = 0, \tag{2.28}$$

the solution of which allows us to determine the so-called dispersion relations $\omega = \omega(k)$ for the longitudinal and transverse waves in the relaxed micromorphic continuum (figure 1).[5] The solutions of the eigenvalue problem obtained via the proposed decomposition are the same as the ones obtained via the standard formulation shown in appendix Aa with the full 12×12 matrix; for more details, see [34]. For estimates on the isotropic moduli, we refer the reader to [17,33] and, for a comparison with other micromorphic models, to [32,36]. For solutions $\omega = \omega(k)$ of (2.28), we define

$$\textit{phase velocity: } v = \frac{\omega}{k} \quad \text{and} \quad \textit{group velocity: } \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}. \tag{2.29}$$

Real wavenumbers $k \in \mathbb{R}$ correspond to propagating waves, whereas complex values of k are associated with waves whose amplitude either grows or decays along the coordinate X. In linear

[5]The formal limit $\eta \to +\infty$ shows no dispersion at all giving two pseudo-acoustic linear curves, longitudinal and transverse with slopes $c_p = \sqrt{(2\mu_e + \lambda_e)/\rho}$ and $c_s = \sqrt{(\mu_e + \mu_c)/\rho}$, respectively.

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elasticity, phase velocity and group velocity coincide, because there is no dispersion, and both are real; see section 3.

Because, in this paper, we are interested only in real k (outside the band-gap region), the wave velocity (phase velocity) is real, if and only if ω is real.

Because ω^2 appears on the diagonal only, the problem (2.28) can be analogously expressed as an eigenvalue problem

$$\left.\begin{aligned} &\det(B_1(k)-\omega^2\,\mathbb{1})=0, \quad \det(B_2(k)-\omega^2\,\mathbb{1})=0 \\ \text{and}\quad &\det(B_3(k)-\omega^2\,\mathbb{1})=0, \quad \det(B_4(k)-\omega^2\,\mathbb{1})=0, \end{aligned}\right\} \tag{2.30}$$

where

$$B_1(k)=\begin{pmatrix} c_p^2k^2 & \frac{2\sqrt{6}}{3}\frac{k\mu_e}{\sqrt{\rho\eta}} & \frac{\sqrt{3}}{3}\frac{k(2\mu_e+3\lambda_e)}{\sqrt{\rho\eta}} \\ \frac{2\sqrt{6}}{3}\frac{k\mu_e}{\sqrt{\rho\eta}} & \frac{1}{3}k^2c_m^2+\omega_s^2 & -\frac{\sqrt{2}}{3}k^2c_m^2 \\ \frac{\sqrt{3}}{3}\frac{k(2\mu_e+3\lambda_e)}{\sqrt{\rho\eta}} & -\frac{\sqrt{2}}{3}k^2c_m^2 & +\frac{2}{3}k^2c_m^2+\omega_p^2 \end{pmatrix}, \tag{2.31}$$

$$B_2(k)=B_3(k)=\begin{pmatrix} k^2c_s^2 & \frac{k\sqrt{2}\mu_e}{\sqrt{\rho\eta}} & -\frac{k\sqrt{2}\mu_c}{\sqrt{\rho\eta}} \\ \frac{k\sqrt{2}\mu_e}{\sqrt{\rho\eta}} & \frac{c_m^2}{2}k^2+\omega_s^2 & \frac{c_m^2}{2}k^2 \\ -\frac{k\sqrt{2}\mu_c}{\sqrt{\rho\eta}} & \frac{c_m^2}{2}k^2 & \frac{c_m^2}{2}k^2+\omega_r^2 \end{pmatrix} \tag{2.32}$$

and

$$B_4(k)=\begin{pmatrix} c_m^2k^2+\omega_s^2 & 0 & 0 \\ 0 & c_m^2k^2+\omega_r^2 & 0 \\ 0 & 0 & c_m^2k^2+\omega_s^2 \end{pmatrix}. \tag{2.33}$$

Note that $B_1(k)$, $B_2(k)$, $B_3(k)$ and $B_4(k)$ are real symmetric matrices and, therefore, the resulting eigenvalues ω^2 are real. Obtaining real wave velocities is tantamount to having $\omega^2\geq 0$ for all solutions of (2.30).

(c) Necessary and sufficient conditions for real wave propagation

We show next that all the eigenvalues ω^2 of $B_1(k)$, $B_2(k)$ and $B_3(k)$ are real and positive for every $k\neq 0$ and non-negative for $k=0$, provided certain conditions on the material coefficients are satisfied. Sylvester's criterion states that a Hermitian matrix M is positive definite if and only if the leading principal minors are positive [38]. For the matrix B_1, the three principal minors are

$$(B_1)_{11}=\frac{2\mu_e+\lambda_e}{\rho}, \tag{2.34}$$

$$\begin{aligned}(\mathrm{Cof}\,(B_1))_{33}&=\frac{k^2}{3\eta\rho}[6(2\mu_e+\lambda_e)\mu_{\text{micro}}+6\mu_e\kappa_e+(2\mu_e+\lambda_e)\mu_e\,L_c^2k^2]\\ &=\frac{k^2}{3\eta\rho}[2(4\mu_{\text{macro}}+3\kappa_e)(\mu_e+\mu_{\text{micro}})+(2\,\mu_e+\lambda_e)\mu_e\,L_c^2k^2]\end{aligned} \tag{2.35}$$

and

$$\det(B_1) = \frac{k^2}{\eta^2 \rho}[6\kappa_e \kappa_{micro}(\mu_e + \mu_{micro}) + 8\mu_e \mu_{micro}(\kappa_e + \kappa_{micro}) + (2\mu_e + \lambda_e)(2\mu_{micro} + \lambda_{micro})\mu_e L_c^2 k^2]$$
$$= \frac{k^2}{\eta^2 \rho}[6(\kappa_e + \kappa_{micro})(\mu_e + \mu_{micro})(2\mu_{macro} + \lambda_{macro}) + (2\mu_e + \lambda_e)(2\mu_{micro} + \lambda_{micro})\mu_e L_c^2 k^2]. \tag{2.36}$$

The three principal minors of B_1 are clearly positive for $k \neq 0$ if[6]

$$\left.\begin{aligned} &\mu_e > 0, \quad \mu_{micro} > 0, \quad \kappa_e + \kappa_{micro} > 0, \quad 2\mu_{macro} + \lambda_{macro} > 0 \\ \text{and} \quad &4\mu_{macro} + 3\kappa_e > 0, \quad 2\mu_e + \lambda_e > 0, \quad 2\mu_{micro} + \lambda_{micro} > 0. \end{aligned}\right\} \tag{2.37}$$

Similarly, for the matrix B_2, the three principal minors are

$$(B_2)_{11} = \frac{\mu_e + \mu_c}{\rho}, \tag{2.38}$$

$$(\mathrm{Cof}(B_2))_{33} = \frac{k^2}{2\eta\rho}[4(\mu_e \mu_c + \mu_{micro}(\mu_e + \mu_c)) + (\mu_e + \mu_c)\mu_e L_c^2 k^2] \tag{2.39}$$

and

$$\det(B_2) = \frac{k^2}{\eta^2 \rho}[4\mu_{micro}\mu_c \mu_e + (\mu_e + \mu_c)\mu_{micro}\mu_e L_c^2 k^2]. \tag{2.40}$$

For the matrix $B_2(k) = B_3(k)$, considering positive η, ρ and separating terms in the brackets by looking at large and small values of k, we can state the *necessary* and *sufficient* conditions for strict positive definiteness of $B_2(k)$ at arbitrary $k \neq 0$,

$$\mu_e > 0, \quad \mu_{micro} > 0 \quad \text{and} \quad \mu_c \geq 0. \tag{2.41}$$

Because $B_4(k)$ is diagonal, it is easy to show that positive definiteness is tantamount to the set of *necessary* and *sufficient* conditions for $k \neq 0$

$$\mu_e > 0, \quad \mu_e + \mu_{micro} > 0, \quad \mu_c \geq 0. \tag{2.42}$$

On the other hand, considering the case $k = 0$, we obtain that the matrices reduce to

$$B_1(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_s^2 & 0 \\ 0 & 0 & \omega_p^2 \end{pmatrix}, \quad B_2(0) = B_3(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_s^2 & 0 \\ 0 & 0 & \omega_r^2 \end{pmatrix} \quad \text{and} \quad B_4(0) = \begin{pmatrix} \omega_s^2 & 0 & 0 \\ 0 & \omega_r^2 & 0 \\ 0 & 0 & \omega_s^2 \end{pmatrix}. \tag{2.43}$$

Because the matrices are diagonal for $k = 0$, it is easy to show that positive semi-definiteness is tantamount to the set of *necessary* and *sufficient* conditions

$$\mu_e \geq 0, \quad \mu_e + \mu_{micro} \geq 0, \quad \mu_c \geq 0 \quad \text{and} \quad \kappa_e + \kappa_{micro} \geq 0. \tag{2.44}$$

Hence, we can state a simple *sufficient* condition for real wave velocities for all real k

$$\left.\begin{aligned} &\mu_e > 0, \quad \mu_{micro} > 0, \quad \kappa_e + \kappa_{micro} > 0, \quad 2\mu_{macro} + \lambda_{macro} > 0 \\ \text{and} \quad &4\mu_{macro} + 3\kappa_e > 0, \quad 2\mu_e + \lambda_e > 0, \quad 2\mu_{micro} + \lambda_{micro} > 0. \end{aligned}\right\} \tag{2.45}$$

In order to see a set of global necessary conditions for positivity at arbitrary $k \neq 0$, we consider first large and small values of $k \neq 0$ separately. For $k \to +\infty$, we must have

$$2\mu_e + \lambda_e > 0, \quad (2\mu_e + \lambda_e)\mu_e L_c^2 > 0, \quad (2\mu_e + \lambda_e)(2\mu_{micro} + \lambda_{micro})\mu_e L_c^2 > 0, \tag{2.46}$$

or analogously

$$2\mu_e + \lambda_e > 0, \quad \mu_e L_c^2 > 0 \quad \text{and} \quad 2\mu_{micro} + \lambda_{micro} > 0, \tag{2.47}$$

[6]We note here that $4\mu_{macro} + 3\kappa_e > 0 \iff 2\mu_e + \lambda_e > \frac{4}{3}(\mu_e - \mu_{macro}) \iff 2\mu_{macro} + \lambda_{macro} > \kappa_{macro} - \kappa_e$. Furthermore, if $\mu_e + \mu_{micro} > 0$ and $\kappa_e + \kappa_{micro} > 0$, we have $3(2\mu_e + \lambda_e) \geq 4\mu_{macro} + 3\kappa_e \geq 3(2\mu_{macro} + \lambda_{macro})$; see appendix A.

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while, for $k \to 0$, we must have

$$\left.\begin{aligned} &2\mu_e + \lambda_e > 0, \quad (4\mu_{\text{macro}} + 3\kappa_e)(\mu_e + \mu_{\text{micro}}) > 0 \\ \text{and} \quad &(\kappa_e + \kappa_{\text{micro}})(\mu_e + \mu_{\text{micro}})(2\mu_{\text{macro}} + \lambda_{\text{macro}}) > 0. \end{aligned}\right\} \tag{2.48}$$

Because from (2.41) we have necessarily $\mu_e > 0$, $\mu_{\text{micro}} > 0$, and from (2.44) we get $\kappa_e + \kappa_{\text{micro}} \geq 0$, and considering together the two limits for k, we obtain the necessary condition

$$\left.\begin{aligned} &2\mu_e + \lambda_e > 0, \quad 2\mu_{\text{micro}} + \lambda_{\text{micro}} > 0, \quad 4\mu_{\text{macro}} + 3\kappa_e > 0, \quad \kappa_e + \kappa_{\text{micro}} > 0 \\ \text{and} \quad &\mu_e > 0, \quad \mu_{\text{micro}} > 0, \quad \mu_c \geq 0, \quad 2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0. \end{aligned}\right\} \tag{2.49}$$

Inspection shows that (2.49) is our proposed sufficient condition (2.37). From $\mu_e > 0$ and $\mu_{\text{micro}} > 0$, it follows that $\mu_{\text{macro}} > 0$. Therefore, condition (2.49) is *necessary* and *sufficient*. We have shown our main proposition as follows.

Proposition (real wave velocities). *The dynamic relaxed micromorphic model* (*equation* (2.15)) *admits real planar waves if and only if*

$$\left.\begin{aligned} &\mu_c \geq 0,\ \mu_e > 0, \quad 2\mu_e + \lambda_e > 0, \\ &\mu_{\text{micro}} > 0, \quad 2\mu_{\text{micro}} + \lambda_{\text{micro}} > 0, \\ &(\mu_{\text{macro}} > 0), \quad 2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0 \\ \text{and} \quad &\kappa_e + \kappa_{\text{micro}} > 0, \quad 4\mu_{\text{macro}} + 3\,\kappa_e > 0. \end{aligned}\right\} \tag{2.50}$$

In (2.50), the requirement $\mu_{\text{macro}} > 0$ is redundant, because it is already assumed that $\mu_e, \mu_{\text{micro}} > 0$. It is clear that positive definiteness of the elastic energy (2.4) implies (2.50). We remark that, as shown in appendix Aa, the set of inequalities (2.50) is already implied by

$$\boxed{\mu_e > 0, \quad \mu_{\text{micro}} > 0, \quad \mu_c \geq 0, \quad \kappa_e + \kappa_{\text{micro}} > 0 \quad \text{and} \quad 2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0.} \tag{2.51}$$

Finally letting $\mu_{\text{micro}} \to +\infty$ and $\kappa_{\text{micro}} \to +\infty$ (or $\mu_{\text{micro}} \to +\infty$ and $\lambda_{\text{micro}} >$ const.) generates the limit condition for real wave velocities ($\mu_e \to \mu_{\text{macro}}$)

$$\mu_{\text{macro}} > 0, \quad \mu_c \geq 0 \quad \text{and} \quad 2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \tag{2.52}$$

which coincides, up to μ_c, with the strong ellipticity condition in isotropic linear elasticity (see §3) and it coincides fully with the condition for real wave velocities in micropolar elasticity; see §4. A condition similar to (2.52) can be found in [10, equation 8.14 p. 26] where Mindlin requires that $\mu_{\text{macro}} > 0$, $2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0$ (in our notation),[7] which are obtained from the requirement of positive *group velocity* at $k = 0$

$$\frac{\mathrm{d}\omega_{\text{acoustic, long}}(0)}{\mathrm{d}k} > 0 \quad \text{and} \quad \frac{\mathrm{d}\omega_{\text{acoustic, trans}}(0)}{\mathrm{d}k} > 0. \tag{2.53}$$

Let us emphasize that our method is not easily generalized to two immediate extensions. First, one could be interested in the isotropic-relaxed micromorphic model with weighted inertia contributions and weighted curvatures [34]. Second, one could be interested in the anisotropic setting [20]. In the second case, the block structure of the problem will be lost, and one has to deal with the full 12×12 case; see equation (A 23) in appendix A. Nonetheless, we expect positive definiteness to always imply real wave propagation.

[7] Mindlin explains that such parameters 'are less than those that would be calculated from the strain-stiffnesses [of the unit cell]. This phenomenon is due to the compliance of the unit cell and has been found in a theory of crystal lattices by Gazis & Wallis [39]'.

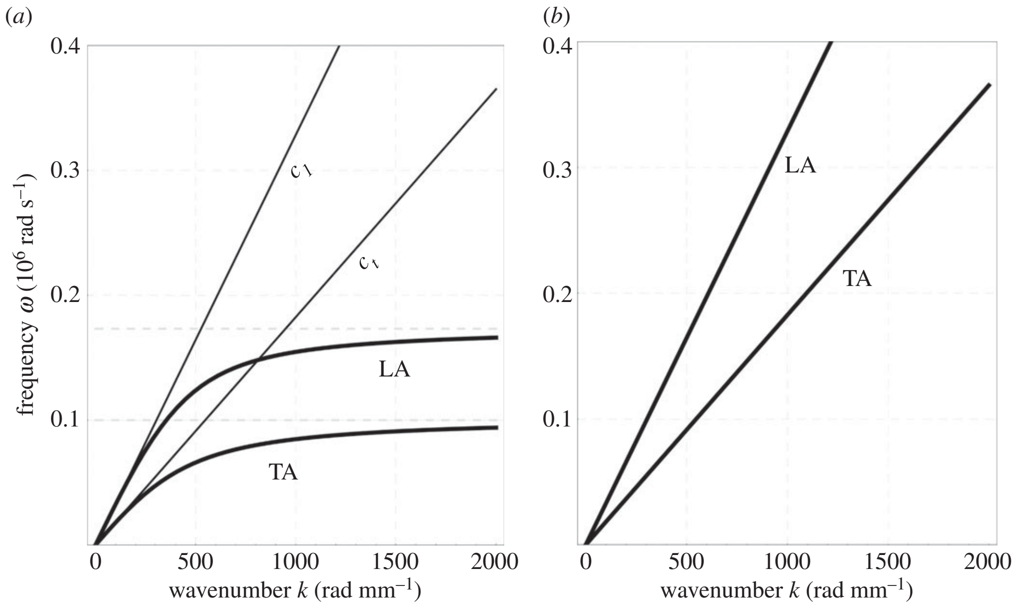

Figure 2. Dispersion relations $\omega = \omega(k)$ for the longitudinal acoustic wave LA, and the transverse acoustic TA in the *relaxed micromorphic model* (*a*) and in a classical Cauchy medium (*b*).

In [34], we show that the tangents of the acoustic branches in $k = 0$ in the dispersion curves are

$$c_l = \frac{\mathrm{d}\omega_{\text{acoustic, long}}(0)}{\mathrm{d}k} = \sqrt{\frac{2\mu_{\text{macro}} + \lambda_{\text{macro}}}{\rho}} \quad \text{and} \quad c_t = \frac{\mathrm{d}\omega_{\text{acoustic, trans}}(0)}{\mathrm{d}k} = \sqrt{\frac{\mu_{\text{macro}}}{\rho}}. \tag{2.54}$$

The tangents coincide with the classical linear elastic response if the latter has Lamé constants μ_{macro} and λ_{macro}, as shown in figure 2.

3. A comparison: classical isotropic linear elasticity

For classical linear elasticity with isotropic energy density and kinetic energy density

$$W(\nabla u) = \mu_{\text{macro}} \| \operatorname{sym} \nabla u \|^2 + \frac{\lambda_{\text{macro}}}{2} (\operatorname{tr}(\nabla u))^2, \quad J = \frac{\rho}{2} \| u_{,t} \|^2. \tag{3.1}$$

The positive definiteness of the energy is equivalent to

$$\mu_{\text{macro}} > 0, \quad 2\mu_{\text{macro}} + 3\lambda_{\text{macro}} > 0. \tag{3.2}$$

It is easy to see that our homogenization formula (2.6) implies (3.2) under the condition of positive definiteness of the relaxed micromorphic model.

The dynamic formulation is obtained by defining a joint Hamiltonian and assuming stationary action. The dynamic equilibrium equations are

$$\rho u_{,tt} = \operatorname{Div}[2\mu_{\text{macro}} \operatorname{sym}(\nabla u) + \lambda_{\text{macro}} \operatorname{tr}(\nabla u) \mathbb{1}]. \tag{3.3}$$

As before, in our study of wave propagation in micromorphic media, we limit ourselves to the case of plane waves travelling in an infinite domain. We suppose that the space dependence of all introduced kinematic fields is limited to a direction defined by a unit vector $\tilde{\xi} \in \mathbb{R}^3$, which is the direction of propagation of the wave. Therefore, we look for solutions of (3.3) in the form

$$u(x,t) = \hat{u}\, \mathrm{e}^{\mathrm{i}(k\langle \tilde{\xi}, x\rangle_{\mathbb{R}^3} - \omega t)}, \quad \hat{u} \in \mathbb{C}^3, \quad \|\tilde{\xi}\|^2 = 1. \tag{3.4}$$

Because our formulation is isotropic, we can, without loss of generality, specify the direction $\tilde{\xi} = e_1$. Then, $X = \langle e_1, x\rangle_{\mathbb{R}^3} = x_1$, and we obtain

$$u(x,t) = \hat{u}\, \mathrm{e}^{\mathrm{i}(kX - \omega t)} \quad \text{and} \quad \hat{u} \in \mathbb{C}^3. \tag{3.5}$$

With this ansatz, it is possible to write (3.3) as

$$A_5(e_1, \omega, k) \cdot \hat{u} = 0 \iff (\mathcal{B}(e_1, k) - \omega^2\, \mathbb{1}) \cdot \hat{u} = 0, \tag{3.6}$$

where

$$A_5(e_1,\omega,k) = \begin{pmatrix} \dfrac{2\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}}}{\rho} k^2 - \omega^2 & 0 & 0 \\ 0 & \dfrac{\mu_{\mathrm{macro}}}{\rho} k^2 - \omega^2 & 0 \\ 0 & 0 & \dfrac{\mu_{\mathrm{macro}}}{\rho} k^2 - \omega^2 \end{pmatrix} \tag{3.7}$$

and

$$\mathcal{B}(e_1,k) = \frac{k^2}{\rho} \begin{pmatrix} 2\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}} & 0 & 0 \\ 0 & \mu_{\mathrm{macro}} & 0 \\ 0 & 0 & \mu_{\mathrm{macro}} \end{pmatrix}. \tag{3.8}$$

Here, we observe that $A_5(e_1, \omega, k)$ is already diagonal and real. Requesting real wave velocities means $\omega^2 \geq 0$. For $k \neq 0$, this leads to the classical so-called *strong ellipticity condition*

$$\mu_{\mathrm{macro}} > 0, \quad 2\,\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}} > 0, \tag{3.9}$$

which is implied by positive definiteness of the energy (3.2).

In classical (linear or nonlinear) elasticity, the condition of real wave propagation (3.9) is equivalent to *strong ellipticity* and *rank-one convexity*. Indeed, rank-one convexity amounts to set ($\xi = k\tilde{\xi}$ with $\|\xi\|^2 = 1$)

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} W(\nabla u + t\hat{u} \otimes \xi) \geq 0 \iff \langle \mathbb{C}(\hat{u} \otimes \xi), \hat{u} \otimes \xi\rangle_{\mathbb{R}^{3\times 3}} \geq 0, \tag{3.10}$$

where $\mathbb{C}$ is the fourth-order elasticity tensor. Condition (3.10) then reads

$$0 \leq 2\mu_{\mathrm{macro}}\|\,\mathrm{sym}\,(\hat{u} \otimes \xi)\|^2 + \lambda_{\mathrm{macro}}(\mathrm{tr}(\hat{u} \otimes \xi))^2 = \mu_{\mathrm{macro}}\|\hat{u}\|^2\|\xi\|^2 + (\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}})\langle \hat{u}, \xi\rangle^2_{\mathbb{R}^3}. \tag{3.11}$$

We may express (3.11) given $\xi \in \mathbb{R}^3$ as a quadratic form in $\hat{u} \in \mathbb{R}^3$, which results in

$$\mu_{\mathrm{macro}}\|\hat{u}\|^2\|\xi\|^2 + (\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}})\langle \hat{u}, \xi\rangle^2_{\mathbb{R}^3} = \langle \mathcal{D}(\xi)\hat{u}, \hat{u}\rangle_{\mathbb{R}^3}, \tag{3.12}$$

where the components of the symmetric and real 3×3 matrix $\mathcal{D}(\xi)$ read

$$\mathcal{D}(\xi) = \begin{pmatrix} (2\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}})\xi_1^2 + \mu_{\mathrm{macro}}(\xi_2^2 + \xi_3^2) & (\lambda_{\mathrm{macro}} + \mu_{\mathrm{macro}})\xi_1\xi_2 \\ (\lambda_{\mathrm{macro}} + \mu_{\mathrm{macro}})\xi_1\xi_2 & (2\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}})\xi_2^2 + \mu_{\mathrm{macro}}(\xi_1^2 + \xi_3^2) \\ (\lambda_{\mathrm{macro}} + \mu_{\mathrm{macro}})\xi_1\xi_3 & (\lambda_{\mathrm{macro}} + \mu_{\mathrm{macro}})\xi_1\xi_2 \\ (\lambda_{\mathrm{macro}} + \mu_{\mathrm{macro}})\xi_1\xi_3 & \\ (\lambda_{\mathrm{macro}} + \mu_{\mathrm{macro}})\xi_2\xi_3 & \\ (2\mu_{\mathrm{macro}} + \lambda_{\mathrm{macro}})\xi_3^2 + \mu_{\mathrm{macro}}(\xi_1^2 + \xi_2^2) & \end{pmatrix}. \tag{3.13}$$

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The three principal invariants are independent of the direction ξ owing to isotropy and are given by

$$\left.\begin{aligned} \operatorname{tr}(\mathcal{D}(\xi)) &= \|\xi\|^2(4\mu_{\text{macro}} + \lambda_{\text{macro}}) = k^2(4\mu_{\text{macro}} + \lambda_{\text{macro}}), \\ \operatorname{tr}(\operatorname{Cof}\mathcal{D}(\xi)) &= \|\xi\|^4\mu_{\text{macro}}(5\mu_{\text{macro}} + 2\lambda_{\text{macro}}) = k^4\mu_{\text{macro}}(5\mu_{\text{macro}} + 2\lambda_{\text{macro}}) \\ \text{and}\qquad \det(\mathcal{D}(\xi)) &= \|\xi\|^6\mu^2_{\text{macro}}(2, \mu_{\text{macro}} + \lambda_{\text{macro}}) = k^6\mu^2_{\text{macro}}(2\mu_{\text{macro}} + \lambda_{\text{macro}}). \end{aligned}\right\} \tag{3.14}$$

Because $\mathcal{D}(\xi)$ is real and symmetric, its eigenvalues are real. The eigenvalues of the matrix $\mathcal{D}(\xi)$ are $k^2(2\mu_{\text{macro}} + \lambda_{\text{macro}})$ and $k^2\mu_{\text{macro}}$ (of multiplicity 2) such that positivity at $k \neq 0$ is satisfied, if and only if[8]

$$\mu_{\text{macro}} > 0, \quad 2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \tag{3.15}$$

which are the usual strong ellipticity conditions. We note here that the latter calculations also show that $\mathcal{B}(e_1) = (1/\rho)k^2\mathcal{D}(e_1)$. Alternatively, one may directly form the so-called *acoustic tensor* $B(\xi) \in \mathbb{R}^{3\times 3}$ by

$$B(\xi)\cdot\hat{u} := [\mathbb{C}(\hat{u}\otimes\xi)]\cdot\xi, \quad \forall\hat{u}\in\mathbb{R}^3; \tag{3.16}$$

in indices we have $(B(\xi))_{ij} = \mathbb{C}_{ikjl}\hat{u}_k\hat{u}_l \neq \mathbb{C}(\xi\otimes\xi)$. With (3.16), we obtain[9]

$$\begin{aligned} \langle\hat{u}, B(\xi)\cdot\hat{u}\rangle_{\mathbb{R}^3} &= \langle\underbrace{[\mathbb{C}(\hat{u}\otimes\xi)]}_{=:\hat{B}\in\mathbb{R}^{3\times 3}}\xi, \cdot\hat{u}\rangle_{\mathbb{R}^3} = \langle\hat{B}\cdot\xi, \hat{u}\rangle_{\mathbb{R}^3} = \langle\hat{B}\cdot(\xi\otimes\hat{u}), \mathbb{1}\rangle_{\mathbb{R}^{3\times 3}} = \langle\hat{B}, (\xi\otimes\hat{u})^{\mathrm{T}}\rangle_{\mathbb{R}^{3\times 3}} \\ &= \langle\hat{B}, \hat{u}\otimes\xi\rangle_{\mathbb{R}^{3\times 3}} = \langle\mathbb{C}(\hat{u}\otimes\xi), \hat{u}\otimes\xi\rangle_{\mathbb{R}^{3\times 3}}, \end{aligned} \tag{3.17}$$

and we see that strong ellipticity $\langle\mathbb{C}(\hat{u}\otimes\xi), \hat{u}\otimes\xi\rangle_{\mathbb{R}^{3\times 3}} > 0$ is equivalent to the positive definiteness of the acoustic tensor $B(\xi)$.

4. A further comparison: the linear Cosserat model

In the isotropic hyperelastic case, the elastic energy density and the kinetic energy density of the Cosserat model read

$$\left.\begin{aligned} W &= \mu_{\text{macro}}\|\operatorname{sym}\nabla u\|^2 + \mu_{\text{c}}\|\operatorname{skew}(\nabla u - A)\|^2 + \frac{\lambda_{\text{macro}}}{2}(\operatorname{tr}(\nabla u))^2 \\ &\quad + \frac{\mu_{\text{macro}}L_{\text{c}}^2}{2}\|\operatorname{Curl}A\|^2 \\ \text{and}\qquad J &= \frac{\rho}{2}\|u_{,t}\|^2 + \frac{\eta}{2}\|A_{,t}\|^2. \end{aligned}\right\} \tag{4.1}$$

Introducing the canonical identification of R^3 with $\mathfrak{so}(3)$, A can be expressed as a function of $a \in \mathbb{R}^3$ as

$$A = \operatorname{anti}(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \tag{4.2}$$

Here, we assume for clarity a uniconstant curvature expression in terms of only $\|\operatorname{Curl}A\|^2$. Strict positive definiteness of the potential energy is equivalent to the following simple relations for the

[8] The eigenvalues of $\mathcal{D}(\xi)$ are independent of the propagation direction $\xi \in \mathbb{R}^3$, which makes sense for the isotropic formulation at hand.

[9] The term $[\mathbb{C}(\hat{u}\otimes\xi)]\cdot(\hat{u}\otimes\xi)$ that in index notation reads $\mathbb{C}_{ijkl}\hat{u}_k\xi_l\hat{u}_j\xi_m$ is different from $\mathbb{C}[(\hat{u}\otimes\xi)\cdot(\hat{u}\otimes\xi)]$, i.e. $\mathbb{C}_{ijkl}\hat{u}_k\xi_m\hat{u}_m\xi_l$.

introduced parameters:

$$2\mu_{\text{macro}} + 3\lambda_{\text{macro}} > 0, \quad \mu_{\text{macro}} > 0, \quad \mu_{\text{c}} > 0 \quad \text{and} \quad L_{\text{c}} > 0. \tag{4.3}$$

The dynamic formulation is obtained by defining a joint Hamiltonian and assuming stationary action. The dynamic equilibrium equations are

$$\left.\begin{aligned} \rho u_{,tt} &= \text{Div}[2\mu_{\text{macro}}\,\text{sym}\,(\nabla u - A) + 2\mu_{\text{c}}\,\text{skew}(\nabla u - A) \\ &\quad + \lambda_{\text{macro}}\,\text{tr}\,(\nabla u - A)\mathbb{1}] \\ \text{and} \qquad \eta A_{,tt} &= -\mu_{\text{macro}} L_{\text{c}}^2\,\text{skew}(\text{Curl}\,\text{Curl}\,A) + 2\,\mu_{\text{c}}\,\text{skew}(\nabla u - A); \end{aligned}\right\} \tag{4.4}$$

see also [40–43] for formulations in terms of axial vectors. Note that, for zero Cosserat couple modulus $\mu_{\text{c}} = 0$, the coupling of the two fields (u, A) is absent, in opposition to the relaxed micromorphic model (equation (2.15)). Considering plane and stationary waves of amplitudes $\hat{u}$ and $\hat{a}$, it is possible to express this system as

$$A_6(\omega, k) \cdot (\hat{u}_1 \quad \hat{a}_1)^{\text{T}} = 0, \quad A_7(\omega, k) \cdot (\hat{u}_2 \quad -\hat{a}_3)^{\text{T}} = 0, \quad A_7(\omega, k) \cdot (\hat{u}_3 \quad \hat{a}_2)^{\text{T}} = 0, \tag{4.5}$$

where

$$A_6(\omega, k) = \begin{pmatrix} \dfrac{k^2(2\mu_{\text{macro}} + \lambda_{\text{macro}})}{\rho - \omega^2} & 0 \\ 0 & \dfrac{(2\mu_{\text{macro}} L_{\text{c}}^2 k^2 + 2\mu_{\text{c}})}{\eta - \omega^2} \end{pmatrix} \tag{4.6}$$

and

$$A_7(\omega, k) = \begin{pmatrix} \dfrac{k^2(\mu_{\text{macro}} + \mu_{\text{c}})}{\rho - \omega^2} & -\dfrac{2\mathrm{i}k\mu_{\text{c}}}{\rho} \\ \dfrac{\mathrm{i}k\mu_{\text{c}}}{\eta} & \dfrac{(k^2\mu_{\text{macro}} L_{\text{c}}^2 + 4\mu_{\text{c}})}{(2\eta)} - \omega^2 \end{pmatrix}. \tag{4.7}$$

As done in the case of the relaxed micromorphic model, it is possible to express equivalently the problem with $A_6(\omega, k)$ and the following symmetric matrix:

$$\bar{A}_7(k) = \text{diag}_7 \cdot A_7(\omega, k) \cdot \text{diag}_7^{-1} = \begin{pmatrix} \dfrac{k^2(\mu_{\text{macro}} + \mu_{\text{c}})}{\rho - \omega^2} & \dfrac{\sqrt{2}k\mu_{\text{c}}}{\sqrt{\rho\eta}} \\ \dfrac{\sqrt{2}k\mu_{\text{c}}}{\sqrt{\rho\eta}} & \dfrac{(k^2\mu_{\text{macro}} L_{\text{c}}^2 + 4\mu_{\text{c}})}{(2\eta) - \omega^2} \end{pmatrix}, \tag{4.8}$$

where

$$\text{diag}_7 = \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & \mathrm{i}\sqrt{2\eta} \end{pmatrix}. \tag{4.9}$$

Because ω^2 appears only on the diagonal, the problem can be analogously expressed as the following eigenvalue problems:

$$\det(B_6(k) - \omega^2\mathbb{1}) = 0 \quad \text{and} \quad \det(B_7(k) - \omega^2\mathbb{1}) = 0, \tag{4.10}$$

where

$$B_6(k) = \begin{pmatrix} \dfrac{k^2(2\mu_{\text{macro}} + \lambda_{\text{macro}})}{\rho} & 0 \\ 0 & \dfrac{(2\mu_{\text{macro}} L_{\text{c}}^2 k^2 + 2\mu_{\text{c}})}{\eta^2} \end{pmatrix} \tag{4.11}$$

and

$$B_7(k) = \begin{pmatrix} \dfrac{k^2(\mu_{\text{macro}} + \mu_c)}{\rho} & \dfrac{\sqrt{2}k\mu_c}{\sqrt{\rho\eta}} \\ \dfrac{\sqrt{2}k\mu_c}{\sqrt{\rho\eta}} & \dfrac{(k^2\mu_{\text{macro}}L_c^2 + 4\mu_c)}{(2\eta)} \end{pmatrix} \tag{4.12}$$

are the blocks of the *acoustic tensor B*

$$B(k) = \begin{pmatrix} B_6 & 0 & 0 \\ 0 & B_7 & 0 \\ 0 & 0 & B_7 \end{pmatrix}. \tag{4.13}$$

The eigenvalues of the matrix $B_6(k)$ are simply the elements of the diagonal; therefore, we have

$$\omega_{\text{acoustic, long}}(k) = k\sqrt{\frac{2\mu_{\text{macro}} + \lambda_{\text{macro}}}{\rho}} \quad \text{and} \quad \omega_{\text{optic, long}}(k) = \sqrt{\frac{2\mu_{\text{macro}}L_c^2k^2 + 2\mu_c}{\eta}}, \tag{4.14}$$

whereas for $B_7(k)$ it is possible to find

$$\omega_{\text{acoustic, trans}}(k) = \sqrt{a(k) - \sqrt{a(k)^2 - b^2k^2}} \quad \text{and} \quad \omega_{\text{optic, trans}}(k) = \sqrt{a(k) + \sqrt{a(k)^2 - b^2k^2}}, \tag{4.15}$$

where we have set

$$a(k) = \frac{4\mu_c + \mu_{\text{macro}}L_c^2k^2}{\eta} + 2\frac{\mu_{\text{macro}} + \mu_c}{\rho}k^2 \quad \text{and} \quad b^2 = 8\frac{\mu_{\text{macro}}(4\mu_c + k^2L_c^2(\mu_{\text{macro}} + \mu_c))}{\rho\eta}. \tag{4.16}$$

The acoustic branches are those curves $\omega = \omega(k)$ as solutions of (4.9) that satisfy $\omega(0) = 0$. We note here that the acoustic branches of the longitudinal and transverse dispersion curves have, as tangent in $k = 0$,[10]

$$c_l = \frac{d\omega_{\text{acoustic, long}}(0)}{dk} = \sqrt{\frac{2\mu_{\text{macro}} + \lambda_{\text{macro}}}{\rho}} \quad \text{and} \quad c_t = \frac{d\omega_{\text{acoustic, trans}}(0)}{dk} = \sqrt{\frac{\mu_{\text{macro}}}{\rho}}, \tag{4.17}$$

respectively. Moreover, the longitudinal acoustic branch is non-dispersive, i.e. a straight line with slope $(4.17)_1$. The matrix $B_6(k)$ is positive definite for arbitrary $k \neq 0$ if

$$2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \quad \mu_{\text{macro}} > 0, \quad \mu_c \geq 0. \tag{4.18}$$

Using the Sylvester criterion, $B_7(k)$ is positive definite, if and only if the principal minors are positive, namely

$$\left.\begin{aligned} (B_7)_{11} &= k^2\frac{(\mu_{\text{macro}} + \mu_c)}{\rho} > 0 \\ \text{and} \qquad \det(B_7) &= \frac{k^2}{2\eta\rho}(4\mu_{\text{macro}}\mu_c + k^2\mu_{\text{macro}}L_c^2(\mu_{\text{macro}} + \mu_c)) > 0, \end{aligned}\right\} \tag{4.19}$$

from which we obtain the condition

$$\mu_{\text{macro}} + \mu_c > 0, \quad \mu_{\text{macro}} > 0 \quad \text{and} \quad \mu_c \geq 0. \tag{4.20}$$

[10] To obtain the slopes in 0, it is possible to search for a solution of the type $\omega = a\,k$ and then evaluate the limit for $a \to 0$; see [34] for a thorough explanation in the relaxed micromorphic case.

Considering these two sets of conditions, it is possible to state a *necessary* and *sufficient* condition for the positive definiteness of $B_6(k)$ and $B_7(k)$ and therefore of the acoustic tensor $B(k)$

$$2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \quad \mu_{\text{macro}} > 0 \quad \text{and} \quad \mu_c \geq 0, \tag{4.21}$$

which are implied by the positive definiteness of the energy (4.3). Eringen [12, p. 150] also obtains correctly (4.18) *and* (4.20) (in his notation $\mu_c = \kappa/2$, $\mu_{\text{macro}} = \mu_{\text{Eringen}} + \kappa/2$).

In [44,45], strong ellipticity for the Cosserat micropolar model is defined and investigated. In this respect, we note that ellipticity is connected to acceleration waves, whereas our investigation concerns real wave velocities for planar waves. Similarly to [46], it is established in [44,45] that strong ellipticity for the micropolar model holds if and only if (the uniconstant curvature case in our notation)

$$2\mu_{\text{macro}} + \lambda_{\text{macro}} > 0 \quad \text{and} \quad \mu_{\text{macro}} + \mu_c > 0. \tag{4.22}$$

We conclude that, for micropolar material models (and therefore also for micromorphic materials), strong ellipticity (4.22) is too weak to ensure real planar waves because it is implied by, but does not imply, (4.21). This fact seems to have been appreciated also in the study of the Cosserat model [47–51].

5. Conclusion

In this paper, we derive the set of necessary and sufficient conditions that have to be imposed on the constitutive parameters of the relaxed micromorphic model in order to guarantee

— positive definiteness;
— real wave velocity; and
— Legendre–Hadamard strong ellipticity condition.

We show that if, on the one hand, definite positiveness implies real wave propagation, on the other hand, real wave propagation is not guaranteed by the strong ellipticity condition.

We conclude that in strong contrast to the case of classical isotropic linear elasticity, where the three concepts are known to be equivalent, in the case of the relaxed micromorphic continua only definite positiveness of the strain energy density can be considered to be a good criterion to guarantee real wave speeds in the considered media. The proposed considerations can be extended to all generalized continua where the equivalence between the three notions is far from being straightforward.

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Appendix A

(a) Inequality relations between material parameters

The formulae in §2a are based on the harmonic mean of two numbers κ_e and κ_{micro} (or μ_e and μ_{micro}). If the two numbers are positive, it is easy to see that

$$\kappa_{\text{macro}} \leq \min(\kappa_e, \kappa_{\text{micro}}). \tag{A 1}$$

Here, we show that the same conclusion still holds if we merely assume that $\kappa_e + \kappa_{\text{micro}} > 0$. This allows for either $\kappa_e < 0$ or $\kappa_{\text{micro}} < 0$. Therefore, considering that $\kappa_e + \kappa_{\text{micro}} > 0$, even if the energy is not strictly positive, it is possible to derive that

$$\left.\begin{aligned} \kappa_{\text{macro}} &= \frac{\kappa_{\text{micro}}\kappa_{\text{e}}}{\kappa_{\text{e}}+\kappa_{\text{micro}}} = \frac{\kappa_{\text{micro}}\kappa_{\text{e}}+\kappa_{\text{e}}^2-\kappa_{\text{e}}^2}{\kappa_{\text{e}}+\kappa_{\text{micro}}} = \kappa_{\text{e}}\frac{\kappa_{\text{micro}}+\kappa_{\text{e}}}{\kappa_{\text{e}}+\kappa_{\text{micro}}} - \frac{\kappa_{\text{e}}^2}{\kappa_{\text{e}}+\kappa_{\text{micro}}} \\ &= \kappa_{\text{e}} \underbrace{-\frac{\kappa_{\text{e}}^2}{\kappa_{\text{e}}+\kappa_{\text{micro}}}}_{\leq 0} \leq \kappa_{\text{e}} \\ \text{and}\quad \kappa_{\text{macro}} &= \frac{\kappa_{\text{micro}}\kappa_{\text{e}}}{\kappa_{\text{e}}+\kappa_{\text{micro}}} = \frac{\kappa_{\text{micro}}\kappa_{\text{e}}+\kappa_{\text{micro}}^2-\kappa_{\text{micro}}^2}{\kappa_{\text{e}}+\kappa_{\text{micro}}} = \kappa_{\text{micro}}\frac{\kappa_{\text{micro}}+\kappa_{\text{e}}}{\kappa_{\text{e}}+\kappa_{\text{micro}}} - \frac{\kappa_{\text{micro}}^2}{\kappa_{\text{e}}+\kappa_{\text{micro}}} \\ &= \kappa_{\text{micro}} \underbrace{-\frac{\kappa_{\text{micro}}^2}{\kappa_{\text{e}}+\kappa_{\text{micro}}}}_{\leq 0} \leq \kappa_{\text{micro}}. \end{aligned}\right\} \tag{A 2}$$

Considering similarly $\mu_{\text{e}}+\mu_{\text{micro}}>0$, it is possible to obtain

$$\left.\begin{aligned} \mu_{\text{macro}} &= \frac{\mu_{\text{micro}}\mu_{\text{e}}}{\mu_{\text{e}}+\mu_{\text{micro}}} = \frac{\mu_{\text{micro}}\mu_{\text{e}}+\mu_{\text{e}}^2-\mu_{\text{e}}^2}{\mu_{\text{e}}+\mu_{\text{micro}}} = \mu_{\text{e}}\frac{\mu_{\text{micro}}+\mu_{\text{e}}}{\mu_{\text{e}}+\mu_{\text{micro}}} - \frac{\mu_{\text{e}}^2}{\mu_{\text{e}}+\mu_{\text{micro}}} \\ &= \mu_{\text{e}} \underbrace{-\frac{\mu_{\text{e}}^2}{\mu_{\text{e}}+\mu_{\text{micro}}}}_{\leq 0} \leq \mu_{\text{e}} \\ \text{and}\quad \mu_{\text{macro}} &= \frac{\mu_{\text{micro}}\mu_{\text{e}}}{\mu_{\text{e}}+\mu_{\text{micro}}} = \frac{\mu_{\text{micro}}\mu_{\text{e}}+\mu_{\text{micro}}^2-\mu_{\text{micro}}^2}{\mu_{\text{e}}+\mu_{\text{micro}}} \\ &= \mu_{\text{micro}}\frac{\mu_{\text{micro}}+\mu_{\text{e}}}{\mu_{\text{e}}+\mu_{\text{micro}}} - \frac{\mu_{\text{micro}}^2}{\mu_{\text{e}}+\mu_{\text{micro}}} = \mu_{\text{micro}} \underbrace{-\frac{\mu_{\text{micro}}^2}{\mu_{\text{e}}+\mu_{\text{micro}}}}_{\leq 0} \leq \mu_{\text{micro}}. \end{aligned}\right\} \tag{A 3}$$

Therefore, if $\mu_{\text{e}}+\mu_{\text{micro}}>0$ and $\kappa_{\text{e}}+\kappa_{\text{micro}}>0$, the macroscopic parameters are less than or equal to the respective microscopic parameters, namely

$$\kappa_{\text{e}} \geq \kappa_{\text{macro}}, \quad \kappa_{\text{micro}} \geq \kappa_{\text{macro}}, \quad \mu_{\text{e}} \geq \mu_{\text{macro}}, \quad \mu_{\text{micro}} \geq \mu_{\text{macro}}, \tag{A 4}$$

and it is possible to show that

$$\left.\begin{aligned} & 2\mu_{\text{e}}+\lambda_{\text{e}} = \tfrac{1}{3}(4\mu_{\text{e}}+3\kappa_{\text{e}}) \geq \tfrac{1}{3}(4\mu_{\text{macro}}+3\kappa_{\text{macro}}) = 2\mu_{\text{macro}}+\lambda_{\text{macro}} > 0, \\ & 2\mu_{\text{micro}}+\lambda_{\text{micro}} = \tfrac{1}{3}(4\mu_{\text{micro}}+3\kappa_{\text{micro}}) \geq \tfrac{1}{3}(4\mu_{\text{macro}}+3\kappa_{\text{macro}}) = 2\mu_{\text{macro}}+\lambda_{\text{macro}} > 0, \\ & (2\mu_{\text{e}}+\lambda_{\text{e}})+(2\mu_{\text{micro}}+\lambda_{\text{micro}}) \geq 2(2\mu_{\text{macro}}+\lambda_{\text{macro}}) > 0 \\ \text{and}\quad & 4\mu_{\text{macro}}+3\kappa_{\text{e}} \geq 4\mu_{\text{macro}}+3\kappa_{\text{macro}} = 3(2\mu_{\text{macro}}+\lambda_{\text{macro}}) > 0. \end{aligned}\right\} \tag{A 5}$$

Therefore, the set of inequalities (2.50) is implied from the smaller set

$$\boxed{\mu_{\text{e}}>0, \quad \mu_{\text{micro}}>0, \quad \mu_{\text{c}}\geq 0, \quad \kappa_{\text{e}}+\kappa_{\text{micro}}>0 \quad \text{and} \quad 2\mu_{\text{macro}}+\lambda_{\text{macro}}>0.} \tag{A 6}$$

We note here that $3(2\mu_{\text{e}}+\lambda_{\text{e}}) \geq 4\mu_{\text{macro}}+3\kappa_{\text{e}} \geq 3(2\mu_{\text{macro}}+\lambda_{\text{macro}})$, because

$$3(2\mu_{\text{e}}+\lambda_{\text{e}}) = 4\mu_{\text{e}}+3\kappa_{\text{e}} \geq 4\mu_{\text{macro}}+3\kappa_{\text{e}} \geq 4\mu_{\text{macro}}+3\kappa_{\text{macro}} = 3(2\mu_{\text{macro}}+\lambda_{\text{macro}}). \tag{A 7}$$

(b) The 12 × 12 acoustic tensor for arbitrary direction

We suppose that the space dependence of all introduced kinematic fields is limited to a direction defined by a unit vector ξ, which is the direction of propagation of the wave. Therefore, we look for solutions of

$$\left.\begin{aligned} \rho u_{,tt} &= \operatorname{Div}[2\mu_e \operatorname{sym}(\nabla u - P) + 2\mu_c \operatorname{skew}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\mathbb{1}] \\ \text{and} \qquad \eta P_{,tt} &= -\mu_e L_c^2 \operatorname{Curl}\operatorname{Curl} P + 2\mu_e \operatorname{sym}(\nabla u - P) + 2\mu_c \operatorname{skew}(\nabla u - P) \\ &\quad + \lambda_e \operatorname{tr}(\nabla u - P)\mathbb{1} - [2\mu_{\text{micro}} \operatorname{sym} P + \lambda_{\text{micro}} \operatorname{tr}(P)\mathbb{1}], \end{aligned}\right\} \tag{A 8}$$

in the form

$$\left.\begin{aligned} u(x,t) &= \hat{u}\, \underbrace{e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)}}_{s(x,t)\in\mathbb{R}/\mathbb{C} \text{ scalar}}, \quad \hat{u} \in \mathbb{C}^3, \quad \|\xi\|^2 = 1 \\ \text{and} \qquad P(x,t) &= \hat{P}\, \underbrace{e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)}}_{s(x,t)\in\mathbb{R}/\mathbb{C} \text{ scalar}}, \quad \hat{P} \in \mathbb{C}^{3\times 3}, \end{aligned}\right\} \tag{A 9}$$

where $\hat{u}$ is the polarization vector and $\hat{P}$ is the polarization matrix. We start by remarking that, considering $A, B \in \mathbb{R}^{3\times 3}$, we have

$$\operatorname{Curl}(A \cdot B) = L_B(\nabla A) + A \cdot \operatorname{Curl}(B), \tag{A 10}$$

where $L_B : \mathbb{R}^{27} \to \mathbb{R}^{3\times 3}$ is a linear operator with constant coefficients defined by the appropriate product rule of differentiation. Therefore, we obtain

$$\operatorname{Curl}(\hat{P}s(x,t)) = \operatorname{Curl}(\hat{P} \cdot \mathbb{1}s(x,t)) = \hat{P} \cdot \operatorname{Curl}(\mathbb{1}s(x,t)), \tag{A 11}$$

where

$$\operatorname{Curl}(\mathbb{1}s(x,t)) = \begin{pmatrix} 0 & \partial_3 s(x,t) & \partial_2 s(x,t) \\ -\partial_3 s(x,t) & 0 & \partial_1 s(x,t) \\ \partial_2 s(x,t) & -\partial_1 s(x,t) & 0 \end{pmatrix} \in \mathfrak{so}(3). \tag{A 12}$$

The derivatives of $s(x,t)$ can be evaluated by considering

$$\nabla_x s(x,t) = \begin{pmatrix} \partial_1 s(x,t) \\ \partial_2 s(x,t) \\ \partial_3 s(x,t) \end{pmatrix} = e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)} \begin{pmatrix} ik\xi_1 \\ ik\xi_2 \\ ik\xi_3 \end{pmatrix} = e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)} ik\xi = ik\xi\, s(x,t). \tag{A 13}$$

It can be noted that

$$\operatorname{Curl}(s(x,t)\,\mathbb{1}) = \operatorname{anti}(\nabla s(x,t)) = e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)} ik \operatorname{anti}(\xi) = s(x,t) ik \operatorname{anti}(\xi). \tag{A 14}$$

Therefore, it is possible to evaluate the Curl Curl P as

$$\begin{aligned} \operatorname{Curl}\operatorname{Curl}(\hat{P}s(x,t)) &= \operatorname{Curl}(\hat{P} \cdot \underbrace{\operatorname{anti}(\xi)}_{\in\mathfrak{so}(3)} iks(x,t)) = ik \operatorname{Curl}([\hat{P} \cdot \operatorname{anti}(\xi)] \cdot \mathbb{1}s(x,t)) \\ &= ik\hat{P} \cdot \operatorname{anti}(\xi) \operatorname{Curl}(\mathbb{1}s(x,t)) \\ &= ikik\hat{P} \cdot \operatorname{anti}(\xi) \cdot \operatorname{anti}(\xi) s(x,t) = -k^2\, \hat{P} \cdot \operatorname{anti}(\xi) \cdot \operatorname{anti}(\xi)\, e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)}. \end{aligned} \tag{A 15}$$

On the other hand, the second derivative of P with respect to time is

$$P_{,tt} = \partial_t^2 (\hat{P}\, e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)}) = -\omega^2 \hat{P}\, e^{i(k\langle \xi, x\rangle_{\mathbb{R}^3} - \omega t)}) = -\omega^2 \hat{P}s(x,t). \tag{A 16}$$

Analogously for u, it is possible to evaluate the gradient and the derivatives with respect to time as

$$\nabla_x u = iks(x,t)\hat{u} \otimes \xi, \quad u_{,tt} = -\omega^2 \hat{u}s(x,t). \tag{A 17}$$

The sym, skew and tr of $\nabla u - P$ can then be expressed as

$$\left.\begin{aligned} \operatorname{sym}(\nabla u - P) &= \operatorname{sym}(ik\hat{u} \otimes \xi - \hat{P})s(x,t) = (ik \operatorname{sym}(\hat{u} \otimes \xi) - \operatorname{sym}\hat{P})s(x,t), \\ \operatorname{skew}(\nabla u - P) &= \operatorname{skew}(ik\hat{u} \otimes \xi - \hat{P})s(x,t) = (ik \operatorname{skew}(\hat{u} \otimes \xi) - \operatorname{skew}\hat{P})s(x,t) \\ \text{and} \qquad \operatorname{tr}(\nabla u - P) &= \operatorname{tr}(ik\hat{u} \otimes \xi - \hat{P})s(x,t) = (ik\langle \hat{u}, \xi\rangle_{\mathbb{R}^3} - \operatorname{tr}\hat{P})s(x,t). \end{aligned}\right\} \tag{A 18}$$

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Therefore, we have

$$\left.\begin{aligned}
\operatorname{Div}\operatorname{sym}(\nabla u - P) &= \operatorname{Div}[(ik\operatorname{sym}(\hat{u}\otimes\xi) - \operatorname{sym}\hat{P})s(x,t)]\\
&= (ik\operatorname{sym}(\hat{u}\otimes\xi) - \operatorname{sym}\hat{P})\cdot\nabla_x s(x,t)\\
&= (ik\operatorname{sym}(\hat{u}\otimes\xi) - \operatorname{sym}\hat{P})\cdot(ik\xi s(x,t))\\
&= -(k^2\operatorname{sym}(\hat{u}\otimes\xi)\cdot\xi + ik\operatorname{sym}\hat{P}\cdot\xi)s(x,t),\\
\operatorname{Div}\operatorname{skew}(\nabla u - P) &= \operatorname{Div}[(ik\operatorname{skew}(\hat{u}\otimes\xi) - \operatorname{skew}\hat{P})s(x,t)]\\
&= (ik\operatorname{skew}(\hat{u}\otimes\xi) - \operatorname{skew}\hat{P})\cdot\nabla_x s(x,t)\\
&= (ik\operatorname{skew}(\hat{u}\otimes\xi) - \operatorname{skew}\hat{P})\cdot(ik\xi s(x,t))\\
&= -(k^2\operatorname{skew}(\hat{u}\otimes\xi)\cdot\xi + ik\operatorname{skew}\hat{P}\cdot\xi)s(x,t)\\
\text{and}\qquad \operatorname{Div}(\operatorname{tr}(\nabla u - P)\,\mathbb{1}) &= \operatorname{Div}[((ik\langle\hat{u},\xi\rangle_{\mathbb{R}^3} - \operatorname{tr}\hat{P})\,\mathbb{1})s(x,t)]\\
&= (ik\langle\hat{u},\xi\rangle_{\mathbb{R}^3} - \operatorname{tr}\hat{P})\,\mathbb{1}\cdot\nabla_x s(x,t)\\
&= (ik\langle\hat{u},\xi\rangle_{\mathbb{R}^3} - \operatorname{tr}\hat{P})\,\mathbb{1}\cdot(ik\xi s(x,t))\\
&= -(k^2\langle\hat{u},\xi\rangle_{\mathbb{R}^3} + ik\operatorname{tr}\hat{P})\xi s(x,t).
\end{aligned}\right\}\tag{A 19}$$

Here, we have considered that, given a generic $B\in\mathbb{R}^{3\times3}$ and a scalar $s(x,t)$, we have

$$\operatorname{Div}[B\,s(x,t)] = \underbrace{\operatorname{Div}[B]}_{=0}s(x,t) + B\cdot\nabla_x s(x,t).\tag{A 20}$$

With all the formulae obtained, it is possible to write (A 8) simplifying $s(x,t)$ everywhere as

$$\left.\begin{aligned}
-\rho\omega^2\hat{u} &= -[2\mu_e(k^2\operatorname{sym}(\hat{u}\otimes\xi)\cdot\xi + ik\operatorname{sym}\hat{P}\cdot\xi))\\
&\quad + 2\mu_c(k^2\operatorname{skew}(\hat{u}\otimes\xi)\cdot\xi + ik\operatorname{skew}\hat{P}\cdot\xi)\\
&\quad + \lambda_e(k^2\langle\hat{u},\xi\rangle_{\mathbb{R}^3} + ik\operatorname{tr}\hat{P})\xi]\\
\text{and}\qquad -\eta\omega^2\hat{P} &= \mu_e L_c^2 k^2\hat{P}\operatorname{anti}(\xi)\cdot\operatorname{anti}(\xi) + 2\mu_e(ik\operatorname{sym}(\hat{u}\otimes\xi) - \operatorname{sym}\hat{P})\\
&\quad + 2\mu_c(ik\operatorname{skew}(\hat{u}\otimes\xi) - \operatorname{skew}\hat{P})\\
&\quad + \lambda_e(ik\langle\hat{u},\xi\rangle_{\mathbb{R}^3} - \operatorname{tr}\hat{P})\mathbb{1} - [2\mu_{\text{micro}}\operatorname{sym}\hat{P} + \lambda_{\text{micro}}\operatorname{tr}(\hat{P})\mathbb{1}],
\end{aligned}\right\}\tag{A 21}$$

or analogously

$$\left.\begin{aligned}
&-\rho\omega^2\hat{u} + k^2(2\mu_e\operatorname{sym}(\hat{u}\otimes\xi)\cdot\xi + 2\mu_c\operatorname{skew}(\hat{u}\otimes\xi)\cdot\xi + \lambda_e\langle\hat{u},\xi\rangle_{\mathbb{R}^3}\xi)\\
&\quad + ik(2\mu_e\operatorname{sym}\hat{P}\cdot\xi + 2\mu_c\operatorname{skew}\hat{P}\cdot\xi + \lambda_e\operatorname{tr}\hat{P}\xi) = 0\\
\text{and}\qquad &-\eta\omega^2\hat{P} - \mu_e L_c^2 k^2\hat{P}\operatorname{anti}(\xi)\cdot\operatorname{anti}(\xi)\\
&\quad + 2(\mu_e + \mu_{\text{micro}})\operatorname{sym}\hat{P} + 2\mu_c\operatorname{skew}\hat{P} + (\lambda_e + \lambda_{\text{micro}})\operatorname{tr}(\hat{P})\mathbb{1}\\
&\quad - 2\mu_e ik\operatorname{sym}(\hat{u}\otimes\xi) - 2\mu_c ik\operatorname{skew}(\hat{u}\otimes\xi) - \lambda_e ik\langle\hat{u},\xi\rangle_{\mathbb{R}^3}\mathbb{1} = 0.
\end{aligned}\right\}\tag{A 22}$$

At given $\xi \in \mathbb{R}^3$, this is a linear system in $(\hat{u}, \hat{P}) \in \mathbb{C}^{12}$ which can be written in 12×12 matrix format as

$$\left(\tilde{A}(\xi,\omega,k)\right)\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\\ \hat{P}_{11}\\ \hat{P}_{12}\\ \hat{P}_{13}\\ \hat{P}_{21}\\ \hat{P}_{22}\\ \hat{P}_{23}\\ \hat{P}_{31}\\ \hat{P}_{32}\\ \hat{P}_{33}\end{pmatrix}=0,\quad \left(\tilde{\mathcal{B}}(\xi,k)-\omega^2\,\mathbb{1}\right)\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\\ \hat{P}_{11}\\ \hat{P}_{12}\\ \hat{P}_{13}\\ \hat{P}_{21}\\ \hat{P}_{22}\\ \hat{P}_{23}\\ \hat{P}_{31}\\ \hat{P}_{32}\\ \hat{P}_{33}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}. \tag{A 23}$$

Here, $\tilde{\mathcal{B}}(\xi,k)$ is the 12×12 acoustic tensor. The columns of $\tilde{A}$ are

$$\tilde{A}_{i1}=\begin{pmatrix}\rho\omega^2-k^2(\lambda_e+2\mu_e)\xi_1^2-k^2(\mu_c+\mu_e)(\xi_2^2+\xi_3^2)\\ -k^2(\lambda_e-\mu_c+\mu_e)\xi_1\xi_2\\ -k^2(\lambda_e-\mu_c+\mu_e)\xi_1\xi_3\\ ik(\lambda_e+2\mu_e)\xi_1\\ ik(\mu_c+\mu_e)\xi_2\\ ik(\mu_c+\mu_e)\xi_3\\ -ik(\mu_c-\mu_e)\xi_2\\ ik\lambda_e\xi_1\\ 0\\ -ik(\mu_c-\mu_e)\xi_3\\ 0\\ ik\lambda_e\xi_1\end{pmatrix},\quad \tilde{A}_{i2}=\begin{pmatrix}-k^2(\lambda_e-\mu_c+\mu_e)\xi_1\xi_2\\ \rho\omega^2-k^2(\lambda_e+2\mu_e)\xi_2^2-k^2(\mu_c+\mu_e)(\xi_1^2+\xi_3^2)\\ -k^2(\lambda_e-\mu_c+\mu_e)\xi_2\xi_3\\ ik\lambda_e\xi_2\\ -ik(\mu_c-\mu_e)\xi_1\\ 0\\ ik(\mu_c+\mu_e)\xi_1\\ ik(\lambda_e+2\mu_e)\xi_2\\ ik(\mu_c+\mu_e)\xi_3\\ 0\\ -ik(\mu_c-\mu_e)\xi_3\\ ik\lambda_e\xi_2\end{pmatrix},$$

$$\tilde{A}_{i3}=\begin{pmatrix}-k^2(\lambda_e-\mu_c+\mu_e)\xi_1\xi_3\\ -k^2(\lambda_e-\mu_c+\mu_e)\xi_2\xi_3\\ \rho\omega^2-k^2(\lambda_e+2\mu_e)\xi_3^2-k^2(\mu_c+\mu_e)(\xi_1^2+\xi_2^2)\\ ik\lambda_e\xi_3\\ 0\\ -ik(\mu_c-\mu_e)\xi_1\\ 0\\ ik\lambda_e\xi_3\\ -ik(\mu_c-\mu_e)\xi_2\\ ik(\mu_c+\mu_e)\xi_1\\ ik(\mu_c+\mu_e)\xi_2\\ ik(\lambda_e+2\mu_e)\xi_3\end{pmatrix},\quad \tilde{A}_{i4}=\begin{pmatrix}-ik(\lambda_e+2\mu_e)\xi_1\\ -ik\lambda_e\xi_2\\ -ik\lambda_e\xi_3\\ \eta\omega^2-(2(\mu_e+\mu_{\text{micro}})+\lambda_e+\lambda_{\text{micro}})\\ \quad -k^2\mu_e L_c^2(\xi_2^2+\xi_3^3)\\ k^2\mu_e L_c^2\xi_1\xi_2\\ k^2\mu_e L_c^2\xi_1\xi_3\\ 0\\ -(\lambda_e+\lambda_{\text{micro}})\\ 0\\ 0\\ 0\\ -(\lambda_e+\lambda_{\text{micro}})\end{pmatrix},$$

$$\tilde{A}_{i5}=\begin{pmatrix}-ik(\mu_c+\mu_e)\xi_2\\ ik(\mu_c-\mu_e)\xi_1\\ 0\\ k^2\mu_e L_c^2\xi_1\xi_2\\ \eta\omega^2-(\mu_c+\mu_e+\mu_{\text{micro}})-k^2\mu_e L_c^2(\xi_1^2+\xi_3^2)\\ k^2\mu_e L_c^2\xi_1\xi_2\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix},\quad \tilde{A}_{i6}=\begin{pmatrix}-ik(\mu_c+\mu_e)\xi_3\\ 0\\ ik(\mu_c-\mu_e)\xi_1\\ k^2\mu_e L_c^2\xi_1\xi_3\\ k^2\mu_e L_c^2\xi_2\xi_3\\ \eta\omega^2-(\mu_c+\mu_e+\mu_{\text{micro}})-k^2\mu_e L_c^2(\xi_1^2+\xi_2^2)\\ 0\\ 0\\ 0\\ \mu_c-\mu_e-\mu_{\text{micro}}\\ 0\\ 0\end{pmatrix},$$

$$\tilde{A}_{i7} = \begin{pmatrix} ik(\mu_c - \mu_e)\xi_2 \\ -ik(\mu_c + \mu_e)\xi_1 \\ 0 \\ 0 \\ \mu_c - \mu_e - \mu_{micro} \\ 0 \\ \eta\omega^2 - (\mu_c + \mu_e + \mu_{micro}) - k^2\mu_e L_c^2(\xi_2^2 + \xi_3^2) \\ k^2\mu_e L_c^2\xi_1\xi_2 \\ k^2\mu_e L_c^2\xi_1\xi_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{A}_{i8} = \begin{pmatrix} -ik\lambda_e\xi_1 \\ -ik(2\mu_e + \lambda_e)\xi_2 \\ -ik\lambda_e\xi_3 \\ -\lambda_e - \lambda_{micro} \\ 0 \\ 0 \\ k^2\mu_e L_c^2\xi_1\xi_2 \\ \eta\omega^2 - (\mu_c + \mu_e + \mu_{micro}) - k^2\mu_e L_c^2(\xi_1^2 + \xi_3^2) \\ k^2\mu_e L_c^2\xi_2\xi_3 \\ 0 \\ 0 \\ -\lambda_e - \lambda_{micro} \end{pmatrix},$$

$$\tilde{A}_{i9} = \begin{pmatrix} 0 \\ -ik(\mu_c + \mu_e)\xi_3 \\ ik(\mu_c + \mu_e)\xi_2 \\ 0 \\ 0 \\ 0 \\ k^2\mu_e L_c^2\xi_1\xi_3 \\ k^2\mu_e L_c^2\xi_2\xi_3 \\ \eta\omega^2 - (\mu_c + \mu_e + \mu_{micro}) - k^2\mu_e L_c^2(\xi_1^2 + \xi_2^2) \\ 0 \\ \mu_c - \mu_e - \mu_{micro} \\ 0 \end{pmatrix}, \quad \tilde{A}_{i10} = \begin{pmatrix} ik(\mu_c - \mu_e)\xi_3 \\ 0 \\ -ik(\mu_c + \mu_e)\xi_1 \\ 0 \\ 0 \\ \mu_c - \mu_e - \mu_{micro} \\ 0 \\ 0 \\ 0 \\ \eta\omega^2 - (\mu_c + \mu_e + \mu_{micro}) - k^2\mu_e L_c^2(\xi_2^2 + \xi_3^2) \\ k^2\mu_e L_c^2\xi_1\xi_2 \\ k^2\mu_e L_c^2\xi_1\xi_3 \end{pmatrix},$$

$$\tilde{A}_{i11} = \begin{pmatrix} 0 \\ ik(\mu_c - \mu_e)\xi_3 \\ -ik(\mu_c + \mu_e)\xi_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mu_c - \mu_e - \mu_{micro} \\ k^2\mu_e L_c^2\xi_1\xi_2 \\ \eta\omega^2 - (\mu_c + \mu_e + \mu_{micro}) - k^2\mu_e L_c^2(\xi_1^2 + \xi_3^2) \\ k^2\mu_e L_c^2\xi_2\xi_3 \end{pmatrix}, \quad \tilde{A}_{i12} = \begin{pmatrix} -ik\lambda_e\xi_1 \\ -ik\lambda_e\xi_2 \\ -ik(\lambda_e + 2\mu_e)\xi_3 \\ -\lambda_e - \lambda_{micro} \\ 0 \\ 0 \\ 0 \\ -\lambda_e - \lambda_{micro} \\ 0 \\ k^2\mu_e L_c^2\xi_1\xi_3 \\ k^2\mu_e L_c^2\xi_2\xi_3 \\ \eta\omega^2 - (\mu_c + \mu_e + \mu_{micro}) - k^2\mu_e L_c^2(\xi_1^2 + \xi_2^2) \end{pmatrix}.$$

It is clear that, even with the aid of up-to-date computer algebra systems, it is practically impossible to determine the positive definiteness of the 12×12 acoustic tensor $\tilde{\mathcal{B}}$ as dependent on the given material parameters. In the main body of our paper, we succeed by choosing immediately the propagation direction $\xi = e_1$ and by considering a set of new variables (2.16). This allows us to obtain a certain pre-factorization of $\tilde{\mathcal{B}}(e_1, k)$ in 3×3 blocks. Because the formulation is isotropic, choosing $\xi = e_1$ is no restriction, as argued before.

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