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Invariant measures for unipotent translations on homogeneous spaces

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ABSTRACT It is shown that all Borel probability measures invariant under unipotent translations on homogeneous spaces of connected Lie groups are algebraic.

There has been a great deal of interest in the classification of measures invariant under unipotent translations on homogeneous spaces (see refs. 1–9, †). The question is whether all such measures arise algebraically.

More specifically, let $\mathbf{G}$ be a real Lie group (all groups in this paper are assumed to be second countable), Γ a discrete subgroup of $\mathbf{G}$, and $\pi : \mathbf{G} \to \Gamma\backslash\mathbf{G}$ the projection $\pi(\mathbf{g}) = \Gamma\mathbf{g}$. The group $\mathbf{G}$ acts by right translations on $\Gamma\backslash\mathbf{G}$, $(x, \mathbf{g}) \to x\mathbf{g}$, $x \in \Gamma\backslash\mathbf{G}$, $\mathbf{g} \in \mathbf{G}$. Let μ be a Borel probability measure on $\Gamma\backslash\mathbf{G}$. Define

$$\Lambda(\mu) = \Lambda(\mathbf{G}, \Gamma, \mu) = \{\mathbf{g} \in \mathbf{G} : \text{the action of } \mathbf{g} \text{ preserves } \mu\}.$$

The set $\Lambda(\mu)$ is a closed subgroup of $\mathbf{G}$. The measure μ is called *algebraic* if there exists $\mathbf{x} = \mathbf{x}(\mu) \in \mathbf{G}$ such that $\mu(\pi(\mathbf{x})\Lambda(\mu)) = 1$. In this case $\mathbf{x}\Lambda(\mu)\mathbf{x}^{-1} \cap \Gamma$ is a lattice in $\mathbf{x}\Lambda(\mu)\mathbf{x}^{-1}$.

Definition 1: Let $\mathbf{U}$ be a subgroup of $\mathbf{G}$. We say that the action of $\mathbf{U}$ on $\Gamma\backslash\mathbf{G}$ is *measure rigid* if every ergodic $\mathbf{U}$-invariant Borel probability measure on $\Gamma\backslash\mathbf{G}$ is algebraic. The group $\mathbf{U}$ is called *measure rigid* in $\mathbf{G}$ if its action on $\Gamma\backslash\mathbf{G}$ is measure rigid for every lattice $\Gamma \subset \mathbf{G}$. An element $\mathbf{u} \in \mathbf{G}$ is measure rigid if the group $\{\mathbf{u}^k : k \in \mathbf{Z}\}$ is measure rigid. $\mathbf{U} \subset \mathbf{G}$ and $\mathbf{u} \in \mathbf{G}$ are called *strictly measure rigid* if their action on $\Gamma\backslash\mathbf{G}$ is measure rigid for *every* discrete subgroup Γ of $\mathbf{G}$.

A subset $A \subset \Gamma\backslash\mathbf{G}$ is said to be *homogeneous* if there exists $\mathbf{x} \in \mathbf{G}$ and a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{x}\mathbf{H}\mathbf{x}^{-1} \cap \Gamma$ is a lattice in $\mathbf{x}\mathbf{H}\mathbf{x}^{-1}$ and $A = \pi(\mathbf{x})\mathbf{H}$.

Definition 2: A subgroup $\mathbf{U} \subset \mathbf{G}$ is called *orbit rigid* if, for every lattice Γ in $\mathbf{G}$, every compact minimal $\mathbf{U}$-invariant subset of $\Gamma\backslash\mathbf{G}$ is homogeneous.

Definition 3: A subgroup $\mathbf{U} \subset \mathbf{G}$ is called *topologically rigid* if, given any lattice $\Gamma \subset \mathbf{G}$ and any $x \in \Gamma\backslash\mathbf{G}$, the closure of the orbit $x\mathbf{U}$ in $\Gamma\backslash\mathbf{G}$ is homogeneous.

It is clear that topological rigidity implies orbit rigidity.

A natural question arising in this context is what groups are rigid. It has been shown in refs. 5 and 7 that if $\mathbf{G}$ is nilpotent then every subgroup of $\mathbf{G}$ is both measure and topologically rigid. On the other hand, it is a fact that not every subgroup of a semisimple Lie group is rigid. However, certain unipotent subgroups of semisimple Lie groups are rigid (see refs. 2, 3, 6, and 9). A subgroup $\mathbf{U}$ of $\mathbf{G}$ is called unipotent if for each $\mathbf{u} \in \mathbf{U}$ the map $\mathrm{Ad}_{\mathbf{u}}$ is a unipotent linear transformation of the Lie algebra of $\mathbf{G}$. Here are two related conjectures.

Conjecture 1: Every unipotent subgroup of a connected Lie group $\mathbf{G}$ is topologically rigid.

Conjecture 2: Every unipotent subgroup of a connected Lie group $\mathbf{G}$ is measure rigid.

The first conjecture is due to M. S. Raghunathan and the second naturally comes to mind as a measure theoretic version of the first. Various versions of *Conjecture 2* have been stated in ref. 3 and footnote †. I call it Raghunathan's measure conjecture.

It has been shown in refs. 2 and 6 that for $\mathbf{G} = \mathrm{SL}(2, R)$ both conjectures are true. Recently, Dani and Margulis showed in ref. 10 that certain unipotent subgroups of $\mathrm{SL}(3, R)$ are topologically rigid. To the best of my knowledge these are the only cases of semisimple Lie groups for which the conjectures have been settled.

The purpose of this paper is to announce the following result.

MAIN THEOREM. *Every unipotent subgroup of a connected Lie group* $\mathbf{G}$ *is strictly measure rigid.*

To prove this theorem it suffices to show that every unipotent element of a connected Lie group $\mathbf{G}$ is strictly measure rigid. The proof of this consists of three parts. To state the results we need to introduce some notations.

Let $\mathbf{Z}(\mathbf{G})$ denote the center of $\mathbf{G}$ and for a closed subgroup Λ of $\mathbf{G}$ let $\mathfrak{L}(\Lambda)$ and Λ^0 denote the Lie algebra and the identity component of Λ, respectively. Note that $\mathfrak{L}(\Lambda)$ might be trivial. In part 1, the following is proved.

THEOREM 1. *Let* $\mathbf{G}$ *be a Lie group with the Lie algebra* $\mathfrak{G}$, Γ *a discrete subgroup of* $\mathbf{G}$ *(not necessarily a lattice),* μ *a Borel probability measure on* $\Gamma\backslash\mathbf{G}$, *and* $\Lambda = \Lambda(\mathbf{G}, \Gamma, \mu)$. *Let* $\mathbf{H}$ *be a connected simply connected unipotent subgroup of* $\mathbf{G}$ *with the Lie algebra* $\mathfrak{H}$ *such that* $[\mathfrak{G}, \mathfrak{H}] \cap \mathfrak{L}(\Lambda) \subset \mathfrak{H}$. *Suppose that* $\mathbf{u} = \boldsymbol{\alpha}\, exp\, \mathrm{u} \in \Lambda = \Lambda(\mathbf{G}, \Gamma, \mu)$ *for some* $\boldsymbol{\alpha} \in \mathbf{Z}(\mathbf{G})$, $\mathrm{u} \in \mathfrak{H}$ *and the action of* $\mathbf{u}$ *on* $(\Gamma\backslash\mathbf{G}, \mu)$ *is ergodic. Then* μ *is algebraic. In particular, there are* $\mathrm{x} \in \Gamma\backslash\mathbf{G}$, $\mathrm{n} \in \mathbf{Z}^+$ *such that* $\mathrm{x}\Lambda = \cup\{\mathrm{x}\Lambda^0\mathbf{u}^{\mathrm{k}} : \mathrm{k} = 0, \ldots, \mathrm{n}-1\}$*—a disjoint union, where* $\mu(\mathrm{x}\Lambda^0\mathbf{u}^{\mathrm{k}}) = 1/\mathrm{n}$, $\mathrm{k} = 0, \ldots, \mathrm{n}-1$.

COROLLARY 1. (i) *Every unipotent subgroup of a connected solvable Lie group* $\mathbf{G}$ *is strictly measure rigid;* (ii) *if* $\mathbf{G}$ *is a Lie group and* $\mathbf{u} \in \mathbf{Z}(\mathbf{G})$ *then* $\mathbf{u}$ *is strictly measure rigid.*

COROLLARY 2. *Let* $\mathbf{G}$ *be a Lie group,* Γ *a discrete subgroup of* $\mathbf{G}$, *and* μ *a Borel probability measure on* $\Gamma\backslash\mathbf{G}$. *Suppose that* $\Lambda^0 = \Lambda^0(\mathbf{G}, \Gamma, \mu)$ *is contained in a connected simply connected unipotent subgroup* $\mathbf{H} \subset \mathbf{G}$, $\mathbf{u} = \boldsymbol{\alpha}\mathbf{h} \in \Lambda$, $\boldsymbol{\alpha} \in \mathbf{Z}(\mathbf{G})$, $\mathbf{h} \in \mathbf{H}$ *and the action of* $\mathbf{u}$ *on* $(\Gamma\backslash\mathbf{G}, \mu)$ *is ergodic. Then* μ *is algebraic. In particular, there are* $\mathrm{x} \in \Gamma\backslash\mathbf{G}$, $\mathrm{n} \in \mathbf{Z}^+$ *such that* $\mathrm{x}\Lambda$ *has the form as in* Theorem 1.

COROLLARY 3. *Let* $\mathbf{G}$, Γ *and* μ *be as in* Corollary 2. *Suppose that the action of a unipotent element* $\mathbf{u} = exp\, \mathrm{u} \in \Lambda = \Lambda(\mathbf{G}, \Gamma, \mu)$, $\mathrm{u} \in \mathfrak{G}$ *on* $(\Gamma\backslash\mathbf{G}, \mu)$ *is ergodic and* (i) $\mathfrak{L}(\Lambda) = \{0\}$ *or* (ii) $\mathfrak{L}(\Lambda) = \{\mathrm{tu} : \mathrm{t} \in \mathrm{R}\}$. *Then* μ *is supported on a closed orbit of* $\mathbf{u}$ *in case* i *and on a closed orbit of* $\mathbf{u}(\mathrm{t}) = exp\, \mathrm{tu}$, $\mathrm{t} \in \mathrm{R}$ *in case* ii.

To prove *Theorem 1* in part 1 the ideas and techniques used throughout the entire proof of the *Main Theorem* are developed. Using polynomial divergence of unipotent orbits we derive a dynamical property of unipotent group actions,

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†Margulis, G. A. (1989) Proceedings of the Conference in Honor of Professor A. Selberg, Number Theory, Trace Formulas, and Discrete Groups, Oslo, July 14–24, 1987.

called the R-property, which plays a central role in the method. It is a generalization of the property for unipotent flows introduced in refs. 8 and 11. (In ref. 8 it is called the H-property and in ref. 11 the Ratner property. Recently, a topological version of this property has been exploited by Margulis in footnote†.) The R-property states, roughly speaking, that if $\mathbf{U}_1$ and $\mathbf{U}_2$ are large pieces of a simply connected unipotent subgroup of $\mathbf{G}$ such that the distance $d(\mathbf{x}\mathbf{U}_1, \mathbf{y}\mathbf{U}_2)$ is small for some $\mathbf{x}, \mathbf{y} \in \mathbf{G}$ then the distance $d(\mathbf{x}\mathbf{u}, \mathbf{y}\mathbf{V}_2)$ is close to $d(\mathbf{x}, \mathbf{y})$ for all $\mathbf{u} \in \mathbf{V}_1 \subset \mathbf{U}_1$ with $\mathbf{V}_i$, $i = 1, 2$ being a large subset of $\mathbf{U}_i$ whose Haar measure is proportional to that of $\mathbf{U}_i$. Also, in part 1, an ergodic theory of measure preserving actions of simply connected nilpotent Lie groups is developed that enables us to trace the behavior of most of the individual orbits of such actions.

In part 2 semisimple $\mathbf{G}$ is studied. We say that the $\mathbf{g}$-orbit of $\pi(\mathbf{x}) \in \Gamma\backslash\mathbf{G}$, $\mathbf{x} \in \mathbf{G}$, $\mathbf{g} \in \mathbf{G}$ *diverges* when $n \to \infty$ if there are $\mathbf{e} \neq \gamma_n \in \Gamma$, $n = 1, 2, \ldots$ such that $(\mathbf{x}\mathbf{g}^n)^{-1}\gamma_n(\mathbf{x}\mathbf{g}^n) \to \mathbf{e}$, when $n \to \infty$. Here $\mathbf{e}$ denotes the identity element of $\mathbf{G}$. Define

$$D(\mathbf{g}) = \{x \in \Gamma\backslash\mathbf{G} : \text{the } \mathbf{g}\text{-orbit of } x \text{ diverges when } n \to \infty\}.$$

It is clear that if $D(\mathbf{g}) \neq \emptyset$ for some $\mathbf{g} \in \mathbf{G}$ then $\Gamma\backslash\mathbf{G}$ is not compact.

We call an element $u \in \mathfrak{G}$ nilpotent if $\mathrm{ad}_u : \mathfrak{G} \to \mathfrak{G}$, $\mathrm{ad}_u(v) = [v, u]$ is a nilpotent endomorphism of $\mathfrak{G}$. An element $g \in \mathfrak{G}$ is called diagonalizable (over R) if $\mathfrak{G}$ is the direct sum of one-dimensional invariant subspaces of ad_g. We introduce the following definition.

Definition 4: Let $\mathbf{G}$ be a Lie group with the Lie algebra $\mathfrak{G}$ and Γ a discrete subgroup of $\mathbf{G}$. (*i*) A nilpotent element $u \in \mathfrak{G}$ is called *horocyclic* if there is a diagonalizable element $g \in \mathfrak{G}$ and a nilpotent element $u^* \in \mathfrak{G}$ such that $\mathrm{ad}_{u^*}(u) = g$, $\mathrm{ad}_g(u) = -2u$, $\mathrm{ad}_g(u^*) = 2u^*$. In this case we say that u is "*horocyclic for* g" and "g is diagonal for u." (*ii*) An element $\mathbf{u} \in \mathbf{G}$ is called *horocyclic* if $\mathbf{u} = \exp u$ for some horocyclic element $u \in \mathfrak{G}$. (*iii*) An element $\mathbf{u} \in \mathbf{G}$ is called Γ-horocyclic if $\mathbf{u} = \mathbf{z} \exp u$ for some horocyclic $u \in \mathfrak{G}$ and some $\mathbf{z} \in \mathbf{Z}(\mathbf{G})$ with $\mathbf{z}^k \in \Gamma$ for some $k \in \mathbf{Z}$. An element $\mathbf{g} \in \mathbf{G}$ is diagonal for $\mathbf{u}$ if $\mathbf{g} = \exp g$ with g being diagonal for u.

The terminology in *Definition 4* is motivated by the fact that u, g, and u^* generate a Lie subalgebra $\mathrm{sl}_2(u, g)$ of $\mathfrak{G}$ isomorphic to $\mathrm{sl}(2, R)$. It is a fact that if $\mathbf{G}$ is a connected semisimple Lie group then all nontrivial elements of one-parameter unipotent subgroups of $\mathbf{G}$ are horocyclic. Also if Γ is a lattice in $\mathbf{G}$ which projects densely into the maximal compact factor of $\mathbf{G}$ (such Γ is called *compatible* with $\mathbf{G}$) then every noncentral unipotent element of $\mathbf{G}$ is Γ-horocyclic.

Let g be a diagonalizable element of $\mathfrak{G}$, $\mathbf{g}_p = \exp pg$, $\mathbf{g} = \mathbf{g}_1$, $\mathfrak{E}_\lambda$ the eigenspace of ad_g with the eigenvalue λ, $\mathfrak{E}^-(g) = \Sigma\{\mathfrak{E}_\lambda, \lambda < 0\}$ and $\mathbf{E}^- = \mathbf{E}^-(\mathbf{g}) = \exp \mathfrak{E}^-(g)$. It is clear that if g is diagonal for u, then so is $\mathbf{c}g\mathbf{c}^{-1}$ for every $\mathbf{c} \in \mathbf{C}(u)$—the centralizer of $\{\exp tu, t \in R\}$ in $\mathbf{G}$. We show that an element $\bar{g} \in \mathfrak{G}$ is diagonal for u if and only if $\bar{g} = \mathbf{c}g\mathbf{c}^{-1}$ for some $\mathbf{c} \in \mathbf{E}^-(\mathbf{g}) \cap \mathbf{C}(u)$. In this case $\mathfrak{E}^-(\bar{g}) = \mathfrak{E}^-(g)$. It is clear that $D(\mathbf{g}) = D(\bar{\mathbf{g}})$ and $x\mathbf{E}^-(\mathbf{g}) \subset D(\mathbf{g})$ whenever $x \in D(\mathbf{g})$. We prove the following theorem.

THEOREM 2. *Let* $\mathbf{G}$ *be a Lie group and* Γ *a discrete subgroup of* $\mathbf{G}$ *(not necessarily a lattice). Let* $\mathbf{u} = \mathbf{z} \exp \mathrm{u}$, $\mathbf{z} \in \mathbf{Z}(\mathbf{G})$, $\mathrm{u} \in \mathfrak{G}$ *be a* Γ*-horocyclic element of* $\mathbf{G}$ *and* $\mathrm{g} \in \mathfrak{G}$ *a diagonal element for* u, $\mathbf{g}_\mathrm{p} = \exp \mathrm{pg}$, $\mathbf{g}_1 = \mathbf{g}$. *Let* μ *be a Borel probability measure on* $\Gamma\backslash\mathbf{G}$ *such that* $\mathbf{u} \in \Lambda = \Lambda(\mu)$ *and the action of* $\mathbf{u}$ *on* $(\Gamma\backslash\mathbf{G}, \mu)$ *is ergodic. Then either* (i) $\mu(\mathrm{D}(\mathbf{g})) = 1$ *or* (ii) $\mathbf{c}\mathbf{g}_\mathrm{p}\mathbf{c}^{-1} \in \Lambda$ *for some* $\mathrm{p} \in \mathrm{R}$, $\mathbf{c} \in \mathbf{E}^-(\mathbf{g})$. *In this case* $\mathrm{u} \in \mathfrak{L}(\Lambda)$, $\mathbf{c}sl_2(\mathrm{u}, \mathrm{g})\mathbf{c}^{-1} \subset \mathfrak{L}(\Lambda)$ *and* μ *is algebraic. Also* $\mathrm{x}\Lambda = \mathrm{x}\Lambda^0$, *where* $\mathrm{x} = \pi(\mathbf{x}(\mu))$.

COROLLARY 4. *Let* $\mathbf{G}$ *be a connected semisimple Lie group. Then* (i) Theorem 1 *holds for all nontrivial elements* $\mathbf{u}$ *of one-parameter unipotent subgroups of* $\mathbf{G}$; (ii) *if* Γ *is a compatible lattice in* $\mathbf{G}$ *then* Theorem 1 *holds for all noncentral unipotent elements of* $\mathbf{G}$.

COROLLARY 5. *Let* $\mathbf{G}$ *be a Lie group and* Γ *a uniform lattice in* $\mathbf{G}$. *Let* $\mathbf{H}$ *be a closed subgroup of* $\mathbf{G}$ *such that* $\mathbf{H} \cap \Gamma$ *is a lattice in* $\mathbf{H}$. *Suppose that* $\mathbf{H}$ *contains a* Γ*-horocyclic element of* $\mathbf{G}$. *Then the Lie algebra* $\mathfrak{H}$ *of* $\mathbf{H}$ *is not trivial and* $sl_2(\mathrm{u}, \mathrm{g}) \subset \mathfrak{H}$ *for some horocyclic* $\mathrm{u} \in \mathfrak{G}$ *and a diagonal* $\mathrm{g} \in \mathfrak{G}$.

COROLLARY 6. *Let* $\mathbf{G}$ *be a connected Lie group and* Γ *a uniform lattice in* $\mathbf{G}$. *Then the action of every* Γ*-horocyclic element of* $\mathbf{G}$ *on* $\Gamma\backslash\mathbf{G}$ *is measure rigid. If, in addition,* $\mathbf{G}$ *is semisimple and* Γ *is compatible with* $\mathbf{G}$ *then the action of every unipotent subgroup of* $\mathbf{G}$ *on* $\Gamma\backslash\mathbf{G}$ *is measure rigid.*

In part 3 the proof of the *Main Theorem* is completed using induction on the dimension of $\mathbf{G}$. First we show using *Theorems 1* and *2* that if $\mathfrak{L}(\Lambda)$ is amenable then μ is algebraic. [$\mathfrak{L}(\Lambda)$ is called amenable if the quotient space of Λ^0 modulo its radical is compact.] Then we assume $\mathfrak{L}(\Lambda)$ is not amenable. Then it contains a horocyclic element h and a diagonal element g for h. By *Theorem 2* every ergodic component of the action of $\mathbf{h} = \exp h$ on $(\Gamma\backslash\mathbf{G}, \mu)$ is algebraic. Then we take quotients modulo these components to lower the dimension of the underlying group and use the inductive hypothesis.

The main theorem provides some important ergodic theoretic consequences. Namely, it allows us to classify up to an isomorphism all ergodic joinings of two unipotent translations as well as factors of such translations. More specifically, let $\mathbf{G}_i$, $i = 1, 2$ be a Lie group, Γ_i a lattice in $\mathbf{G}_i$, ν_i a $\mathbf{G}_i$-invariant Borel probability measure on $\Gamma_i\backslash\mathbf{G}_i = X_i$, $\mathbf{u}^{(i)} \in \mathbf{G}$, $\mathbf{u} = \mathbf{u}^{(1)} \times \mathbf{u}^{(2)}$. A $\mathbf{u}$-invariant Borel probability measure μ on $X = X_1 \times X_2$ is called a joining of $\mathbf{u}^{(1)}$ on (X_1, ν_1) and $\mathbf{u}^{(2)}$ on (X_2, ν_2) if $\mu(A \times X_2) = \nu_1(A)$, $\mu(X_1 \times B) = \nu_2(B)$ for all Borel subsets $A \subset X_1$, $B \subset X_2$. The joining $\nu_1 \times \nu_2$ will be called the trivial joining. It has been shown (in part 2) that if $\mathbf{G}_1$ and $\mathbf{G}_2$ are connected and a joining μ is algebraic then the groups $\Lambda_1(\mu)$ and $\Lambda_2(\mu)$ defined by

$$\Lambda_1(\mu) = \{\mathbf{h} \in \mathbf{G}_1 : (\mathbf{h}, \mathbf{e}) \in \Lambda(\mu)\},$$

$$\Lambda_2(\mu) = \{\mathbf{h} \in \mathbf{G}_2 : (\mathbf{e}, \mathbf{h}) \in \Lambda(\mu)\}$$

are closed normal subgroups of $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. Here $\Lambda(\mu) \subset \mathbf{G}_1 \times \mathbf{G}_2$, $\mu(x(\mu)\Lambda(\mu)) = 1$, $x(\mu) \in X = X_1 \times X_2$. For $\mathbf{c} \in \mathbf{G}_2$ write $\Gamma_2^\xi = \{\gamma\Lambda_2(\mu) : \gamma \in \mathbf{c}^{-1}\Gamma_2\mathbf{c}\}$ and for $z \in X_1$ let

$$\xi_\mu(z) = \{y \in X_2 : (z, y) \in x(\mu)\Lambda(\mu)\}.$$

The set $\xi_\mu(z)$ is called the z-fiber of μ.

THEOREM 3. *Let* $\mathbf{G}_\mathrm{i}$ *be a connected Lie group,* Γ_i *a lattice in* $\mathbf{G}_\mathrm{i}$ *and* $\mathbf{u}^{(\mathrm{i})} \in \mathbf{G}_\mathrm{i}$, $\mathrm{i} = 1, 2$. *Let* μ *be an ergodic algebraic joining of* $\mathbf{u}^{(1)}$ *on* $(\mathrm{X}_1 = \Gamma_1\backslash\mathbf{G}_1, \nu_1)$ *and* $\mathbf{u}^{(2)}$ *on* $(\mathrm{X}_2 = \Gamma_2\backslash\mathbf{G}_2, \nu_2)$. *Then there is* $\mathbf{c} \in \mathbf{G}_2$ *and a continuous surjective homomorphism* $\alpha : \mathbf{G}_1 \to \mathbf{G}_2/\Lambda_2(\mu)$ *with kernel* $\Lambda_1(\mu)$, $\alpha(\mathbf{u}^{(1)}) = \mathbf{u}^{(2)}\Lambda_2(\mu)$ *such that*

$$\xi_\mu(\Gamma_1\mathbf{h}) = \{\Gamma_2\mathbf{c}\beta_\mathrm{i}\alpha(\mathbf{h}) : \mathrm{i} = 1, \ldots, \mathrm{n}\}$$

for all $\mathbf{h} \in \mathbf{G}_1$, *where the intersection* $\Gamma_0 = \alpha(\Gamma_1) \cap \Gamma_2^\xi$ *is of finite index in* $\alpha(\Gamma_1)$ *and in* Γ_2^ξ, $\mathrm{n} = |\Gamma_0\backslash\alpha(\Gamma_1)|$ *and* $\alpha(\Gamma_1) = \{\Gamma_0\beta_\mathrm{i} : \mathrm{i} = 1, \ldots, \mathrm{n}\}$.

COROLLARY 7 (The Joinings Theorem). (i) *Let* $\mathbf{G}_\mathrm{i}$ *be a connected Lie group,* Γ_i *a lattice in* $\mathbf{G}_\mathrm{i}$ *and* $\mathbf{u}^{(\mathrm{i})}$ *a unipotent element of* $\mathbf{G}_\mathrm{i}$, $\mathrm{i} = 1, 2$. *Let* μ *be an ergodic joining of* $\mathbf{u}^{(1)}$ *and* $\mathbf{u}^{(2)}$. *Then* μ *is algebraic and the fibers of* μ *are given by* Theorem 3. (ii) *If in addition* $\mathbf{G}_\mathrm{i}$ *is simple,* $\mathrm{i} = 1, 2$, *and* μ *is nontrivial then every fiber of* μ *is finite and* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are locally isomorphic.*

Corollary 7 generalizes the joinings theorem for $\mathbf{G}_i = \mathrm{SL}(2, R)$, $i = 1, 2$ obtained in ref. 8. Some restricted results of this nature were also obtained in ref. 12. As in corollary 4 of ref. 8 we obtain the following.

COROLLARY 8 (The Rigidity Theorem). *Let* $\mathbf{G}_i$ *be a connected Lie group,* Γ_i *a lattice in* $\mathbf{G}_i$ *containing no nontrivial normal subgroups of* $\mathbf{G}_i$ *and* $\mathbf{u}^{(i)}$ *a unipotent element of* $\mathbf{G}_i$, i = 1, 2. *Suppose that the action of* $\mathbf{u}^{(1)}$ *on* (X_1, ν_1) *is ergodic and there is a measure preserving map* $\psi : (X_1, \nu_1) \to (X_2, \nu_2)$ *such that* $\psi(x\mathbf{u}^{(1)}) = \psi(x)\mathbf{u}^{(2)}$ *for* ν_1*-almost every* $x \in X_1$. *Then there is* $\mathbf{c} \in \mathbf{G}_2$ *and a surjective homomorphism* $\alpha : \mathbf{G}_1 \to \mathbf{G}_2$ *such that* $\alpha(\Gamma_1) \subset \mathbf{c}^{-1}\Gamma_2\mathbf{c}$ *and* $\psi(\Gamma_1\mathbf{h}) = \Gamma_2\mathbf{c}\alpha(\mathbf{h})$ *for* ν_1*-almost every* $\Gamma_1\mathbf{h} \in X_1$. *Also* α *is a local isomorphism whenever* ψ *is finite to one or* $\mathbf{G}_1$ *is simple and it is an isomorphism whenever* ψ *is one-to-one or* $\mathbf{G}_1$ *is simple with trivial center.*

This corollary generalizes the rigidity theorem for SL(2, *R*) presented in ref. 13. It was previously obtained in refs. 11 and 12 by methods from refs. 8 and 13.

Let $\mathbf{G}$, Γ, ν, and $\mathbf{u} \in \mathbf{G}$ be as above. A $\mathbf{u}$-invariant measurable partition ξ of $(\Gamma\backslash\mathbf{G}, \nu)$ is called a factor of $\mathbf{u}$. We denote by $\xi(x)$ the atom of ξ containing $x \in \Gamma\backslash\mathbf{G} = X$. The factor ξ is called *algebraic* if there is a surjective homomorphism $\alpha : \mathbf{G} \to \mathbf{G}$ such that $\xi(x\mathbf{h}) = \xi(x)\alpha(\mathbf{h})$ for all $\mathbf{h} \in \mathbf{G}$ and ν-almost every $x \in X$. It has been shown in refs. 8 and 14 that if $\mathbf{G} = SL(2, R)$ then every factor of a unipotent element of $\mathbf{G}$ is algebraic. In general, algebraicity of factors of unipotent translations is rather an exception. Indeed, it was shown in ref. 8 that if $\mathbf{u}$ is the *n*-fold cartesian product $\mathbf{u}_1 \times \dots \times \mathbf{u}_n$ of unipotent elements $\mathbf{u}_i \in \mathbf{G}_i = SL(2, R)$, $i = 1, \dots, n$ acting ergodically on $(\Gamma\backslash\mathbf{G}, \nu^n)$ with $\mathbf{G} = \mathbf{G}_1 \times \dots \times \mathbf{G}_n$, $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$, $\nu^n = \nu_1 \times \dots \times \nu_n$, then every factor of this action has the form $\mathbf{H}\backslash\mathbf{G}/L$, where $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$, containing Γ and L is a closed group of affine maps on $\mathbf{H}\backslash\mathbf{G}$ centralized by $\mathbf{u}$. Recently, Witte (15) showed using the *Main Theorem* that this is true for general $\mathbf{G}$ and $\mathbf{u}$.

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