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Regularities in the sequences of the number of nucleons in the revolving clusters for the ground-state energy bands of the even–even nuclei with neutron number equal to or greater than 126

(nuclear rotational bands/deformed nuclei/radius of revolution of clusters)

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ABSTRACT **Values of *m*, the number of nucleons in the revolving cluster, and of *R*, the radius of revolution of the cluster about the center of mass of the spherical part of the nucleus, are calculated from the observed values of the energy for the ground-state bands of all nuclei with neutron number $N \geq 126$ on the basis of the assumptions (*i*) that both *m* and *R* change in a reasonable way with increase in the angular momentum quantum number *J* and with change in the proton number *Z* and the neutron number *N*, (*ii*) that *m* is usually an even integer, (*iii*) that certain clusters are especially stable, and (*iv*) that there is a special stability of the doubly magic sphere $p^{82}n^{126}$.**

Historical Comparison of the Development of Theories of the Structure of Molecules and of Atomic Nuclei

Beginning about 1860, chemists striving to interpret observations on the composition and properties of chemical compounds developed a powerful empirical chemical theory of molecular structure, comprising concepts of valence and of orientation in space of the valence bonds formed by an atom (the four single bonds of carbon and some other elements directed toward the corners of a tetrahedron, and the four or six bonds of some transition metals towards the corners of a square or octahedron). Various refinements of the theory have greatly increased its usefulness. The discovery of quantum mechanics led to the recognition that the Schrödinger wave equation provides a sound and essentially complete basis for chemical structure. The practical difficulties in solving the Schrödinger equation for larger molecules are so great, however, that in fact almost all chemical discoveries continue to be made with use of the empirical structure theory rather than the Schrödinger equation.

The history of the theory of nuclear structure has been, however, quite different. Little information about the properties of the 1500 different atomic nuclei had been gathered before quantum mechanics was discovered. When new experimental results began to be obtained, beginning around 1930, theoretical physicists began to apply quantum mechanics to determine the law of force between nucleons and to calculate theoretical values of nuclear properties. The 1500 nuclei, built of two kinds of nucleons, protons and neutrons, constitute a simpler problem than the millions of known chemical molecules, built of 106 chemically different kinds of atoms. It should be possible to formulate an empirical structure theory of nuclei, which might well have greater usefulness than the quantum mechanical calculations, but in fact the existence of quantum mechanics, which sometimes permits some broad generalizations to be made about nuclear structure and properties, such as the classification of nuclei as spherical or deformed, seems to have had a powerful inhibitory effect, preventing most nuclear physicists from attempting to formulate a useful empirical theory.

William Draper Harkins, then professor of physical chemistry at the University of Chicago, was a pioneer in the nuclear-structure field. In 1920 he made use of the known atomic numbers, *Z*, and mass numbers, *A*, of nuclei to draw the conclusion that a useful picture of an atomic nucleus is that it consists usually of α particles (^{4}He) and tritons (^{3}H), with, in some cases, deuterons (^{2}H) or neutrons (*n*) (1). At that time the neutron, the deuteron, and the triton had not been discovered as independent particles. Harkins, who was the first scientist to discuss the neutron in a published paper, proposed neutron as its name (2).

Later work on the α-particle model, the shell model, and the liquid-drop model has formed the basis for the present conventional nuclear theory.

After thinking about the structure and properties of nuclei for many years, I published in 1965 a paper based on the assumption that the spherons (α particles and tritons), with diameter about 3.27 fm, are closely packed in the nuclei (3). The larger nuclei have alternating sequences in which there is either a helion (α particle) or a tetrahedral complex of four helions (^{16}O) at the center, surrounded by successive layers of spherons. The composition of the layers is determined by both the usual shell model and close packing. This spheron-packing model provides a structural interpretation of some observed nuclear properties.

Revolving Clusters in Nuclei

In 1950 Rainwater (4) pointed out that the large observed values of the electric quadrupole moments of some nuclei could be accounted for by the assumption that the nuclei have a prolate spheroidal shape rather than a spherical shape, and that an extension of the shell model leads to the prediction that for values of the number of protons, *Z*, and the number of neutrons, *N*, differing from the magic numbers, this spheroidal deformation would be stable. He also solved the Schrödinger wave equation for a particle in a spheroidal box. Bohr and Mottelson (5) then used this model to account for the observed bands of energy levels for many nuclei. They took into consideration the fact that the inner completed shells and subshells do not contribute to the angular momentum by assuming that the ellipsoidal nucleus does not rotate as a rigid body, but instead the angular momentum results from the motion of a wave over its surface.

In 1969 I was struck by the observation that the excitation energy to the 2^+ level for ${}^{206}_{82}\mathrm{Pb}_{124}$ (${}^{A}_{Z}\mathrm{M}_N$), 0.803 MeV, is almost exactly twice that for ${}^{204}_{80}\mathrm{Hg}_{124}$, with the same ratio for other similar pairs. This fact suggested that ${}^{204}_{80}\mathrm{Hg}_{124}$ has an α-particle hole revolving in the sphere of the doubly magic ${}^{208}_{82}\mathrm{Pb}_{126}$, whereas ${}^{206}_{82}\mathrm{Pb}_{124}$ has a dineutron hole (6). If the

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α-particle hole and the dineutron hole revolve at the same radius, R, from the center of mass, the moments of inertia would be in the ratio 2:1, accounting for the observation and suggesting that such an analysis could be used to evaluate R.

The study of many values of the $J = 2^+$ level led to the conclusion that the value of R in the region near $A = 208$ is about 7 fm, rising to about 11 fm at $A = 242$. This fact suggested that the other energy levels of the bands be used to obtain values of m and R and thus to permit some understanding to be developed about the nature of the structural changes that cause the sometimes apparently erratic changes in the values of the energy E as a function of J and also changes in E from nucleus to adjacent nucleus for the same value of J. So far as I am aware, no previous analysis of these values of E has provided any detailed understanding of their origin.

Values of the Radius of Revolution

All of the values of $E(J)$ used in this study are from the 1984 compilation for even–even nuclei made by Sakai (7). For the heavy nuclei there are about eight isotopes per element and an average of about 10 levels per ground-state band. The present analysis is limited to the 11 elements from $_{80}$Hg to $_{100}$Fm.

Values of the energy levels $E(J)$ relative to $E(0^+) = 0$, in MeV, are given in ref. 7 and are not reproduced in this paper. Values of R, in fm, are given by Eq. 1, with μ being the reduced mass (in daltons) of the cluster and the rest of the nucleus.

$$R = \{20.9\, J(J+1)/\mu E(J)\}^{1/2}. \quad [1]$$

In applying this equation the assumption is made that m is usually an even integer such as to make $R(J)$ as close as possible to the expected value. Moreover, for Z or N close to or equal to a magic number, the value of m for $J = 2^+$ can usually be taken to be 2 or 4, as for the Hg–Pb example mentioned above.

Deviation of the Number of Nucleons in the Revolving Cluster from an Even Integer

The shell model leads to the expectation that the number of nucleons in the revolving cluster should be an even integer. For example, for jj coupling the configuration $(d3/2)^2$ gives the allowed states $J = 0^+$ (moments coupled) and $J = 2^+$ (moments uncoupled), and $(d5/2)^2$ gives $J = 0^+, 2^+$, and 4^+. In Tables 3 and 5, however, there are a few examples of odd-integral values of J, assigned to give regularity to the sequences of values of R. It is likely that these levels represent resonance between two structures with adjacent even-integral values of m. If the off-diagonal energy terms (resonance energy) in the secular equation are small compared with the separation of levels in the band, resonance of $m - 2$, m, and $m + 2$ will lead to an energy level close to that for the even-integral value m except when the pure-state energies of $m - 2$ and m or of m and $m + 2$ are nearly equal, in which cases the two structures will contribute nearly equally, giving a resonance state approximated by the intermediate odd-integral value of m.

Correction of the Radius of Revolution of the Cluster for Centrifugal Stretching

For some nuclei the values of $E(J)$ are nearly proportional to $J(J + 1)$. Some examples are given in Table 1. For these nuclei the increase of the moment of inertia $I(8^+)$ over $I(2^+)$ decreases from 6.5% for $A = 232$ to 2.5% for $A = 250$. A discussion of this decrease is given in a following section.

Table 1. Values of the moment of inertia, $I(J^+)$, calculated by Eq. 1 from the energy values $E(J^+)$ from ref. 7, for several nuclei

	I, Da·fm^2				
J^+	$^{232}_{90}$Th$_{142}$	$^{234}_{92}$U$_{142}$	$^{240}_{94}$Pu$_{146}$	$^{246}_{96}$Cm$_{150}$	$^{250}_{98}$Cf$_{152}$
2^+	2540.1	2884.1	2928.2	2926.4	2935.3
4^+	2578.3	2916.6	2950.2	2943.5	2946.0
6^+	2634.5	2965.2	2982.5	2977	2963.0
8^+	2702.1	3030.2	3024.1	3007	3009.6

Increase of $I(8^+)$ over $I(2^+)$ is by 6.5, 5.1, 3.3, 2.8, and 2.5%, respectively.

These increases are small enough to permit the assumption that they result from centrifugal stretching.

With a Hooke's law potential function between the cluster and the sphere (the rest of the nucleus), the expression for the energy for the zeroth vibrational state relative to $E(0^+)$ is given by Eq. 2 (see ref. 8, p. 271):

$$E(J) = \frac{J(J+1)h^2}{8\pi^2 I_0} - \frac{J^2(J+1)^2 h^4}{128\pi^6 \nu_0^2 I_0^3} + \text{further terms.} \quad [2]$$

Here ν_0 is the frequency of vibration of the cluster and sphere, equal to $(2\pi)^{-1/2}(k/\mu)^{1/2}$, in which k is the vibrational force constant and μ is the reduced mass. We may expect ν_0 to be nearly the same for different nuclei and also for the same nucleus with different values of m, by the following argument. The cluster usually has the form of a cap, an approximately planar (somewhat bowl-shaped) layer of spherons, on the surface of the sphere. The value of k is determined by the interaction of the spherons in the cap with those in the outer layer of the sphere; hence it is proportional to the number of spherons in the cap. The mass m of the cap is also proportional to the number of spherons in the cap, and hence to the nucleon number m. The difference between m and μ affects only the further terms in Eq. 2; hence ν_0 in the second term of this equation can be taken to be constant.

Replacement of I_0^2 in the second term by its approximate value given by the first term and by μR_0^2 then leads, on expansion, to the following equation:

$$I(J^+) = I(0^+)/\{1 - cJ(J+1)\} + \text{further terms.} \quad [3]$$

To this approximation, Eq. 3 can also be derived from the equation for the energy corresponding to the assumption of the Morse function for the vibrational energy (see ref. 8, p. 274). Only the further terms are different from those for the Hooke's law equation.

Evidence for the Assumption That the Nucleon Number of the Revolving Cluster Is Usually Close to an Even Integer

Sakai in his table (7) gives for the ground-state band the ratio of the energy of the second excited state to that of the first, $r = E(4^+)/E(2^+)$. When this ratio is close to 3.33, it may be concluded that the nucleon number of the cluster is the same in the two states, and it is often found that m remains constant for the whole band, with only a steady decrease in I with increasing J. Observed values of r are often much less than 3.33, even as small as 1.1, corresponding to $\mu(4^+) = 3\mu(2^+)$. An example is $^{212}_{86}$Rn$_{126}$, with $r = 1.179$; I assign to it the values $m(2^+) = 2$ and $m(4^+) = 6$.

The value 2 for Δm, equal to $m(J + 2) - m(J)$, occurs often. An example of a band with six values of 2 for Δm (following one value of 4) is given in Table 2. For this nucleus the composition p^4n^6 is expected for $J = 2^+$, because it has a good

Table 2. Values of the effective moment of inertia, I, assigned number of nucleons in the revolving cluster, m, and the radius of revolution, R, for the ground-state band of $^{222}_{90}Th_{132}$

J	$I(J)$, Da·fm^2	m	R, fm
2^+	634.1	10	8.46
4^+	950.4	14	8.51
6^+	1170.4	16	8.88
8^+	1376	18	9.12
10^+	1573	20	9.30
12^+	1761	22	9.43
14^+	1942	24	9.52
16^+	2113	26	9.59

neutron/proton ratio and gives a magic number of neutrons, 126, for the sphere; moreover, the value of $R(2^+)$ fits in a sequence with the values for neighboring nuclei.

Many nuclei show irregularities in the sequence of values of $I(J)$, indicating values of Δm other than 0, 2, and 4. Some examples are given in Table 3.

Nuclei with Neutron-Number N Equal to or Greater Than 126

Values of R for the ground-state bands of all nuclei with $N \geq 126$ are given in Tables 3, 4, and 5. Table 3 gives these values for all of those nuclei for which m increases with increase in J, except for one, $^{206}_{80}Hg_{126}$, which has $m = 2$, $R = 7.70$ fm for $J = 2^+$ and $m = 4$, $R = 7.12$ fm for $J = 4^+$. These values fit in well with those for adjacent nuclei.

We see from Table 3 that for $N = 126$ to 132, the values of $R(J)$ are nearly independent of both J and N, as well as of Z, ranging from 6.30 to 7.32 fm, median being 6.91 fm. For larger values of N or of A there is a steady increase in R to about 10 fm for $^{224}_{88}Ra_{136}$ and $^{224}_{90}Ra_{134}$. It can be seen from Table 4 that this increase continues, reaching 11.17 fm for $^{234}_{92}U_{142}$.

Table 3. Values of the nucleon number m and the radius of revolution R of isotopes of $_{82}Pb$, $_{84}Po$, $_{86}Rn$, $_{88}Ra$, and $_{90}Th$ for levels with $J = 2^+$, 4^+, 6^+, and 8^+ (some not reported)

	$J = 2^+$		$J = 4^+$		$J = 6^+$		$J = 8^+$	
Isotope	m	R, fm	m	R, fm	m	R, fm	m	R, fm
$_{82}Pb$								
Pb_{128}	4	6.32	8	7.03	16	7.05	26	7.19
Pb_{130}	4	6.30	8	6.97	16	6.82	26	7.03
$_{84}Po$								
Po_{126}	2	7.32	6	7.09	12	7.16	22	7.06
Po_{128}	4	6.63	8	6.92	14	7.04	24	6.92
Po_{130}	4	7.24						
Po_{132}	5	7.08						
$_{86}Rn$								
Rn_{126}	2	7.05	6	6.91	12	6.88	20	7.05
Rn_{128}	4	6.78	8	6.90	14	6.82	20	6.88
Rn_{130}	6	6.80						
Rn_{132}	8	7.08	14	6.99				
Rn_{134}	8	7.39	12	7.27				
Rn_{136}	10	8.73	14	8.43				
$_{88}Ra$								
Ra_{126}	2	6.77	6	6.62	14	6.68	20	6.68
Ra_{128}	4	6.82	8	6.84	14	6.67	22	6.68
Ra_{130}	8	6.80	12	7.05	16	6.96	22	7.01
Ra_{132}	16	6.91	20	6.96				
Ra_{134}	18	8.26	22	8.36				
Ra_{136}	16	10.00	18	10.03				
$_{90}Th$								
Th_{132}	10	8.46	14	8.51	16	8.88	18	9.12
Th_{134}	14	10.13	16	10.02				

Values of m and R were calculated from the values of $E(J^+)$ given in ref. 7, with values of m assigned to make R nearly constant for each nucleus.

Table 4. Values of the radius of revolution, $R(2^+)$, for isotopes of $_{88}Ra$, $_{90}Th$, and $_{92}U$ with the number of nucleons, m, equal to $A - 208$ for each value of J in the ground-state band, the doubly magic sphere being $p^{82}n^{126}$

	$R(2^+)$ values of isotopes, fm				
Element	$A = 226$, $m = 18$	$A = 228$, $m = 20$	$A = 230$, $m = 22$	$A = 232$, $m = 24$	$A = 234$, $m = 26$
$_{88}Ra$	10.58	10.58			
$_{90}Th$	10.33	10.91	10.88	10.87	
$_{92}U$		10.79	11.04	11.07	11.17

Expected Compositions of Clusters

The expected composition of a cluster is determined by several factors. An important one is having a sphere that is doubly magic or semimagic. For the very heavy nuclei this sphere is the $^{208}_{82}Pb_{126}$ structure. In Table 4 there are given values of $R(2^+)$ for 10 nuclei for which the cluster has $m = A - 208$ for all values of J. Also, nuclei with $A = 210$ have $m = 2$, and those with $A = 212$ usually have $m = 4$ (7).

The second factor is the neutron/proton ratio of the cluster, which for maximum stability is equal to that of the sphere, and hence to N/Z (1.54 for $^{208}_{82}Pb_{126}$ and $^{244}_{92}U_{142}$, and 1.56 for $^{256}_{100}Fm_{156}$). The clusters in Table 4 have values between 1.6 and 2.0.

A third factor is the number of spherons in the cluster. Centered rings are especially stable: rings of 4, 5, 6, or 7 spherons around a central spheron, corresponding to 10, 12, 14, or 16 neutrons, are often found.

Expected Values of the Radius of Revolution

Values of the radius of revolution of the cluster have been found to increase rapidly by about one spheron diameter

Table 5. Values of $I(2^+)$ [from $E(2^+)$ by use of Eq. 1], the number of nucleons in the revolving cluster, m, and $R(2^+)$ for 24 nuclei with m not equal to $A - 208$

Z	A	$I(2^+)$, Da·fm^2	m	$R(2^+)$, fm
$_{88}Ra$	228	1964.6	18	10.89
$_{90}Th$	234	2530.8	24	10.84
$_{92}U$	236	2771.8	26	10.95
	238	2791.9	26	10.98
	240	2790	26	10.97
$_{94}Pu$	236	2812	26	11.02
	238	2844.8	26	11.08
	240	2928.2	26	11.24
	242	2815	25	11.20
	244	2726	24	11.22
	246	2730	24	11.23
$_{96}Cm$	242	2977.2	26	11.33
	244	2923	26	11.22
	246	2926.4	26	11.22
	248	2889.4	26	11.14
	250	2920	26	11.20
$_{98}Cf$	244	3130	28	11.24
	248	3016	27	11.20
	250	2935.3	26	11.23
	252	2742.8	24	11.24
$_{100}Fm$	250	2850	26	11.1
	252	2790	26	10.9
	254	2787.4	26	10.93
	256	2720	26	10.79

(3.27 fm) from nuclei with N close to a magic number to nuclei with larger values of N (9). This increase has been interpreted as involving the change from revolution of the cluster in the mantle of the nucleus to coasting over the surface, when the mantle becomes tightly packed (9). The smallest values found are about 6.9 fm (Table 3), whereas the largest values of $R(2^+)$ are about 11.2 fm (Tables 4 and 5).

Transfer of Momentum by Collision of One Cluster with Another

The difference between the two values of R in the preceding paragraph, 4.3 fm, is larger than expected. It is likely that the value 6.63 fm is lower than the correct value because of the phenomenon of transfer of momentum by collision of one cluster with another, which transfers not only the momentum but also the decoupling of the nucleon pair.

It is known that the mantle is loosely packed at $N = 82$, becoming tightly packed at $N = 90$, and a similar change probably occurs at about $N = 134$. For ${}_{84}Po_{128}$ the proton pair and neutron pair that constitute the revolving α particle do not occupy shell-model orbitals $1h\frac{9}{2}$ and $2g\frac{9}{2}$, respectively, but rather localized hybrids of these orbitals with others, both odd and even. There are similar hybrid orbitals occupied by coupled pairs, constituting nonrevolving α particles in the mantle, and collisions with transfer of momentum and of decoupling can occur. Each collision decreases the length of the path over which the momentum is quantized by the effective collision diameter of the spheron. Thus, if R is 7.67 fm and two collisions occur per revolution with collision diameter 3.27 fm, the apparent value of R would be 6.63 fm.

Momentum-transferring collisions have smaller effects for larger clusters and are not expected for clusters coasting over the surface of the mantle.

Values of the Nucleon Number and Radius of Revolution for Heavy Nuclei for Which the Nucleon Number of the Cluster Is Less Than $A - 208$

With the assumption that the value of the radius of revolution should change by only a small amount from one nucleus to a neighboring nucleus, as in Table 4, it is found that the experimental values of $I(2^+)$, calculated by Eq. **1**, indicate that for many nuclei in this region the value of m is smaller than $A - 208$. The likely values of m and the corresponding values of $R(2^+)$ are given in Table 5 for the 24 nuclei in this class (all of the isotopes of ${}_{94}Pu$, ${}_{96}Cm$, ${}_{98}Cf$, and ${}_{100}Fm$ and all of those of Ra, Th, and U with N greater than the values in Table 4).

For ${}^{228}_{88}Ra_{140}$ the value $m = 18$ corresponds to the cluster p^6n^{12}—that is, to six tritons. To have the sphere doubly magic would require adding a dineutron to the cluster, which would decrease the triton–α particle resonance energy. The same explanation of the limit on m applies to ${}^{234}_{90}Th_{144}$, with $m = 24$ and cluster p^8n^{16}.

For ${}^{234}_{92}U_{142}$ (Table 4) the revolving cluster is $p^{10}n^{16}$. This cluster, with eight spherons (a centered 7-membered ring) and n/p ratio 1.60, close to $N/Z = 1.57$, is especially stable. A cluster $p^{10}n^{18}$ in ${}^{236}U_{144}$ would be less stable, in that a centered 8-membered ring is too crowded. It is accordingly not unexpected that for many of the heavier nuclei the revolving cluster is found to have $m = 26$ and presumably the composition $p^{10}n^{16}$.

For Pu with $A = 242$, 244, and 246 the values $m = 25$, 24, and 24 are assigned, leading to $R = 11.20$, 11.22, and 11.33 fm, respectively, which fit well into the sequence of values for other nuclei; $m = 26$ would give 11.01, 10.83, and 10.84 fm. Also, for the last five nuclei in Table 5 the values of R show a decrease that could be avoided by taking lower values of m.

The Centrifugal Stretching Constant

It was suggested in an earlier section that effects of the centrifugal force and the Hooke's law constant for similar nuclei, such as those in Table 1, on the centrifugal stretching constant should lead to a nearly constant value of this constant, which in fact decreases by about 50% from ${}^{232}_{90}Th_{142}$ to ${}^{250}_{98}Cf_{152}$. The explanation of this decrease may be that, with the cluster about the same size in this series, there is an increase in the value of the Hooke's law constant. In ${}^{232}Th$ and ${}^{234}U$ the cluster is coasting over the surface of the $p^{82}n^{126}$ sphere, and the force holding it to the sphere is the interaction with the eight adjacent spherons in the mantle of the sphere. For ${}^{250}Cf$, however, there are 16 additional nucleons in the layer containing the cluster outside the mantle of the sphere, and the interaction of the cluster with them may increase the value of the Hooke's law constant.

The much larger value of the stretching constant for ${}^{222}Th$ indicated by the values of R in Table 2 may be attributed to a change in structure of the sphere as a result of the centrifugal force exerted on it by the cluster. This nucleus is in a region where the structure changes rapidly with change in Z and N, as is illustrated in Table 3.

Conclusion

Analysis of the energy levels of the ground-state band of all nuclei with N equal to or greater than 126 tabulated in ref. 7 has been carried out on the basis of several assumptions: that the angular momentum results from the revolution of a single cluster of m nucleons at radius of revolution R about a sphere; that for each value of the angular momentum quantum number J the value of m is an even integer or, rather rarely, an odd integer; that a cluster that is a centered ring of 5, 6, or 7 spherons, that has n/p ratio close to N/Z, and that consists of all or most of the nucleons outside of the doubly magic sphere $p^{82}n^{126}$ is especially stable; and that the values of m and R form reasonable sequences with those of neighboring nuclei and also change with J in a reasonable manner. This analysis has provided the result that $R(J = 2^+)$ remains constant at about 6.9 fm for nuclei with $N = 126$ to 132 and then increases rapidly to about 11.2 fm for $N \geq 142$. The major part of this increase is attributed to the transition from revolution of the cluster in the outer mantle of the sphere to revolution by coasting over the surface of the sphere, as the mantle becomes crowded. The low value 6.9 fm is the result also in part of an effect that has not before been recognized, resulting from the transfer of momentum by collision from the cluster to a pair of protons or neutrons with zero angular momentum, reducing the effective length of the orbit over which the angular momentum is quantized. Information is also obtained about some other structural features of these nuclei. The success of the analysis illustrates the value of the effort to develop a simple structural theory of atomic nuclei.

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1. Harkins, W. D. (1920) *J. Am. Chem. Soc.* **42,** 1956–1966.
2. Harkins, W. D. (1921) *Philos. Mag.* **42,** 305–312.
3. Pauling, L. (1965) *Science* **150,** 297–305.
4. Rainwater, J. (1950) *Phys. Rev.* **79,** 432–434.
5. Bohr, A. & Mottelson, B. R. (1953) *Phys. Rev.* **89,** 316–317.
6. Pauling, L. (1969) *Proc. Natl. Acad. Sci. USA* **64,** 807–809.
7. Sakai, M. (1984) *Atomic Data Nucl. Data Tables* **31,** 399–426.
8. Pauling, L. & Wilson, E. B., Jr. (1935) *Introduction to Quantum Mechanics* (Dover, New York).
9. Pauling, L. & Robinson, A. B. (1975) *Can. J. Phys.* **53,** 1953–1964.