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Invariants of sets of linear varieties

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ABSTRACT **A minimal set of generators of the ring of invariants for four linear subspaces of dimension n in a vector space of dimension $2n$ is computed, using the symbolic method introduced by Grosshans *et al.* [Grosshans, F., Rota, G.-C. & Stein, J. A. (1987) *Invariant Theory and Superalgebras* (Am. Math. Soc., Providence, RI)].**

Section 1. Introduction

The notation of Grosshans *et al.* (1) is used. Let V be a vector space of dimension n over a field K of characteristic zero. A decomposable skew-symmetric tensor in $\Lambda(V)$ defines a subspace of V or, equivalently, a linear variety in a projective space of dimension $n - 1$. By combining the straightening algorithm, the exchange identity, and the Grassmannian condition for decomposable skew symmetric tensors (1), a complete set of invariants for four linear subspaces of dimension n in a vector space of dimension $2n$ is computed. This problem was left open by Turnbull in 1941 (2), who had attained some partial results. A related problem has also been studied by Gelfand and his school in the 1970s (3).

A complete set of invariants for five linear subspaces of dimension 3 in a vector space of dimension 6 has also been determined, but the full computation is at present 102 typewritten pages long. This result will be announced in a subsequent note.

Section 2. The Bracket Algebra

Let L be a positively signed alphabet. Assign each letter $x \in L$ a positive integer arity(x), called the arity of x. Let $[L]$ be the alphabet whose members are all monomials w in Super[L]. A letter in $[L]$ is denoted by $[w]$. The bracket algebra Bracket[L] of dimension n is the quotient of the free associative algebra Tens[[L]] generated by the alphabet $[L]$, subject to the congruence relations below, where w, w', and w'' are monomials in Super[L]:

(*i*) $[w] = 0$ if Length$(w) \neq n$;

(*ii*) $[w][w'] = (-1)^n[w'][w]$; and

(*iii*) the exchange identity

$$\sum_{w'} [ww'_{(1)}][w'_{(2)}w''] = \sum_{w} (-1)^{\mathrm{Length}(w_{(2)})}[w'w_{(1)}][w_{(2)}w''].$$

An element p in Tens[[L]] will be identified with its image in Bracket[L]. Let $D = (w_1, w_2, \ldots, w_k)$ be a Young diagram with $w_i \in \mathrm{Mon}(L)$. The Young tableau of D is the bracket monomial

$$\mathrm{Young}(D) = [\mathrm{stand}(w_1)][\mathrm{stand}(w_2)] \ldots [\mathrm{stand}(w_k)].$$

If $w_i = x_{i1}x_{i2} \ldots x_{in}$, the word $x_{1j}x_{2j} \ldots x_{kj}$ is called the jth column of D. The Young diagram D is called standard (1) if it has weakly increasing rows and strictly increasing columns. Such a Young tableau Young(D) in Bracket[L] does not vanish only if each of the words $w_1, w_2, \ldots, w_k$ is of length n. This condition will be tacitly assumed below.

PROPOSITION 1 [standard basis theorem for bracket algebra (4)]. *Given a Young diagram* D, *there exist unique standard Young diagrams* $\mathrm{D_i}$ *with the same content as* D *and unique nonzero integer coefficients* $\mathrm{r_i}$ *such that*

$$\mathit{Young}(\mathrm{D}) = \sum \mathrm{r_i}\, \mathit{Young}(\mathrm{D_i}).$$

We will denote $[D]$ as Young(D) when no confusion arises.

Section 3. Symbolic Representation of Invariants for Linear Varieties

Let V be a vector space of dimension n over the field K. A set of decomposable skew symmetric tensors $S_0 = \{t_a, t_b, t_c, \ldots\}$ in $\Lambda(V)$ is associated with a set of subspaces $S = \{a, b, c, \ldots\}$ of V or, equivalently, with a set of linear varieties $S = \{a, b, c, \ldots\}$ in a projective space of dimension $n - 1$, where each subspace a in S is spanned by the vector factors of t_a.

For each $a \in S$, let L_a be an infinite set of positively signed letters

$$L_a = \{a_1, a_2, a_3, \ldots\}.$$

Define arity(a_i) to be the dimension of a as the subspace of V. Letters in L_a are also called a-letters. Define the alphabet L to be

$$L = L_a \cup L_b \cup L_c \cup \ldots.$$

Two letters in L are said to be *equivalent* if they are both in L_a for some $a \in S$. Two Young diagrams D and D' are said to be *equivalent* if $D = (x_{11} \ldots x_{1n_1}, \ldots, x_{k1} \ldots x_{kn_k})$ and $D' = (\pi(x_{11}) \ldots \pi(x_{1n_1}), \ldots, \pi(x_{k1}) \ldots \pi(x_{kn_k}))$ for some permutation π of L such that x and $\pi(x)$ are equivalent for all $x \in L$. We say a monomial p in Bracket[L] has *right content* if cont(p; x) equals to either zero or arity(x) for all x in L. A Young diagram D is said to have *right content* if $[D]$ does.

Let Bracket$[L]_0$ be the K-subspace of Bracket[L] generated by monomials with right contents. It also has an algebra structure such that, for two monomials p and q with right contents, we have

$$pq = 0$$

if the monomial pq does not have a right content.

Definition 1: Define the algebra Linear[L] to be the quotient of the algebra Bracket$[L]_0$ by the ideal I that is generated by the following monomials p in Bracket$[L]_0$:

(*i*) $p \in I$ if $\sum_{x \in L_a} \mathrm{cont}(w; x) > k$ for some equivalence class L_a with arity(a_1) $= k$ and for some bracket factor $[w]$ of p.

(*ii*) $p = [a_1^{(k)}b_1^{(n-k)}] \in I$ for all $a_1 \in L_a$, $b_1 \in L_b$, where L_a and L_b are two equivalence classes with arity(a_1) $= k$, arity(b_1) $= n - k$.

The motivation for condition *i* is the fact that any $k + 1$ vectors in a linear subspace of dimension k are linearly dependent. Condition *ii* is a technical one to facilitate the computation of invariants.

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If a monomial p has right content, it is identified with its image in Linear[L]. In particular, we keep the notation [D] for the image of [D] in Linear[L] if [D] has the right content.

The following proposition is a corollary of the first fundamental theorem (1).

PROPOSITION 2. *Let* J *be the ideal generated by monomials* p *in Bracket*$[L]_0$ *that satisfy Condition* ii *of Definition 1. Let the algebra linear*[L] *be the quotient of Bracket*$[L]_0$ *by the ideal* J. *Then there is a surjective linear mapping* Φ *from linear*[L] *to the ring of joint invariants of the set of decomposable skew-symmetric tensors* t_a *in* S_0.

Moreover $\Phi([D]) = \pm\Phi([D'])$ *if* D *and* D' *are equivalent Young diagrams and the sign can be determined (4).*

Elements in Linear[L] are called a symbolic representation of the invariants or simply invariants. The straightening algorithm and the exchange identity can be carried over to Linear[L], although the uniqueness of the coefficients r_i in *Proposition 1* no longer holds.

A *position* occupied by a letter x in a Young diagram D is a triple (i, j, x) when the letter x occurs in the ith row and the jth column of D. Such a position is sometimes written as $\underline{x}$ when i and j need not to be referred. Let $\underline{X} = (i, j, x_{ij})_{(i,j)\in I}$ be a set of positions in D occupied by a multiset $X = \{x_{ij}\}_{(i,j)\in I}$. Given a permutation π of X, denote by πD the Young diagram obtained from D by replacing each position (i, j, x_{ij}) of $\underline{X}$ by $(i, j, \pi(x_{ij}))$. Suppose D has k rows, let

$$\underline{X} = \underline{X_1} \cup \underline{X_2} \cup \ldots \cup \underline{X_k},$$

where each $\underline{X_i}$ is a set of positions in the ith row of D (with the possibility to be an empty set), occupied by a multiset X_i. We may write X_i as a word in Mon(L). A permutation π of X induces the words πX_i in a natural way. Two permutations π and π' are called *row distinct* if

$$\pi X_1 \otimes \pi X_2 \otimes \ldots \otimes \pi X_k \neq \pi' X_1 \otimes \pi' X_2 \otimes \ldots \otimes \pi' X_k$$

in Super[L] $\otimes \ldots \otimes$ Super[L] (k-times).

A set of positions $\underline{X}$ in D can be *shuffled* if in Linear[L],

$$\sum [\pi D] = 0,$$

where π ranges over all row distinct permutations of X.

PROPOSITION 3. *Given a Young diagram* D *and one of its row* w, *let* $\underline{X_1}$ *be the set of all positions in* w *occupied by* a*-letters for some equivalence class* L_a. *Suppose* X_1 *is not empty. Let* $a_1 \in X_1$, *and let* $\underline{X_2}$ *be the set of all positions in* D *but not in* w *occupied by the letter* a_1. *Then in Linear*[L],

$$[D] = \sum r_\pi[\pi D],$$

where the sum ranges over the set of row distinct permutations π *of* $X_1 \cup X_2$ *such that* πX_1 *contains only the letter* a_1 *and where* r_π *are integers.*

The proof is carried out by induction on the multiplicity of a_1 in X_1.

PROPOSITION 4. *Let* w *be a row in a Young diagram* D. *If* w *contains* k *equivalent letters of arity* k *as well as* n − k *equivalent letters of arity* n − k, *then* [D] *vanishes in Linear*[L].

This is a corollary of *Proposition 3*.

PROPOSITION 5. *Given a Young diagram* D, *a set of positions* $\underline{X}$ *in* D *can be shuffled if one of the following holds for the multiset* X:

(i) X *consists of* k + 1 *equivalent letters of arity* k;

(ii) X *consists of* k *equivalent letters of arity* k *as well as* n − k *equivalent letters of arity* n − k;

(iii) X *consists of any* n + 1 *letters.*

The proof of this proposition follows from the exchange identity and from *Proposition 3*.

Definition 2: A Young diagram $D_a = (w_1, w_2, \ldots, w_k)$ is called a *block* (or an a-*block* when the space a is relevant) relative to a space $a \in S$ when for $1 \le i \le k$, we have $w_i = \underbrace{a_i a_i \ldots a_i}_{k\text{ times}} u_i$, where k = arity(a_1) and $u_i \in$ Mon(L). A *block product* is defined as

$$[D] = [D_a][D_b] \ldots [D_c]$$

for some $a, b, \ldots, c \in S$, where $D_a, D_b, \ldots, D_c$ are blocks relative to spaces a, b, and $\ldots$, c, respectively.

PROPOSITION 6. *For a Young tableau* [D] *in Linear*[L] *we have*

(i) [D] *is a sum of block products in Linear*[L], *that is*

$$[D] = \sum_i r_i[D_{ia}][D_{ib}] \ldots [D_{ic}],$$

where r_i *are integers.*

(ii) *If we fix any two equivalence classes* L_a *and* L_b *with the property arity*(a_1) + *arity*(b_1) $\ge$ n, *then in each term of the sum above only the* a*-block contains* a*-letters and only the* b*-block contains* b*-letters.*

The proof of part *i* follows from the exchange identity, and the proof of part *ii* follows from the straightening algorithm.

PROPOSITION 7. *For a block product* $[D_a][D_b] \ldots [D_c]$ *in Linear*[L], *each block can be straightened separately in the following sense:*

$$[D_a][D_b] \ldots [D_c] = \sum_i r_i[D_{ia}][D_{ib}] \ldots [D_{ic}],$$

where for each block D_a *with arity*(a_1) = k, *the Young diagrams* D_{ia} *are* a*-blocks and their rightmost* n − k *columns are standard in some ordering of* L *that may vary for different blocks* $D_a, D_b, \ldots, D_c$, *and where* r_i *are integers.*

The proof depends on the fact that the exchange identity can be applied within each block in the following sense:

$$\sum_{w'} [a_1^{(k)} w w'_{(1)}][a_2^{(k)} w'_{(2)} w''] = \sum_{w} (-1)^{\text{Length}(w_{(2)})}[a_1^{(k)} w' w_{(1)}][a_2^{(k)} w_{(2)} w''].$$

PROPOSITION 8. *In a block product* $[D] = [D_a][D_b] \ldots [D_c]$, *let a block* D_a *have arity*(a_1) = n − k. *Then a set of positions* $\underline{X}$ *in the rightmost* k *columns of* D_a *can be shuffled if one of the following holds for the multiset* X:

(i) X *consists of any* k + 1 *letters;*

(ii) X *consists of any* k *equivalent letters of arity* k.

The proof follows from the above exchange identity within the a-block D_a.

PROPOSITION 9. *In a block product* $[D] = [D_a][D_b] \ldots [D_c]$, *let a block* D_a *have arity*(a_1) = n − k, *and let cont*(D_a; x) = *arity*(x) = k *for some* $x \in L \backslash L_a$. *Then* [D] *vanishes in Linear*[L].

This is a corollary of part *ii* of *Proposition 8*.

Section 4. Invariants of Four Medials

Let V be a vector space of dimension $2n$ over K. An n-dimensional subspace a of V is called a *medial*. We can identify a medial with a projective linear variety of dimension $n - 1$ in a projective space of dimension $2n - 1$. Our problem is to determine the algebraic generators of the ring of invariants of four medials.

Let

$$t_a = \alpha_1\alpha_2 \ldots \alpha_n, \qquad t_b = \beta_1\beta_2 \ldots \beta_n,$$

$$t_c = \gamma_1\gamma_2 \ldots \gamma_n, \qquad t_d = \delta_1\delta_2 \ldots \delta_n$$

be four decomposable skew-symmetric tensors in $\Lambda(V)$, where α_i, β_i, γ_i, and δ_i are vectors of V. Let a, b, c, and d be the corresponding four medials; i.e., $a = \text{span}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, etc.

THEOREM 1. *For the ring of invariants of four medials, the following* n + 4 *invariants form a minimum set of algebraic generators:*

(i) *six single determinants such as* $[\alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_n]$, etc.;

(ii) *any* n − 2 *out of the* n − 1 *shuffle products:*

$$[\alpha_1 \ldots \alpha_n\dot{\gamma}_1 \ldots \dot{\gamma}_i\check{\delta}_{i+1} \ldots \check{\delta}_n] \times [\beta_1 \ldots \beta_n\dot{\gamma}_{i+1} \ldots \dot{\gamma}_n\check{\delta}_1 \ldots \check{\delta}_i]$$

$$= \sum_{\sigma,\tau} sign(\sigma)sign(\tau)[\alpha_1 \ldots \alpha_n\gamma_{\sigma(1)} \ldots \gamma_{\sigma(i)}\delta_{\tau(i+1)} \ldots \delta_{\tau(n)}] \times [\beta_1 \ldots \beta_n\gamma_{\sigma(i+1)} \ldots \gamma_{\sigma(n)}\delta_{\tau(1)} \ldots \delta_{\tau(i)}],$$

where σ *and* τ *range over permutations of the set* $\{1, 2, \ldots, n\}$ *and* $1 \leq i \leq n - 1$.

The symbolic version of *Theorem 1* can be stated as follows.

THEOREM 2. *Let each of* L_a, L_b, L_c, L_d *contain infinite many positive letters of arity* n, *and let* $L = L_a \cup L_b \cup L_c \cup L_d$. *Then any* n − 2 *of the following* n − 1 *equivalence classes* Q_i *of invariants form a minimum set of algebraic generators of the algebra Linear*[L]:

$$Q_i = \{[a_1^{(n)}c_1^{(i)}d_1^{(n-i)}][b_1^{(n)}c_1^{(n-i)}d_1^{(i)}] \quad | \quad a_1 \in L_a, \quad b_1 \in L_b, \quad c_1 \in L_c, \quad d_1 \in L_d\},$$

where $i = 1, 2, \ldots, n - 1$.

The proof is subdivided into five steps, as follows:

Step 1. Prove that the block products $[D] = [D_a][D_b]$ generate the algebra Linear$[L]$. By part *i* of *Proposition 6* the block products

$$[D] = [D_a][D_b][D_c][D_d]$$

generate the algebra Linear$[L]$, with the possibility that the c-block D_c and the d-block D_d may be void. I claim that $[D]$ vanishes unless D_c and D_d are void. By part *ii* of *Proposition 6*, we may assume that the Young diagrams D_c and D_d contain only c-letters and d-letters. If $D_c = (w_1, w_2, \ldots, w_k)$ is not a void c-block, then $w_1 = \underbrace{c_1c_1 \ldots c_1}_{n \text{ times}} u$, where u is a word of length n in Mon(L_d). Thus $[D]$ vanishes by *Proposition 4*. Similarly $[D]$ vanishes if the d-block D_d is not void. Hence the generators of the algebra Linear$[L]$ are block products $[D] = [D_a][D_b]$.

Step 2. Apply *Proposition 7* to straighten separately the rightmost n columns of D_a and D_b by giving the alphabet L an order such that $c_1 < d_1$ for all $c_1 \in L_c$, $d_1 \in L_d$. We obtain four subdiagrams D_{ac}, D_{ad}, D_{bc}, and D_{bd} of D with shapes λ_{ac}, λ_{ad}, λ_{bc}, and λ_{bd}, where D_{ac} is the subdiagram of D_a consisting of all c-letters occurring in D_a and where $\lambda_{ac} = (\lambda_{ac,1}, \lambda_{ac,2}, \ldots, \lambda_{ac,k})$ with $\lambda_{ac,1} \geq \lambda_{ac,2} \geq \ldots \geq \lambda_{ac,k}$, etc. In this step we will prove that $|\lambda_{ad}| = |\lambda_{bc}|$ and $|\lambda_{ac}| = |\lambda_{bd}|$.

Since D_{ac} is standard, its first column must be $c_1c_2 \ldots c_k$, where c_i are different c-letters. I claim that these are the only c-letters occurring in D. To prove this, we apply k times *Proposition 3*, respectively, to the first, the second, . . . , the kth rows of D_{ac} and thereby obtain

$$[D] = \sum r_{\pi_k\pi_{k-1}\ldots\pi_1}[\pi_k\pi_{k-1} \ldots \pi_1 D] = \sum r_\pi[\pi D],$$

where π_i are permutations of c-letters and r_π are integers and where in each term $[\pi D]$ the subdiagram πD_{ac} is of the form

$$\pi D_{ac} = (\underbrace{c_1c_1 \ldots c_1}_{\lambda_{ac,1} \text{ times}}, \underbrace{c_2c_2 \ldots c_2}_{\lambda_{ac,2} \text{ times}}, \ldots, \underbrace{c_kc_k \ldots c_k}_{\lambda_{ac,k} \text{ times}}).$$

Therefore if there is a $(k + 1)$th c-letter c_{k+1} occurring in D, then in each term $[\pi D]$ this letter c_{k+1} must occur n times in the b-block πD_b. Thus $[D]$ vanishes by *Proposition 9*, and we may assume that $c_1, c_2, \ldots, c_k$ are the only c-letters occurring in D. Similarly if the rightmost column of D_{ad} is $d_1d_2 \ldots d_k$, then we may assume that $d_1, d_2, \ldots, d_k$ are the only k of d-letters occurring in D. In conclusion, we obtain

$$\begin{aligned} |\lambda_{ad}| &= \sum_{i=1}^{k} \lambda_{ad,i} \\ &= \sum_{i=1}^{k} \text{cont}(D_a;\, d_i) \\ &= \sum_{i=1}^{k} (n - \text{cont}(D_b;\, d_i)) \\ &= kn - \sum_{i=1}^{k} \text{cont}(D_b;\, d_i) \\ &= kn - \sum_{i=1}^{k} \lambda_{bd,i} \\ &= \sum_{i=1}^{k} \lambda_{bc,i} \\ &= |\lambda_{bc}|. \end{aligned}$$

Similarly we have $|\lambda_{ac}| = |\lambda_{bd}|$.

Step 3. Use induction on $\max(\lambda_{ad}, \lambda_{bc})$ in the dominance order to prove that $[D]$ is a polynomial in the set of invariants $\cup_{i=1}^{n-1} Q_i$.

Suppose that

$$\lambda_{ad} = \max\{\lambda_{ad}, \lambda_{bc}\} = (\lambda_{ad,1}, \lambda_{ad,2}, \ldots, \lambda_{ad,k})$$

is the largest among shapes $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ in the dominance order such that $|\lambda| = |\lambda_{ad}|$, $\lambda_i \leq n$. Then $\lambda_{ad,1} = n$; i.e., the kth row of D_a will contain n letters in L_a as well as n letters in L_d. We can infer that $[D]$ vanishes by virtue of *Proposition 4*.

Suppose next that $[D]$ is a polynomial in the set of invariants $\cup_{i=1}^{n-1} Q_i$ whenever $\max\{\lambda_{ad}, \lambda_{ac}\}$ is larger than a shape λ, where $|\lambda| = |\lambda_{ad}|$ and $\lambda_i \leq n$. We can prove that $[D]$ will also be a polynomial in those invariants when $\max\{\lambda_{ad}, \lambda_{bc}\} = \lambda$. Assume that $\lambda_{ad} = \lambda$. Apply $2k$ times *Proposition 3*, respectively, to the first, the second, . . . , the kth rows of D_{ac} and then to the kth, the $(k - 1)$th, . . . , the first rows of D_{ad} and thereby obtain

$$\begin{aligned} [D] &= \sum r_{\tau_1\ldots\tau_k\pi_k\ldots\pi_1}[\tau_1 \ldots \tau_k\pi_k \ldots \pi_1 D] \\ &= \sum r_{\tau\pi}[\tau\pi D], \end{aligned}$$

where π_i and τ_i are permutations of L_c and L_d, respectively, and $c_{\tau\pi}$ are integers; in each term $[\tau\pi D]$ the subdiagrams $\tau\pi D_{ac}$ and $\tau\pi D_{ad}$ are of the form

$$\tau\pi D_{ac} = (\underbrace{c_1c_1 \ldots c_1}_{\lambda_{ac,1} \text{ times}}, \underbrace{c_2c_2 \ldots c_2}_{\lambda_{ac,2} \text{ times}}, \ldots, \underbrace{c_kc_k \ldots c_k}_{\lambda_{ac,k} \text{ times}}),$$

$$\tau\pi D_{ad} = (\underbrace{d_1d_1 \ldots d_1}_{\lambda_{ad,1} \text{ times}}, \underbrace{d_2d_2 \ldots d_2}_{\lambda_{ad,2} \text{ times}}, \ldots, \underbrace{d_kd_k \ldots d_k}_{\lambda_{ad,k} \text{ times}}).$$

Changing notation, we write $[D]$ instead of $[\tau\pi D]$ from now on. We straighten the rightmost n columns of D_b after giving letters $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k$ the following order:

$$c_k < c_{k-1} < \ldots < c_1 < d_k < d_{k-1} < \ldots < d_1.$$

We then obtain

$$[D] = \sum_i r_i[D_a][D_{ib}],$$

where in each term the b-block D_{ib} is standard in the above order. Let λ_{ibc} be the shape of the subdiagram D_{ibc} of D_{ib} containing all the c-letters occurring in D_{ib}. Then we have

$$\begin{aligned}\lambda_{ibc} &\geq (\mathrm{cont}(D_{ib}; c_k), \mathrm{cont}(D_{ib}; c_{k-1}), \ldots, \mathrm{cont}(D_{ib}; c_1))\\ &= (\mathrm{cont}(D_b; c_k), \mathrm{cont}(D_b; c_{k-1}), \ldots, \mathrm{cont}(D_b; c_1))\\ &= (n - \mathrm{cont}(D_a; c_k), n - \mathrm{cont}(D_a; c_{k-1}), \ldots,\\ &\quad n - \mathrm{cont}(D_a; c_1))\\ &= (\mathrm{cont}(D_a; d_k), \mathrm{cont}(D_a; d_{k-1}), \ldots, \mathrm{cont}(D_a; d_1))\\ &= \lambda_{ad}\\ &= \lambda.\end{aligned}$$

By the induction hypothesis we need consider only the term $[D_a][D_{1b}]$, where

$$\begin{aligned}\lambda_{1bc} &= \lambda\\ &= (\mathrm{cont}(D_{1b}; c_k), \mathrm{cont}(D_{1b}; c_{k-1}), \ldots, \mathrm{cont}(D_{1b}; c_1)).\end{aligned}$$

This term can be factored into

$$[D_a][D_{1b}] = \prod_{i=1}^{k}[a_i^{(n)}c_i^{(\mu_i)}d_i^{(n-\mu_i)}][b_i^{(n)}c_i^{(n-\mu_i)}d_i^{(\mu_i)}],$$

where $(\mu_1, \mu_2, \ldots, \mu_k) = (n - \lambda_k, n - \lambda_{k-1}, \ldots, n - \lambda_1)$. This completes the proof that the set of invariants $\cup_{i=1}^{n-1} Q_i$ generate the algebra Linear$[L]$.

Step 4. Let $j \in \{1, 2, \ldots, n-1\}$, and $I = \{1, 2, \ldots, n-1\}\backslash\{j\}$. Prove that any invariants in Q_j can be expressed by invariants in $\cup_{i \in I} Q_i$.

Consider the following $n + 2$ bracket monomials in the bracket algebra Bracket$[L]$:

$$q_i = [a_1^{(n)}c_1^{(i)}d_1^{(n-i)}][c_1^{(n-i)}d_1^{(i)}b_1^{(n)}], \qquad i = 0, 1, \ldots, n,$$

$$q_{n+1} = [a^{(n)}b^{(n)}][c^{(n)}d^{(n)}].$$

Applying the exchange identity we obtain that in Bracket$[L]$

$$q_{n+1} = (-1)^n \sum_{i=0}^{n} q_i. \qquad [1]$$

Since q_0, q_n, q_{n+1} vanish in Linear$[L]$, we obtain

$$q_j = -\sum_{i \in I} q_i$$

as desired.

Step 5. Prove the minimality of a set of generators $\cup_{i \in I} Q_i$. If such a set of generators were not minimal, then one could prove that there is a linear relation in Bracket$[L]$ among the $n + 1$ bracket monomials $\{q_i | i \in I \cup \{0, n, n+1\}\}$. Together with Eq. **1** we can obtain a linear relation in Bracket$[L]$ among the $n + 1$ bracket monomials $\{q_i | i = 0, 1, \ldots, n\}$ by eliminating q_{n+1} from the two linear relations. But $q_0, q_1, \ldots, q_n$ are standard Young tableaux in the order $a_1 < c_1 < d_1 < b_1$, and therefore they are linearly independent in Bracket$[L]$. This contradiction proves the minimality of the set of generators. q.e.d.

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