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On the role of micro-inertia in enriched continuum mechanics

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In this paper, the role of gradient micro-inertia terms $\bar{\eta}\|\nabla u_{,t}\|^2$ and free micro-inertia terms $\eta\|P_{,t}\|^2$ is investigated to unveil their respective effects on the dynamic behaviour of band-gap metamaterials. We show that the term $\bar{\eta}\|\nabla u_{,t}\|^2$ alone is only able to disclose relatively simplified dispersive behaviour. On the other hand, the term $\eta\|P_{,t}\|^2$ alone describes the full complex behaviour of band-gap metamaterials. A suitable mixing of the two micro-inertia terms allows us to describe a new feature of the relaxed-micromorphic model, i.e. the description of a second band-gap occurring for higher frequencies. We also show that a split of the gradient micro-inertia $\bar{\eta}\|\nabla u_{,t}\|^2$, in the sense of Cartan–Lie decomposition of matrices, allows us to flatten separately the longitudinal and transverse optic branches, thus giving us the possibility of a second band-gap. Finally, we investigate the effect of the gradient inertia $\bar{\eta}\|\nabla u_{,t}\|^2$ on more classical enriched models such as the Mindlin–Eringen and the internal variable ones. We find that the addition of such a gradient micro-inertia allows for the onset of one band-gap in the Mindlin–Eringen model and three band-gaps in the internal variable model. In this last case, however, non-local effects cannot be accounted for, which is a too drastic simplification for most metamaterials. We conclude that, even when adding gradient micro-inertia terms, the relaxed micromorphic model remains the best performing one, among the considered enriched models, for the description of non-local band-gap metamaterials.

1. Introduction

The question of effectively studying the dynamic behaviour of microscopically heterogeneous materials in the simplified framework of continuum mechanics is a major challenge for engineering sciences.

Indeed, it is rather clear from the present state of knowledge that classical Cauchy continuum models are too simplified to describe the behaviour of a huge class of materials in the dynamic regime. As a matter of fact, almost all real materials show dispersive behaviour with respect to wave propagation, especially when considering waves with small wavelengths. More precisely, this means that the speed of propagation of waves is not a constant, as happens for Cauchy continua, but that it depends on the wavelength of the travelling wave. Such a phenomenon is comprehensible if one thinks of the fact that the mechanical properties of materials vary when going down to lower scales. It is then sensible that the speed of propagation of mechanical waves varies when considering waves with wavelengths which are small enough to be comparable to the characteristic size of the underlying heterogeneities.

If Cauchy continuum theories are not rich enough to catch these dispersive behaviours, generalized continuum theories offer the possibility of describing wave dispersion while still remaining in the framework of continuum mechanics. Although various generalized continuum models have been introduced to describe dispersion (see the pioneering works [1,2]; for a review of the use of enriched models in the dynamics of heterogeneous materials we refer to [3]), it is still not completely clear whether such dispersive properties can be attributed to the constitutive assumptions which are made on the strain energy density or to the choice of the micro-inertia terms which can be introduced.

The aforementioned considerations about the dispersive behaviour of materials can be reformulated with renewed awareness when talking about metamaterials.

Metamaterials are man-made artefacts which are conceived by assembling small structural elements in periodic or quasi-periodic patterns in such a way that novel mechanical behaviour emerges. Metamaterials are often studied from both a static (those with enhanced mechanical properties with respect to traditional materials) [4–6] and a dynamic (those exhibiting band-gaps, negative refraction, cloaking, focusing, etc.) point of view [7–14]. The characteristic size of microstructures in such metamaterials usually ranges from micrometres to centimetres, so that they show dispersive behaviours for wavelengths which are relatively large.

More than this, some metamaterials can exhibit dispersive behaviour, which gives rise to unorthodox mechanical properties that are not encountered in natural materials. For example, some metamaterials are able to inhibit wave propagation within certain frequency ranges due to the presence of an underlying microstructure which is able to resonate locally when excited at those frequencies or even to remain completely unperturbed. The energy of the incident wave remains trapped at the level of the microstructure and the macroscopic propagation results to be inhibited. Evidence of this type has been reported in the literature based on both theoretical studies [15,16] and experimental results [10,17].

To capture the complex behaviour exhibited by such metamaterials while remaining in the framework of continuum mechanics, generalized continuum models with enriched kinematics are needed. This means that extra degrees of freedom must be introduced in the spirit of micromorphic theories [1,2], which take into account micro-motions at the level of the microstructure. More particularly, the kinematic unknowns of such micromorphic models are usually the macro-displacements u and the micro-distortion tensor P. Well-adapted constitutive choices must then be introduced for the strain energy density in order to describe accurately the behaviour of the considered metamaterials in the static regime.

As a last point, appropriate inertia terms must be introduced to model the mechanical behaviour of metamaterials in the dynamic regime. It is exactly this point that will be the focus of the present paper: how to choose well-suited micro-inertia terms when dealing with enriched continuum models of the micromorphic type? How does each of these terms affect the dynamic behaviour of real band-gap metamaterials? Some hints on the role of micro-inertia in modelling dispersive behaviour are given in [18] but many fundamental questions remain open.

We will show in this paper that:[1]

- Gradient micro-inertia terms $\bar{\eta}\|\nabla u_{,t}\|^2$ only allow us to describe dispersion either in classical or in enriched continuum models [18].
- Micro-inertia terms involving time derivatives of the extra kinematic degrees of freedom $\eta\|P_{,t}\|^2$ allow us to describe and control optic branches in the dispersion relations of classical and relaxed micromorphic continuum models [1,2,19–26].
- The relaxed micromorphic model with micro-inertia of the type $\eta\|P_{,t}\|^2$ is able to describe the onset of the first band-gaps in mechanical metamaterials [20–24].
- The relaxed micromorphic model with both micro-inertia terms $\eta\|P_{,t}\|^2$ and $\bar{\eta}\|\nabla u_{,t}\|^2$ allows us to account for the first and also for the second band-gap which occurs for higher frequencies.
- Classical Mindlin–Eringen models with full micro-inertia $\eta\|P_{,t}\|^2$ and $\bar{\eta}\|\nabla u_{,t}\|^2$ allow for the description of only the first band-gap.
- Internal variable models with full micro-inertia $\eta\|P_{,t}\|^2$ and $\bar{\eta}\|\nabla u_{,t}\|^2$ allow for the description of three band-gaps, even if some peculiar phenomena related to non-locality cannot be accounted for and the resulting behaviour is thus not versatile enough to model realistic metamaterials.

For the first three points a clear treatise is present in the literature, while the last three points are discussed for the first time in this paper.

Finally, we show that a weighted gradient micro-inertia of the type $\frac{1}{2}\bar{\eta}_1\|\operatorname{dev}\operatorname{sym}\nabla u_{,t}\|^2 + \frac{1}{2}\bar{\eta}_2\|\operatorname{skew}\nabla u_{,t}\|^2 + \frac{1}{6}\bar{\eta}_3\operatorname{tr}(\nabla u_{,t})^2$ allows us to flatten some optic curves independently for longitudinal and transverse waves. More precisely, if the parameter $\bar{\eta}_3$ allows us to flatten one optic curve for longitudinal waves, the parameter $\bar{\eta}_2$ has an analogous effect for transverse waves. Such improved control on the dispersion curves will allow for a more effective fitting procedure on real band-gap metamaterials, since the description of the second band-gap occurring at higher frequencies becomes accessible. The effects of analogous decompositions on the other terms of the energy densities have already been studied in [27].

We have already shown in [20] that the relaxed micromorphic model with free micro-inertia can be successfully used to describe the dynamic behaviour of actual band-gap metamaterials. In that case, we showed that the model is perfectly able to capture experimental results related to the transmission coefficient at an interface between a classical Cauchy material and a specific band-gap metamaterial. The proposed use of the relaxed micromorphic model for the description of that particular physical system is accurate enough to faithfully reproduce the transmission coefficient as a function of frequency, also capturing specific internal resonance phenomena that are characteristic of the targeted metamaterial.

[1] In order to clarify the nomenclature used in this paper, we call 'classical continua' the classical continua of Cauchy for which the strain energy density depends on the first gradient of the displacement u. When we talk about 'enriched continua', we are referring to continua with enriched kinematics, i.e. continua whose motion is defined by the displacement 'u' and the micro-distortion P. Different sub-classes of enriched continua can be introduced depending on the constitutive choice of the strain energy density. For example, we talk about 'classical micromorphic' media when the strain energy depends on ∇u, P and ∇P, while we call 'relaxed micromorphic media' those for which the strain energy density is a function of ∇u, P and $\operatorname{Curl} P$.

Moreover, preliminary studies on other band-gap metamaterials, which will be reported in papers in preparation or already submitted to other journals, allow us to:

— confirm the effectiveness of the use of the relaxed micromorphic model for the description of actual band-gap metamaterials via a restricted number of constitutive parameters (such parameters are true material parameters, i.e. they are constants when fixing the metamaterial and independent of frequency; see [28]),
— perfectly fit the relaxed micromorphic model on both the dispersion curves of the targeted metamaterials and the reflection/transmission spectra at material surfaces embedded in such metamaterials,
— show the specific effect that both free micro-inertia and gradient micro-inertia have on the dispersion patterns of such specific metamaterials. Indeed, as will be shown in subsequent works, both types of micro-inertia are needed when one wants to describe, with sufficient precision, a wide class of realistic band-gap metamaterials.

This paper lays the foundations for the extensive use of enriched continuum models of the micromorphic type for the characterization of the behaviour of a huge class of actual metamaterials. The advantage of the use of such models will become evident when the mechanical behaviour of a consistent number of metamaterials is described with the simple introduction of a few material parameters which are true material constants, independent of frequency and not relying on the usual hypothesis of separation of scale.

2. The relaxed micromorphic model

Our novel relaxed micromorphic model endows the Mindlin–Eringen representation with the second-order dislocation density tensor $\alpha = -\mathrm{Curl}\, P$ instead of the full gradient ∇P.[2] In the isotropic case, the elastic energy reads

$$W = \underbrace{\mu_e \, \|\, \mathrm{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2}(\mathrm{tr}(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \, \|\, \mathrm{skew}(\nabla u - P)\|^2}_{\text{rotational elastic coupling}}$$
$$+ \underbrace{\mu_{\text{micro}} \, \|\, \mathrm{sym}\, P\|^2 + \frac{\lambda_{\text{micro}}}{2}(\mathrm{tr}P)^2}_{\text{micro self-energy}} + \underbrace{\frac{\mu_e L_c^2}{2} \, \|\, \mathrm{Curl}P\|^2}_{\text{isotropic curvature}}, \tag{2.1}$$

where the parameters and the elastic stress are analogous to the standard Mindlin–Eringen micromorphic model. The model is well posed in the static and dynamic cases, including when $\mu_c = 0$ (see [19,25]).

In our relaxed model, the complexity of the general micromorphic model has been decisively reduced and features basically only symmetric gradient micro-like variables and the Curl of the micro-distortion P. However, the relaxed model is still general enough to include the full micro-stretch as well as the full Cosserat micro-polar model (see [26]). Furthermore, well-posedness results for the static and dynamic cases have been provided in [26], making decisive use of recently established new coercive inequalities that generalize Korn's inequality to incompatible tensor fields [29–33].

The relaxed micromorphic model counts six constitutive parameters in the isotropic case (μ_e, λ_e, μ_{micro}, λ_{micro}, μ_c, L_c). The characteristic length L_c is intrinsically related to non-local effects due to the fact that it weights a suitable combination of the first-order space derivatives in the strain energy density (2.1). For a general presentation of the features of the relaxed micromorphic model in the anisotropic setting, we refer to [34].

[2]The dislocation tensor is defined as $\alpha_{ij} = -(\mathrm{Curl}\, P)_{ij} = -P_{ih,k}\epsilon_{jkh}$, where ϵ is the Levi–Civita tensor and the Einstein notation of sum over repeated indices is used.

As for the kinetic energy, we consider in this paper that it takes the following form:[3]

$$J = \underbrace{\tfrac{1}{2}\rho\|u_{,t}\|^2}_{\text{Cauchy inertia}} + \underbrace{\tfrac{1}{2}\eta\|P_{,t}\|^2}_{\text{free micro-inertia}} + \underbrace{\tfrac{1}{2}\bar{\eta}_1\|\operatorname{dev}\operatorname{sym}\nabla u_{,t}\|^2 + \tfrac{1}{2}\bar{\eta}_2\|\operatorname{skew}\nabla u_{,t}\|^2 + \tfrac{1}{6}\bar{\eta}_3\operatorname{tr}(\nabla u_{,t})^2}_{\text{new gradient micro-inertia}}, \tag{2.2}$$

where ρ is the value of the average macroscopic mass density of the considered metamaterial, η is the free micro-inertia density and the $\bar{\eta}_i$, $i=\{1,2,3\}$, are the gradient micro-inertia densities associated with the different terms of the Cartan–Lie decomposition of ∇u.

If the first two terms appearing in equation (2.2) can be directly related to those introduced by Mindlin [2], the last three terms of the gradient micro-inertia are considered here for the first time when dealing with enriched continua of the micromorphic type. In fact, gradient micro-inertia terms are currently used when dealing with second gradient continua [35,36], but never when considering micromorphic models. Nevertheless, based on our first comparisons with experimental results [20,28], we are persuaded that gradient micro-inertia is essential also when considering enriched models of the micromorphic type if the ultimate goal is that of describing the behaviour of actual physical systems.

The associated equations of motion in strong form, obtained by a classical least action principle, take the form [22–25]

$$\rho u_{,tt} - \underbrace{\operatorname{Div}[\mathcal{I}]}_{\text{new augmented term}} = \operatorname{Div}[\tilde{\sigma}], \quad \eta P_{,tt} = \tilde{\sigma} - s - \operatorname{Curl} m, \tag{2.3}$$

where

$$\left.\begin{aligned} \mathcal{I} &= \bar{\eta}_1\operatorname{dev}\operatorname{sym}\nabla u_{,tt} + \bar{\eta}_2\operatorname{skew}\nabla u_{,tt} + \tfrac{1}{3}\bar{\eta}_3\operatorname{tr}(\nabla u_{,tt}),\\ \tilde{\sigma} &= 2\mu_e\operatorname{sym}(\nabla u - P) + \lambda_e\operatorname{tr}(\nabla u - P)\mathbb{1} + 2\mu_c\operatorname{skew}(\nabla u - P),\\ s &= 2\mu_{\text{micro}}\operatorname{sym}P + \lambda_{\text{micro}}\operatorname{tr}(P)\mathbb{1}\\ \text{and}\qquad m &= \mu_e L_c^2\operatorname{Curl}P. \end{aligned}\right\} \tag{2.4}$$

The addition of a gradient micro-inertia to the kinetic energy (2.2) modifies the strong-form PDEs of the relaxed micromorphic model with the addition of the new term $\mathcal{I}$. Of course, boundary conditions would also be modified with respect to the ones presented in [20,24]. The study of the new boundary conditions induced by gradient micro-inertia will be the subject of a subsequent paper, in which the effect of such extra terms on the conservation of energy will also be analysed.

3. Plane-wave propagation

Sufficiently far from a source, dynamic wave solutions may be treated as planar waves. Therefore, we now want to study harmonic solutions travelling in an infinite domain for the differential system (2.3). We suppose that the space dependence of all introduced kinematic fields is limited

[3]The Cauchy inertia and free micro-inertia terms appearing in equation (2.2) are classical and already introduced by Mindlin [2] and Eringen and Suhubi [1], while the gradient micro-inertia terms are introduced here for the first time in a micromorphic framework. Indeed, Mindlin [2] recognized inertia terms which are similar to our gradient micro-inertia terms when considering the particular case of the long-wavelength limit of his micromorphic model. Expression (2.2) of the energy that we propose here is more general (i.e. not restricted to large wavelengths) and indeed the gradient micro-inertia will show its higher effect for relatively small wavelengths (high wavenumbers).

to the scalar component X, which is also the direction of propagation of the wave. To do so, following [20–24,37], we define

$$\left.\begin{aligned} P^S &:= \tfrac{1}{3}\,\mathrm{tr}(P), \quad P_{[ij]} := (\mathrm{skew}\,P)_{ij} = \tfrac{1}{2}(P_{ij} - P_{ji}),\\ P^D &:= P_{11} - P^S, \quad P_{(ij)} := (\mathrm{sym}\,P)_{ij} = \tfrac{1}{2}(P_{ij} + P_{ji})\\ \text{and}\quad P^V &:= P_{22} - P_{33}. \end{aligned}\right\} \tag{3.1}$$

With this decomposition, equations (2.3) can be rewritten as [22,23]

— a set of three equations only involving longitudinal quantities:

$$\left.\begin{aligned} \rho\ddot{u}_1 - \underbrace{\frac{2\bar{\eta}_1 + \bar{\eta}_3}{3}\ddot{u}_{1,11}}_{\text{new augmented terms}} &= (2\,\mu_e + \lambda_e)u_{1,11} - 2\mu_e\,P^D_{,1} - (2\mu_e + 3\lambda_e)P^S_{,1},\\ \eta\ddot{P}^D &= \frac{4}{3}\,\mu_e\,u_{1,1} + \frac{1}{3}\,\mu_e L_c^2\,P^D_{,11} - \frac{2}{3}\,\mu_e L_c^2 P^S_{,11} - 2(\mu_e + \mu_{\text{micro}})P^D\\ \text{and}\quad \eta\ddot{P}^S &= \frac{2\,\mu_e + 3\,\lambda_e}{3}u_{1,1} - \frac{1}{3}\,\mu_e L_c^2 P^D_{,11} + \frac{2}{3}\,\mu_e L_c^2 P^S_{,11}\\ &\quad - (2\,\mu_e + 3\,\lambda_e + 2\,\mu_{\text{micro}} + 3\,\lambda_{\text{micro}})P^S, \end{aligned}\right\} \tag{3.2}$$

— two sets of three equations only involving transverse quantities in the ξth direction, with $\xi = 2, 3$:

$$\left.\begin{aligned} \rho\ddot{u}_\xi - \underbrace{\frac{\bar{\eta}_1 + \bar{\eta}_2}{2}\ddot{u}_{\xi,11}}_{\text{new augmented terms}} &= (\mu_e + \mu_c)u_{\xi,11} - 2\,\mu_e\,P_{(1\xi),1} + 2\,\mu_c\,P_{[1\xi],1},\\ \eta\ddot{P}_{(1\xi)} &= \mu_e\,u_{\xi,1} + \frac{1}{2}\,\mu_e L_c^2\,P_{(1\xi),11} + \tfrac{1}{2}\,\mu_e L_c^2\,P_{[1\xi],11} - 2(\mu_e + \mu_{\text{micro}})\,P_{(1\xi)}\\ \text{and}\quad \eta\ddot{P}_{[1\xi]} &= -\mu_c\,u_{\xi,1} + \frac{1}{2}\,\mu_e L_c^2\,P_{(1\xi),11} + \tfrac{1}{2}\,\mu_e L_c^2 P_{[1\xi],11} - 2\,\mu_c\,P_{[1\xi]}, \end{aligned}\right\} \tag{3.3}$$

— one equation only involving the variable $P_{(23)}$:

$$\eta\ddot{P}_{(23)} = -2(\mu_e + \mu_{\text{micro}})P_{(23)} + \mu_e L_c^2 P_{(23),11}, \tag{3.4}$$

— one equation only involving the variable $P_{[23]}$:

$$\eta\ddot{P}_{[23]} = -2\,\mu_c\,P_{[23]} + \mu_e L_c^2 P_{[23],11}, \tag{3.5}$$

— one equation only involving the variable P^V:

$$\eta\ddot{P}^V = -2(\mu_e + \mu_{\text{micro}})P^V + \mu_e L_c^2 P^V_{,11}. \tag{3.6}$$

Once this simplified system of PDEs is obtained, we look for a waveform solution of the type:

$$\left.\begin{aligned} &\underbrace{\mathbf{v}_1(X,t) = \boldsymbol{\beta}\,\mathrm{e}^{\mathrm{i}(kX-\omega t)}}_{\text{longitudinal}}, \quad \underbrace{\mathbf{v}_\tau(X,t) = \boldsymbol{\gamma}^\tau\,\mathrm{e}^{\mathrm{i}(kX-\omega t)}}_{\text{transverse}}, \quad \tau = 2,3\\ \text{and}\quad &\underbrace{\mathbf{v}_4(X,t) = \boldsymbol{\gamma}^4\,\mathrm{e}^{\mathrm{i}(kX-\omega t)}}_{\text{uncoupled}}, \end{aligned}\right\} \tag{3.7}$$

where we set for compactness

$$\left.\begin{aligned}&\mathbf{v}_1=(u_1,P^D,P^S),\quad \mathbf{v}_\tau=(u_\tau,P_{(1\tau)},P_{[1\tau]}),\quad \tau=2,3\\ \text{and}\quad &\mathbf{v}_4=(P_{(23)},P_{[23]},P^V),\end{aligned}\right\}\tag{3.8}$$

where $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)^{\mathrm{T}}\in\mathbb{C}^3$, $\boldsymbol{\gamma}^\tau=(\gamma_1^\tau,\gamma_2^\tau,\gamma_3^\tau)^{\mathrm{T}}\in\mathbb{C}^3$ and $\boldsymbol{\gamma}^4=(\gamma_1^4,\gamma_2^4,\gamma_3^4)^{\mathrm{T}}\in\mathbb{C}^3$ are the unknown amplitudes of the considered waves,[4] k is the wavenumber and ω is the wave frequency.

Replacing the waveform solution (3.7) in equations (3.2)–(3.6), it is possible to express the system as

$$\mathbf{A}_1\cdot\boldsymbol{\beta}=0,\quad \mathbf{A}_\tau\cdot\boldsymbol{\gamma}^\tau=0,\quad \tau=2,3\quad\text{and}\quad \mathbf{A}_4\cdot\boldsymbol{\gamma}^4=0,\tag{3.9}$$

where

$$\mathbf{A}_1(\omega,k)=\begin{pmatrix}-\omega^2\left(1+k^2\dfrac{2\bar\eta_1+\bar\eta_3}{3\rho}\right)+c_p^2k^2 & \dfrac{\mathrm{i}k2\,\mu_e}{\rho} & \dfrac{\mathrm{i}k(2\,\mu_e+3\,\lambda_e)}{\rho}\\ -\mathrm{i}k\dfrac{4}{3}\,\mu_e/\eta & -\omega^2+\dfrac{1}{3}k^2c_m^2+\omega_s^2 & -\dfrac{2}{3}k^2c_m^2\\ -\dfrac{1}{3}\mathrm{i}k(2\,\mu_e+3\,\lambda_e)/\eta & -\dfrac{1}{3}k^2c_m^2 & -\omega^2+\dfrac{2}{3}k^2c_m^2+\omega_p^2\end{pmatrix},$$

$$\mathbf{A}_2(\omega,k)=\mathbf{A}_3(\omega,k)=\begin{pmatrix}-\omega^2\left(1+k^2\dfrac{\bar\eta_1+\bar\eta_2}{2\rho}\right)+k^2c_s^2 & \dfrac{\mathrm{i}k2\,\mu_e}{\rho} & -\mathrm{i}\dfrac{k}{\rho}\eta\omega_r^2\\ -\dfrac{\mathrm{i}k\,\mu_e}{\eta} & -\omega^2+\dfrac{c_m^2}{2}k^2+\omega_s^2 & \dfrac{c_m^2}{2}k^2\\ \dfrac{i}{2}\omega_r^2k & \dfrac{c_m^2}{2}k^2 & -\omega^2+\dfrac{c_m^2}{2}k^2+\omega_r^2\end{pmatrix},$$

$$\mathbf{A}_4(\omega,k)=\begin{pmatrix}-\omega^2+c_m^2k^2+\omega_s^2 & 0 & 0\\ 0 & -\omega^2+c_m^2k^2+\omega_r^2 & 0\\ 0 & 0 & -\omega^2+c_m^2k^2+\omega_s^2\end{pmatrix}.$$

Here, we have defined:

$$c_m=\sqrt{\frac{\mu_eL_c^2}{\eta}},\quad c_s=\sqrt{\frac{\mu_e+\mu_c}{\rho}},\quad c_p=\sqrt{\frac{2\,\mu_e+\lambda_e}{\rho}},$$
$$\omega_s=\sqrt{\frac{2(\mu_e+\mu_{\mathrm{micro}})}{\eta}},\quad \omega_p=\sqrt{\frac{(2\,\mu_e+3\,\lambda_e)+(2\,\mu_{\mathrm{micro}}+3\,\lambda_{\mathrm{micro}})}{\eta}},\quad \omega_r=\sqrt{\frac{2\,\mu_c}{\eta}}.$$

To have non-trivial solutions of the algebraic systems (3.9), one must impose that

$$\underbrace{\det\mathbf{A}_1(\omega,k)=0,}_{\text{longitudinal}}\quad \underbrace{\det\mathbf{A}_2(\omega,k)=\det\mathbf{A}_3(\omega,k)=0,}_{\text{transverse}}\quad \underbrace{\det\mathbf{A}_4(\omega,k)=0.}_{\text{uncoupled}}\tag{3.10}$$

The solutions $\omega=\omega(k)$ of these algebraic equations are called the dispersion curves of the relaxed micromorphic model for longitudinal, transverse and uncoupled waves, respectively.

In what follows we will present the results obtained for the numerical values of the elastic coefficients shown in table 1 if not otherwise specified.

[4]Here, we understand that having found the (in general, complex) solutions of (3.7) only the real or imaginary parts separately constitute actual wave solutions which can be observed in reality.

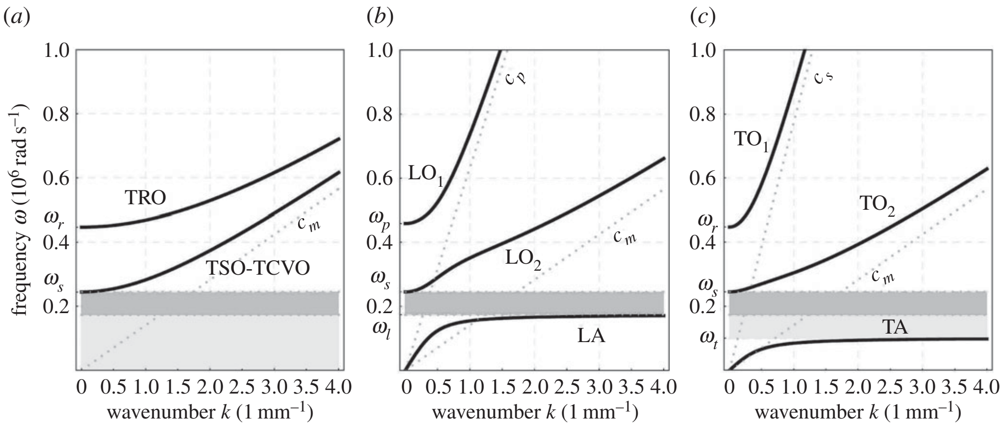

Figure 1. Dispersion relations $\omega = \omega(k)$ for the uncoupled (*a*), longitudinal (*b*) and transverse (*c*) waves of the relaxed micromorphic model with free micro-inertia $\eta = 10^{-2}$ kg m^{-1}. TRO, transverse rotational optic; TSO, transverse shear optic; TCVO, transverse constant-volume optic; LA, longitudinal acoustic; LO_1/LO_2, first and second longitudinal optic; TA, transverse acoustic; TO_1/TO_2, first and second transverse optic.

Table 1. Values of the parameters used in the numerical simulations (top), and corresponding values of the Lamé parameters and of the Young modulus and Poisson ratio (bottom), for the formulas needed to calculate the homogenized macroscopic parameters starting from the microscopic ones (see [34]).

parameter	value	unit
μ_e	200	MPa
$\lambda_e = 2\mu_e$	400	MPa
$\mu_c = 5\mu_e$	1000	MPa
μ_{micro}	100	MPa
λ_{micro}	100	MPa
L_c	1	mm
ρ	2000	kg m^{-13}
λ_{macro}	82.5	MPa
μ_{macro}	66.7	MPa
E_{macro}	170	MPa
ν_{macro}	0.28	—

In the following sections, we will explicitly discuss the effect of each micro-inertia parameter on the dispersion curves of the relaxed micromorphic model. More particularly, we will focus on the following cases:

— vanishing free micro-inertia $\eta = 0$ and non-vanishing gradient micro-inertia,
— both non-vanishing gradient micro-inertia and free micro-inertia.

The remaining case (vanishing gradient micro-inertia $\bar{\eta} = 0$ and non-vanishing free micro-inertia $\eta \neq 0$) is the classical case treated for the relaxed micromorphic model in [20–24,37]. For the sake of completeness, we present in figure 1 the dispersion curves for this case when using the values of the parameters given in table 1.

It can be found that, when considering the free micro-inertia alone, the relaxed micromorphic model is able to predict the first band-gap, which usually occurs at relatively low frequencies.

Moreover, the relaxed micromorphic model is, for the current state of the art, the only continuum model which is able to describe simultaneously band-gaps and non-local behaviour [20].

In the following sections, we will present the new results concerning the effect of the gradient micro-inertia terms on the dispersion curves of the relaxed micromorphic model, as well as the effect of such gradient micro-inertia terms on more classical enriched models (Mindlin, internal variable).

4. Case of vanishing free micro-inertia η and non-vanishing gradient micro-inertia $\bar{\eta}$

In this section, we discuss the effect on the dispersion curves of enriched continuum models of the gradient micro-inertia term alone. We will show that the fact of complementing the macro-inertia $\rho\|u_{,t}\|^2$ only with the gradient micro-inertia $\bar{\eta}\|\nabla u_{,t}\|^2$ is a fundamental modelling limitation since the complexity of the dynamic behaviour of micromorphic models cannot be unveiled. Nevertheless, the gradient micro-inertia allows us to describe some dispersion which is not allowed by classical Cauchy models.

(a) Study of the dispersion curves

In the case in which we consider only the gradient micro-inertia $\bar{\eta}\neq 0$ to be non-vanishing, the matrix associated with the longitudinal dynamic system can be expressed as:[5]

$$\mathbf{A}_1(\omega,k)=\begin{pmatrix} -\omega^2\left(\rho+k^2\dfrac{2\bar{\eta}_1+\bar{\eta}_3}{3}\right)+(2\,\mu_e+\lambda_e)k^2 & \mathrm{i}k2\mu_e & \mathrm{i}k(2\,\mu_e+3\,\lambda_e) \\ -\mathrm{i}k\dfrac{4}{3}\,\mu_e & \dfrac{1}{3}k^2\mu_e L_c^2+2(\mu_e+\mu_{\mathrm{micro}}) & -\dfrac{2}{3}k^2\mu_e L_c^2 \\ -\dfrac{1}{3}\mathrm{i}k(2\,\mu_e+3\,\lambda_e) & -\dfrac{1}{3}k^2\,\mu_e L_c^2 & \dfrac{2}{3}k^2\,\mu_e L_c^2+\omega_p^2 \end{pmatrix}. \tag{4.1}$$

It is possible to remark that the polynomial $\det\mathbf{A}_1(\omega,k)$ is of the second order in ω. This implies that we have a unique positive solution of the equation $\det\mathbf{A}_1(\omega,k)=0$ when considering positive k.[6] In particular, when plotting such a solution in the (ω,k) plane only one acoustic branch can be detected (figure 2).[7]

Comparing the results shown in figure 2 with those presented in figure 1, it can be immediately noticed that the fact of considering the gradient micro-inertia alone significantly constrains the behaviour of the considered enriched continuum. Even if the constitutive expression for the strain energy density W is the same in both figure 2 and figure 1 (see equation (2.1)), the fact of using a gradient micro-inertia $\bar{\eta}\|\nabla u_{,t}\|^2$ instead of a free micro-inertia $\eta\|P_{,t}\|^2$ drastically simplifies the patterns which are found for the dispersion curves. With reference to figure 2, we can remark that a unique acoustic branch is found and that the presence of a non-vanishing micro-inertia parameter $\bar{\eta}_3$ induces a dispersive behaviour. When the gradient micro-inertia parameters are all vanishing ($\bar{\eta}_1=\bar{\eta}_2=\bar{\eta}_3=0$), this means that only a macro-inertia $\rho\|u_{,t}\|^2$ is present and this corresponds to an almost constant speed of the travelling waves, which is what happens for the classical Cauchy case. It can be shown that, considering an adapted choice of the constitutive parameters for the relaxed micromorphic model with macro-inertia $\rho\|u_{,t}\|^2$ alone, the dispersion curve obtained is exactly the straight one obtained with the classical Cauchy model.

[5]We can see from the form of $\mathbf{A}_1(\omega,k)$ that considering an additional micro-inertia $\bar{\eta}$ is equivalent to defining an average macroscopic density depending on the wavelength as $\rho^*(k)=\rho+k^2\bar{\eta}$. The same can be found for the transverse waves.

[6]It can be checked that, when considering elastic parameters which guarantee positive definiteness of the elastic energy, the solutions $\omega=\omega(k)$ of the characteristic polynomials are always real [37].

[7]Here and in the sequel, we will always set $\bar{\eta}_1=0$, since we could not isolate a characteristic effect of such parameters on the dispersion curves.

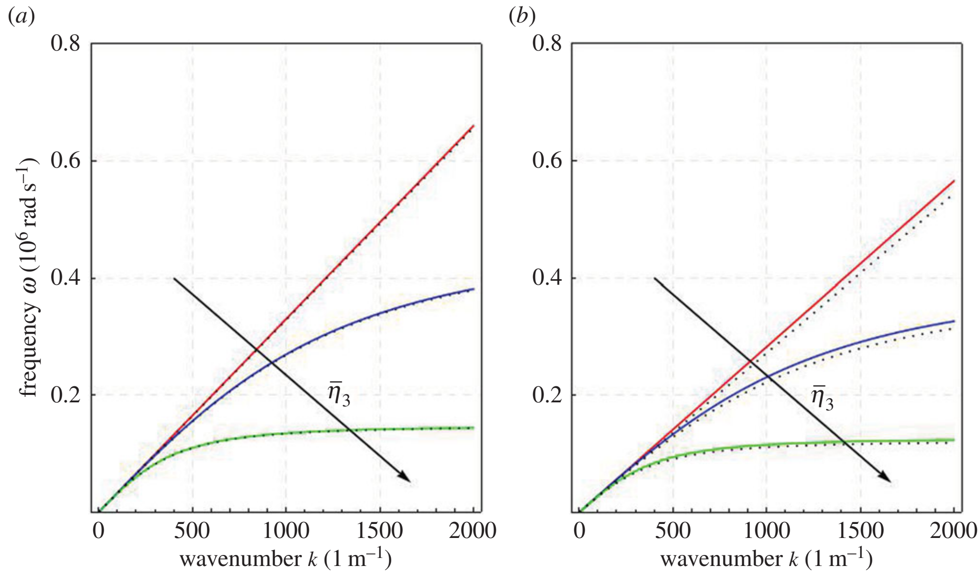

Figure 2. Dispersion relations $\omega = \omega(k)$ for the longitudinal waves of the relaxed micromorphic model with gradient micro-inertia $\bar{\eta}_3 = (0, 3\times 10^{-3},\ 3\times 10^{-2})$ kg m^{-1} and $\eta = 0$. Black dots indicate the dispersion relations for a first gradient model with Lamé parameters μ_{macro} and λ_{macro} and the same inertiae ρ and $\bar{\eta}_3$ (*a*). The same picture obtained imposing $\lambda_{\text{micro}} = 0$ (*b*): a very slight variation with respect to the first gradient case can be detected. (Online version in colour.)

With a similar reasoning to the one made for longitudinal waves, considering the case $\bar{\eta} \neq 0$ for transverse waves, the matrix associated with the transverse dynamic system can be expressed as

$$\mathbf{A}_2(\omega,k) = \begin{pmatrix} -\omega^2\left(\rho + k^2\dfrac{\bar{\eta}_1+\bar{\eta}_2}{2}\right) + k^2(\mu_e+\mu_c) & \mathrm{i}k2\mu_e & -\mathrm{i}k2\mu_c \\ -\mathrm{i}k2\mu_e & \mu_e L_c^2 k^2 + 4(\mu_e+\mu_{\text{micro}}) & \mu_e L_c^2 k^2 \\ \mathrm{i}k2\mu_c & \mu_e L_c^2 k^2 & \mu_e L_c^2 k^2 + 4\mu_c \end{pmatrix}. \tag{4.2}$$

It is possible to see that the new inertia terms $\bar{\eta}_2$ play the same role for the transverse waves that was played by $\bar{\eta}_3$ for the longitudinal waves. The results concerning the solutions $\omega = \omega(k)$ of the characteristic equation $\det \mathbf{A}_2(\omega,k) = 0$ are analogous to the case of longitudinal waves (figure 3).

If the particular case with non-null gradient micro-inertia $\bar{\eta} \neq 0$ and null free micro-inertia $\eta = 0$ is considered, the matrix associated with the uncoupled waves reduces to

$$\mathbf{A}_4(\omega,k) = \begin{pmatrix} \mu_e L_c^2 k^2 + 2(\mu_e+\mu_{\text{micro}}) & 0 & 0 \\ 0 & \mu_e L_c^2 k^2 + 2\mu_c & 0 \\ 0 & 0 & \mu_e L_c^2 k^2 + 2(\mu_e+\mu_{\text{micro}}) \end{pmatrix}, \tag{4.3}$$

from which it is not possible to derive any dispersion curve, due to the absence of inertia terms.

(b) A first conclusion on the effect of gradient micro-inertia on enriched continuum models

— When considering a macro-inertia term $\rho\|u_{,t}\|^2$ alone, only one acoustic branch is present that has an almost constant speed of propagation. Such behaviour is strongly dictated by the macro-inertia term since the difference on the associated dispersion curves between a simple Cauchy energy $W(\nabla u)$ and an enriched model $W = W(\nabla u, P, \operatorname{Curl} P)$ is small and vanishing considering an adapted choice of the constitutive parameters.

— When complementing the macro-inertia $\rho\|u_{,t}\|^2$ with a gradient micro-inertia $\bar{\eta}\|\nabla u_{,t}\|^2$ the speed of propagation of waves is not constant anymore, but it depends on

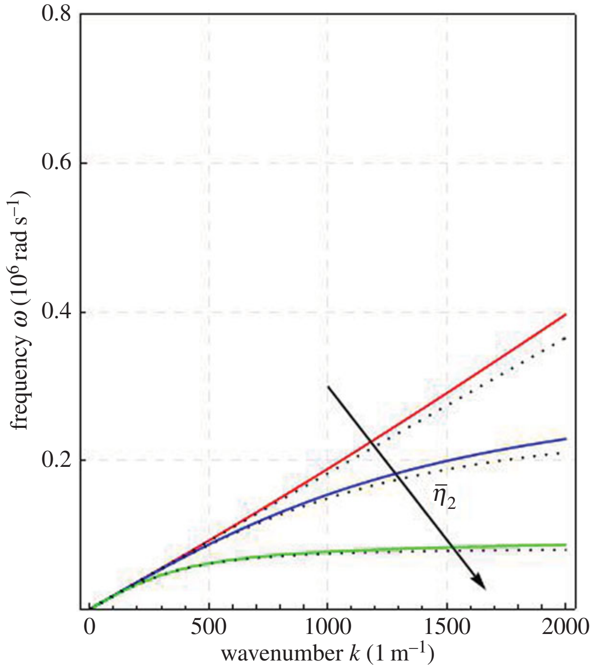

Figure 3. Dispersion relations $\omega = \omega(k)$ for the transverse waves of the relaxed micromorphic model with gradient micro-inertia $\bar{\eta}_2 = (0, 2 \times 10^{-3},\ 2 \times 10^{-2})$ kg m^{-1} and $\eta = 0$. Black dots indicate the dispersion relations for a first gradient model with Lamé parameters μ_{macro} and λ_{macro} and the same inertiae ρ and $\bar{\eta}_t$. (Online version in colour.)

the wavelength of the travelling waves. Nevertheless, only an acoustic branch can be described, independently of the more or less complicated (standard or enriched) kinematics.

— Complementing the macro-inertia $\rho\|u_{,t}\|^2$ with a free micro-inertia $\eta\|P_{,t}\|^2$ allows us to disclose the full rich constitutive behaviour provided by considering an enriched model, as studied in [20–24,37] and reproduced in figure 1. Two optic branches are observed, for both longitudinal and transverse waves, in addition to the acoustic ones already discussed in the previous case (figure 1). The properties of such curves depend both on the constitutive parameters appearing in the expression of the energy (equation (2.1)) and on the free inertia parameter η. In this framework of inertia of the type $\rho\|u_{,t}\|^2 + \eta\|P_{,t}\|^2$, the relaxed micromorphic model is the only non-local, enriched continuum model allowing for the presence of band-gaps [21].

5. Case of both non-vanishing free micro-inertia η and gradient micro-inertia $\bar{\eta}$

In this section, we will discuss the effect of a full inertia $\rho\|u_{,t}\|^2 + \eta\|P_{,t}\|^2$ on the dispersion curves of the relaxed micromorphic model. We will show that the complementation of the macro-inertia with both the gradient and free micro-inertia allows for the description of a new feature of the relaxed micromorphic model, i.e. the onset of a second band-gap occurring at higher frequencies than the first one.

(a) Dispersion relations

Now, we show in figure 4 the results obtained for non-null micro-inertia $\eta \neq 0$ with the addition of gradient micro-inertia $\bar{\eta} \neq 0$. Surprisingly, the combined effect of the traditional micro-inertia η with the gradient micro-inertiae can lead to the onset of a second longitudinal and transverse band-gap. Indeed, the existence of a horizontal asymptote for the first optic branches in figure 4

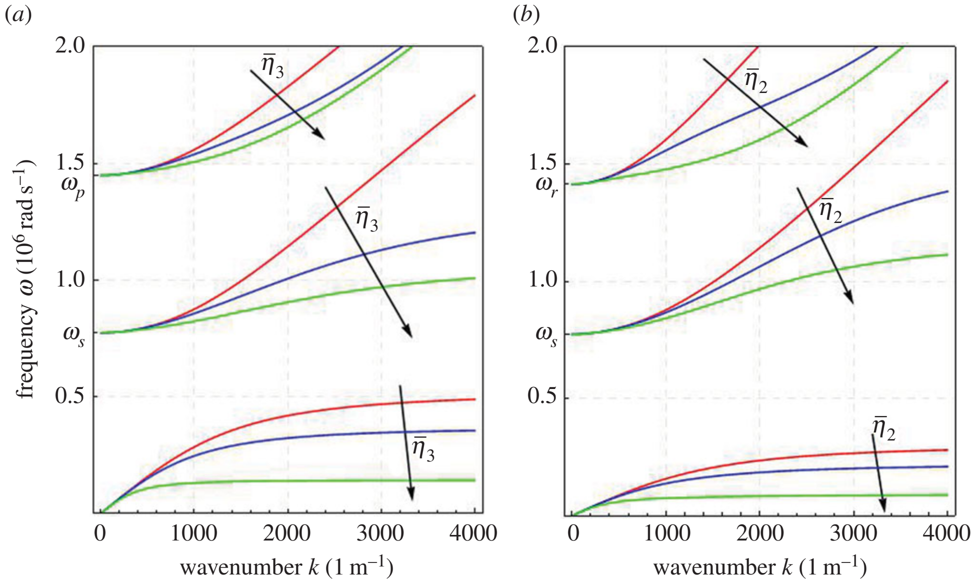

Figure 4. Dispersion relations $\omega = \omega(k)$ of the relaxed micromorphic model for the longitudinal waves with free micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\bar{\eta}_3 = (3 \times 10^{-4}, 3 \times 10^{-3}, 3 \times 10^{-2})$ kg m^{-1} (*a*) and transverse waves with micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\bar{\eta}_2 = (2 \times 10^{-4}, 2 \times 10^{-3}, 2 \times 10^{-2})$ kg m^{-1} (*b*). (Online version in colour.)

could be shown. Nevertheless, an explicit computation of such asymptotes becomes over-burdening. For this reason, we limit ourselves to remark that, for all the metamaterials that we tested up to now, it is always possible to find $\bar{\eta}_1$, $\bar{\eta}_2$ and $\bar{\eta}_3$ that are large enough to have horizontal asymptotes appearing in graphics considering k ranging from 0 (infinite wavelength) to values of k corresponding to wavelengths smaller than the size of the unit cell. Moreover, it is possible to note that the addition of gradient micro-inertiae $\bar{\eta}_1$, $\bar{\eta}_2$ and $\bar{\eta}_3$ has no effect on the cut-off frequencies, which only depend on the free micro-inertia η (and of course on the constitutive parameters).

The uncoupled waves in the relaxed micromorphic model with generalized inertia behave in the same way as in the relaxed micromorphic model, as it is possible to see by analysing the matrix

$$\mathbf{A}_4(\omega, k) = \begin{pmatrix} -\omega^2 + c_m^2 k^2 + \omega_s^2 & 0 & 0 \\ 0 & -\omega^2 + c_m^2 k^2 + \omega_r^2 & 0 \\ 0 & 0 & -\omega^2 + c_m^2 k^2 + \omega_s^2 \end{pmatrix}. \tag{5.1}$$

The resulting dispersion curves are the same as those obtained with the classical relaxed micromorphic model (figure 1*b*).

(b) Cut-offs and asymptotic behaviour

To study the asymptotic behaviour of the dispersion curves for the relaxed micromorphic model with full inertia, let us introduce the following quantities:

$$\omega_v = \sqrt{\frac{(2\,\mu_e + \lambda_e) + (2\,\mu_{\text{micro}} + \lambda_{\text{micro}})}{\eta}}, \quad \omega_l = \sqrt{\frac{2\,\mu_{\text{micro}} + \lambda_{\text{micro}}}{\eta}}, \quad \omega_t = \sqrt{\frac{\mu_{\text{micro}}}{\eta}},$$

$$\omega_{\bar{l}} = \sqrt{\frac{2\,\mu_e + \lambda_e}{\frac{2\bar{\eta}_1 + \bar{\eta}_3}{3}}}, \quad \omega_{\bar{t}} = \sqrt{\frac{2(\mu_c + \mu_e)}{\frac{\bar{\eta}_1 + \bar{\eta}_2}{2}}}.$$

As stated in the previous section, the cut-off frequencies are not modified by the insertion of a gradient micro-inertia term. Therefore, considering the longitudinal waves, we have one acoustic branch of the dispersion curve and two optic branches with cut-off frequencies:

$$\omega_s = \sqrt{\frac{2(\mu_e + \mu_{\text{micro}})}{\eta}} \quad \text{and} \quad \omega_p = \sqrt{\frac{(2\,\mu_e + 3\,\lambda_e) + (2\,\mu_{\text{micro}} + 3\,\lambda_{\text{micro}})}{\eta}}. \tag{5.2}$$

On the other hand, the asymptotic behaviour changes in a radical fashion from the classical relaxed micromorphic model. The horizontal asymptote of the acoustic curve changes and we have the onset of a new horizontal asymptote for one of the optic branches, the values of which are, respectively,

$$\left.\begin{aligned} \omega_{l,\text{acoustic}} &= \sqrt{\frac{\omega_{\bar{l}}^2 + \omega_{\bar{v}}^2 - \sqrt{(\omega_{\bar{l}}^2 + \omega_{\bar{v}}^2)^2 - 4\omega_{\bar{l}}^2\omega_l^2}}{2}} \\ \text{and} \qquad \omega_{l,\text{optic}} &= \sqrt{\frac{\omega_{\bar{l}}^2 + \omega_{\bar{v}}^2 + \sqrt{(\omega_{\bar{l}}^2 + \omega_{\bar{v}}^2)^2 - 4\omega_{\bar{l}}^2\omega_l^2}}{2}}. \end{aligned}\right\} \tag{5.3}$$

No difference is found from the other optic branch that has an asymptote with slope c_m, as in the classical relaxed micromorphic model.

Analogously, considering the transverse waves, we have one acoustic branch and two optic branches with cut-off frequencies,

$$\omega_s = \sqrt{\frac{2(\mu_e + \mu_{\text{micro}})}{\eta}} \quad \text{and} \quad \omega_r = \sqrt{\frac{2\,\mu_c}{\eta}}. \tag{5.4}$$

Once again, the horizontal asymptote of the acoustic curve changes with respect to the classical relaxed case and we have an extra horizontal asymptote for one of the optic branches, the values of which are, respectively,

$$\left.\begin{aligned} \omega_{t,\text{acoustic}} &= \frac{1}{2}\sqrt{\omega_{\bar{t}}^2 + \omega_{\bar{s}}^2 + \omega_{\bar{r}}^2 - \sqrt{(\omega_{\bar{t}}^2 + \omega_{\bar{s}}^2 + \omega_{\bar{r}}^2)^2 - 4\omega_{\bar{t}}^2\omega_t^2}} \\ \text{and} \qquad \omega_{t,\text{optic}} &= \frac{1}{2}\sqrt{\omega_{\bar{t}}^2 + \omega_{\bar{s}}^2 + \omega_{\bar{r}}^2 + \sqrt{(\omega_{\bar{t}}^2 + \omega_{\bar{s}}^2 + \omega_{\bar{r}}^2)^2 - 4\omega_{\bar{t}}^2\omega_t^2}}. \end{aligned}\right\} \tag{5.5}$$

No difference is found from the other optic branch that has an asymptote with slope c_m, as in the classical relaxed micromorphic model.

Finally, no change whatsoever is present in the uncoupled waves that keep having cut-off frequencies ω_s and ω_r and oblique asymptote of slope c_m.

6. Combined effect of the free and gradient micro-inertiae on more classical enriched models (Mindlin–Eringen and internal variable)

In this section, we discuss the effect on the Mindlin–Eringen and the internal variable model of the addition of the gradient micro-inertia $\bar{\eta}\|\nabla u_{,t}\|^2$ to the classical terms $\rho\|u_{,t}\|^2 + \eta\|P_{,t}\|^2$. We will show that the previously discussed effect of the parameters $\bar{\eta}_2$ and $\bar{\eta}_3$ is maintained for both the Mindlin–Eringen case and the internal variable case.

Figure 5 refers to the study of the effects of the parameters $\bar{\eta}_2$ and $\bar{\eta}_3$ on the dispersion curves of the classical Mindlin–Eringen micromorphic model. For the sake of completeness,

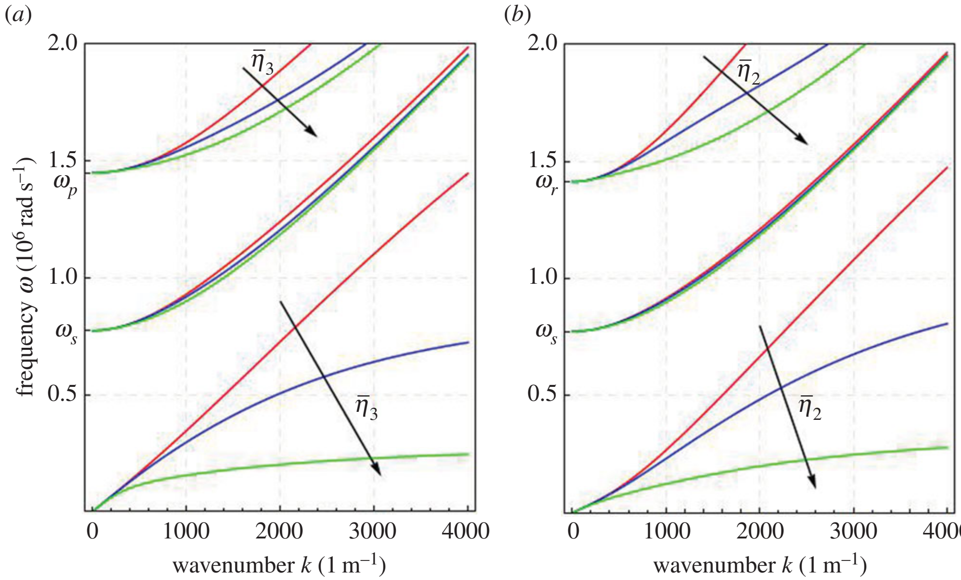

Figure 5. Dispersion relations $\omega = \omega(k)$ of the standard Mindlin–Eringen model for the longitudinal waves with free micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\bar{\eta}_3 = (3 \times 10^{-4},\ 3 \times 10^{-3},\ 3 \times 10^{-2})$ kg m^{-1} (*a*) and transverse waves with micro-inertia $\eta = 10^{-3}$ and gradient micro-inertia $\bar{\eta}_2 = (2 \times 10^{-4},\ 2 \times 10^{-3},\ 2 \times 10^{-2})$ kg m^{-1} (*b*). (Online version in colour.)

we recall that the (simplified) strain energy density for this model in the isotropic case takes the form

$$W = \underbrace{\mu_e \,\| \operatorname{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2}(\operatorname{tr}(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \,\| \operatorname{skew}(\nabla u - P)\|^2}_{\text{rotational elastic coupling}}$$
$$+ \underbrace{\mu_{\text{micro}} \,\| \operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2}(\operatorname{tr} P)^2}_{\text{micro self-energy}} + \underbrace{\frac{\mu_e L_c^2}{2} \,\|\nabla P\|^2}_{\text{isotropic curvature}} . \tag{6.1}$$

Recalling the results of [22], we remark that when the gradient micro-inertia is vanishing ($\bar{\eta}_1 = \bar{\eta}_2 = \bar{\eta}_3 = 0$) the Mindlin–Eringen model does not allow the description of band-gaps, due to the presence of a straight acoustic branch. On the other hand, when switching on the parameters $\bar{\eta}_2$ and $\bar{\eta}_3$, the acoustic branches are flattened (they take a horizontal asymptote), so that the first band-gap can be described. The analogous case for the relaxed micromorphic model (figure 1) allowed instead for the description of two band-gaps.

Figure 6 shows the effect of the addition of the gradient micro-inertia $\bar{\eta} \| \nabla u_{,t}\|^2$ on the behaviour of the internal variable model. We recall [26] that the energy for the internal variable model does not include higher space derivatives of the micro-distortion tensor P and, in the isotropic case, takes the form

$$W = \underbrace{\mu_e \,\| \operatorname{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2}(\operatorname{tr}(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \,\| \operatorname{skew}(\nabla u - P)\|^2}_{\text{rotational elastic coupling}}$$
$$+ \underbrace{\mu_{\text{micro}} \,\| \operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2}(\operatorname{tr} P)^2}_{\text{micro self-energy}} . \tag{6.2}$$

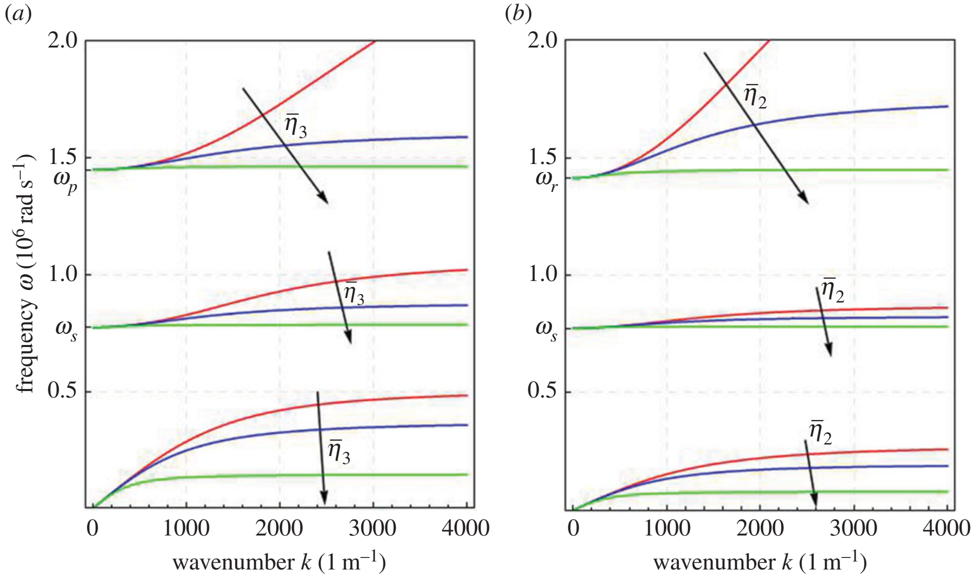

Figure 6. Dispersion relations $\omega=\omega(k)$ of the internal variable model for the longitudinal waves with free micro-inertia $\eta=10^{-3}$ and gradient micro-inertia $\bar{\eta}_3=(3\times10^{-4},\,3\times10^{-3},\,3\times10^{-2})\ \mathrm{kg\,m^{-1}}$ (*a*) and transverse waves with micro-inertia $\eta=10^{-3}$ and gradient micro-inertia $\bar{\eta}_2=(2\times10^{-4},\,2\times10^{-3},\,2\times10^{-2})\ \mathrm{kg\,m^{-1}}$ (*b*). (Online version in colour.)

By direct observation of figure 6, we can note that, by suitably choosing the relative position of ω_r and ω_p, the internal variable model allows us to account for three band-gaps.

We thus have an extra band-gap with respect to the analogous case for the relaxed micromorphic model (figure 1), but we are not able to consider non-local effects. Excluding the possibility of describing non-local effects in metamaterials can be sometimes too restrictive. For example, flattening the curve which originates from ω_r and which is associated with rotational modes of the microstructure is unphysical for the great majority of metamaterials.

7. Conclusion

In this paper, we have discussed the fundamental role of micro-inertia in enriched continuum models of the micromorphic type.

We show that if, on one hand, the free micro-inertia term $\eta\|P_{,t}\|^2$ is strictly necessary to disclose the full rich behaviour of micromorphic media in the dynamic regime, on the other hand the gradient micro-inertia $\bar{\eta}\|\nabla u_{,t}\|^2$ has the macroscopic effect of flattening some of the dispersion curves, so allowing for the description of extra band-gaps. In particular, we have shown that:

— In the case of the relaxed micromorphic model, one band-gap can be described when introducing the free micro-inertia $\eta\|P_{,t}\|^2$ alone. When introducing a mixed micro-inertia $\eta\|P_{,t}\|^2+\bar{\eta}\|\nabla u_{,t}\|^2$ two band-gaps can be accounted for by the same model.

— In the case of the Mindlin–Eringen model, no band-gaps are possible with the term $\eta\|P_{,t}\|^2$ alone, while the onset of a single band-gap can be granted by the addition of the extra term $\bar{\eta}\|\nabla u_{,t}\|^2$.

— In the internal variable model, two band-gaps are possible with the term $\eta\|P_{,t}\|^2$ alone, even if non-localities cannot be accounted for by such a model. When adding the extra term $\bar{\eta}\|\nabla u_{,t}\|^2$ even three band-gaps becomes possible, but the behaviour of the dispersion curves becomes fairly unrealistic for a huge class of real metamaterials.

In conclusion, the results presented in this paper confirm the preceding findings according to which the relaxed micromorphic model is the most suitable enriched model for the simultaneous description of (i) band-gaps and (ii) non-localities in mechanical metamaterials.

Future work will be devoted to the application of the results obtained in this paper for the fitting of the proposed model with enriched micro-inertia on real metamaterials exhibiting multiple band-gaps.

Authors' contributions. All the authors contributed equally to this work.

Competing interests. The authors have no conflict of interests to declare.

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