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Mathematics

Coclosure operators and chromatic polynomials

(closure operators/simplicial complexes/partition lattices/incidence algebras/Hopf algebras)

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ABSTRACT **A theorem about closure operators on partially ordered sets is given, and applications to the counting of colorings of graphs according to partition type are derived.**

In recent papers, the two of us have independently attempted to extend to a broader algebraic context the combinatorial theory of möbius functions as originally developed by Rota. In refs. 1 and 2, the notions of Möbius and zeta functions of a subposet of a partition lattice are generalized to those of *Möbius type function* and *zeta type function*, which serve in enumeration problems not only to count partitions but also to describe the *types*—i.e., the numbers of blocks of each size—of the partitions counted. In the case of the poset of *admissible* partitions of the vertex set of a graph G (which is closely related to the lattice of contractions of G), it is shown that the natural extension of the characteristic polynomial to this setting, called the *umbral chromatic polynomial* of G, counts colorings of the graph according to their partition type. This is a substantial generalization of the well-known result (3) that the characteristic polynomial of the lattice of contractions of a graph is equal to the chromatic polynomial of the graph (modulo a simple multiplicative factor).

In refs. 4 and 5, the Hopf algebra structure of incidence algebras of partially ordered sets is recognized, and the antipodes of these Hopf algebras become the natural generalization of Möbius functions. Theorems for calculating the antipodes of incidence (Hopf) algebras are then given, generalizing well-known results about Möbius functions. These theorems are significant extensions of their Möbius theoretic counterparts, because computing the antipode is equivalent to computing the convolution inverse of all of the functions of an incidence algebra.

The main result of this paper is a generalization of a powerful theorem that relates the Möbius functions of two posets, when one is the image of the other under a closure operator. This result can be understood as a theorem about computing antipodes of incidence Hopf algebras, but to simplify the presentation here, we state it as a formula for inverting a single, arbitrary element of an incidence algebra, thus avoiding the machinery of Hopf algebras. As an application, we show how this theorem leads to a major improvement in our understanding of the construction of the umbral chromatic polynomial $\chi^{\varphi}(G; \lambda)$ of a graph G, answering the question formulated in refs. 1 and 2 as to why the set of admissible partitions, rather than the lattice of contractions, of G provides the appropriate framework for the construction of $\chi^{\varphi}(G; \lambda)$. For convenience, we have stated our results in terms of *co*closure operators, which are the duals (in the sense of partially ordered sets) of closure operators.

Incidence Algebras and Coclosure Operators

Let P be a locally finite partially ordered set, or *poset* for short, and let R be a ring with identity. The *incidence algebra* $R(P)$ of P (over R) is defined to be the set of functions from the collection of intervals in P, Int(P), to the ring R, with pointwise addition and scalar multiplication; and convolution product defined by

$$f * g(x, y) = \sum_{x \le z \le y} f(x, z)g(z, y),$$

for f, g in $R(P)$ and $x \le y$ in P. The identity δ of $R(P)$ is given by $\delta(x, y) = \delta_{x,y}$, the Kronecker delta, and the *zeta function* ζ satisfies $\zeta(x, y) = 1$, for all $x \le y$ in P. The *Möbius function* μ of P is the convolution inverse of the zeta function.

A *corank* function on a poset P is a map ν from P into the nonnegative integers such that $\nu(x) = 0$ if x is maximal and $\nu(x) = \nu(y) + 1$ whenever y covers x in P. If P has corank function ν and unique minimal element $\hat{0}$, then the *characteristic polynomial* of P is the polynomial $\chi(P; \lambda) = \Sigma_{x \in P}\, \mu(\hat{0}, x)\lambda^{\nu(x)}$.

A *coclosure operator* on P is a function $x \to \bar{x}$ from P into itself such that (*i*) $\bar{x} \le x$, (*ii*) $\bar{\bar{x}} = \bar{x}$, and (*iii*) $x \le y$ implies $\bar{x} \le \bar{y}$, for all $x, y \in P$. An element x of P is *closed* if $\bar{x} = x$. Let P_c denote the subposet of closed elements of P, and let ζ_c and μ_c be the zeta and Möbius functions of P_c. One of the most powerful tools for computing Möbius functions is the following well-known theorem, due to Rota (3), which expresses μ_c in terms of μ.

THEOREM 1. *Let* x $\to \bar{\text{x}}$ *be a coclosure operator on a poset* P. *For all* a $\le$ b *in* P

$$\sum_{x, \bar{x}=a} \mu(x, b) = \begin{cases} \mu_c(a, b) & \text{if } a, b \in P_c, \\ 0 & \text{otherwise.} \end{cases} \qquad [1]$$

Now let φ be any element of $R(P)$ and let $\varphi_c \in R(P_c)$ be the restriction of φ to Int(P_c). We seek a formula analogous to Eq. **1** that expresses φ_c^{-1}, the (convolution) inverse of φ_c in $R(P_c)$, in terms of φ^{-1} and φ, and reduces to Eq. **1** in the special case that φ is the zeta function of P. We must make an additional assumption about the map $x \to \bar{x}$ to do this.

Definition 1: Let $\varphi \in R(P)$. A coclosure operator $x \to \bar{x}$ on P is φ*-factoring* if for all $x, y \in P$ with $x \le \bar{y}$

$$\varphi(x, y) = \varphi(x, \bar{y})\varphi(\bar{y}, y). \qquad [2]$$

THEOREM 2. *Suppose* $\varphi \in$ R(P), *and* x $\to \bar{\text{x}}$ *is a* φ*-factoring coclosure operator on* P. *If* P_c *denotes the subposet of closed elements of* P, *and* $\varphi_c \in R(P_c)$ *is the restriction of* φ *to* Int(P_c), *then for all* a $\le$ b *in* P

$$\sum_{x, \bar{x}=a} \varphi(a, x)\varphi^{-1}(x, b) = \begin{cases} \varphi_c^{-1}(a, b) & \text{if } a, b \in P_c, \\ 0 & \text{otherwise.} \end{cases} \qquad [3]$$

Proof: Define functions $\bar{\varphi}$ and $\hat{\varphi}_c^{-1}$ in $R(P)$ by

$$\bar{\varphi}(x, y) = \begin{cases} \varphi(x, y) & \text{if } \bar{y} = x, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{\varphi}_c^{-1}(x, y) = \begin{cases} \varphi_c^{-1}(x, y) & \text{if } x, y \in P_c, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \le y$ in P. Then the theorem is equivalent to the identity $\overline{\varphi} * \varphi^{-1} = \hat{\varphi}_c^{-1}$ in $R(P)$. Or, by convolving with φ on the right,

$$\overline{\varphi} = \hat{\varphi}_c^{-1} * \varphi. \quad [4]$$

Now the sum in Eq. **3** is empty if a is not closed, so we will assume $a \in P_c$. In this case

$$\hat{\varphi}_c^{-1} * \varphi(a, b) = \sum_{\substack{a \le x \le b \\ x \text{ closed}}} \varphi_c^{-1}(a, x)\varphi(x, b).$$

It follows from the definition of a coclosure operator that $\overline{x} \le b$ if and only if $\overline{x} \le \overline{b}$. Combining this with the fact that $x \to \overline{x}$ is φ-factoring, the sum can be rewritten

$$\sum_{\substack{\overline{a} \le x \le \overline{b} \\ x \text{ closed}}} \varphi_c^{-1}(\overline{a}, \overline{x})\varphi_c(\overline{x}, \overline{b})\varphi(\overline{b}, b).$$

Hence

$$\hat{\varphi}_c^{-1} * \varphi(a, b) = \begin{cases} \varphi(\overline{b}, b) & \text{if } \overline{a} = \overline{b}, \\ 0 & \text{otherwise,} \end{cases}$$

which is equal to $\overline{\varphi}(a, b)$, since we have assumed $a = \overline{a}$. Thus Eq. **4** holds, and the proof is complete. □

Note that any coclosure operator on a poset is ζ-factoring; so *Theorems 1* and *2* are identical in the case that $\varphi = \zeta$.

Posets of Partitions

Let V be a finite set. A *partition* of V is a set σ of nonempty, disjoint subsets of V, called *blocks* of σ, whose union is V. $\Pi(V)$ will denote the poset of all partitions of V ordered by refinement; that is, $\sigma \le \pi$ in $\Pi(V)$ if and only if every block of σ is contained in some block of π. $\Pi(V)$ has maximal element $\hat{1} = \{V\}$ and minimal element $\hat{0} = \{\{x\} | x \in V\}$. If $|\sigma|$ denotes the number of blocks of the partition σ, then setting $\nu(\sigma) = |\sigma| - 1$ defines a corank function on $\Pi(V)$.

Let Φ be the polynomial ring $Z[\varphi_1, \varphi_2, \ldots]$ in infinitely many variables. Define the function ζ^φ in the incidence algebra $\Phi(\Pi(V))$ by setting $\zeta^\varphi(\sigma, \pi) = \varphi_1^{k_1}\varphi_2^{k_2} \ldots$, for $\sigma \le \pi$ in $\Pi(V)$, where k_i is the number of blocks of π that are unions of exactly $i + 1$ blocks of σ. The monomial $\zeta^\varphi(\sigma, \pi)$ is called the *type* of the interval $[\sigma, \pi]$. The monomial $\zeta^\varphi(\hat{0}, \pi)$ is the *type* of the partition π. Note that this differs from the usual notion of the type of a partition found in the literature, in that the function ζ^φ does not count blocks of size one.

Now if P is any subposet of $\Pi(V)$, the restriction of ζ^φ to Int(P) will be called the zeta type function of P, and the convolution inverse of ζ^φ in $\Phi(P)$ will be called the Möbius type function of P and denoted μ_c^φ. The Möbius type function of P indeed exists; it is given by

$$\mu_c^\varphi(\sigma, \pi) = \sum_k \sum_{\sigma = \sigma_0 < \cdots < \sigma_k = \pi} (-1)^k \zeta^\varphi(\sigma_0, \sigma_1) \cdots \zeta^\varphi(\sigma_{k-1}, \sigma_k)$$

for all $\sigma \le \pi$ in P, where the inner sum is taken over all chains from σ to π whose elements all belong to P. It is easy to see that this defines a convolution inverse for ζ^φ in $\Phi(P)$. When P is equal to $\Pi(V)$, we write μ^φ instead of μ_c^φ for the Möbius type function. The subposets P of $\Pi(V)$ that we will consider here are subposets of closed elements corresponding to coclosure operators on $\Pi(V)$; hence, we use the notation μ_c^φ for the Möbius type function of P to be consistent with the notation used in *Theorem 2*.

If P contains the element $\hat{0}$ of $\Pi(V)$ and has corank function ν then we define the *characteristic type polynomial* of P by

$$\chi^\varphi(P; \lambda) = \sum_{\sigma \in P} \mu_c^\varphi(\hat{0}, \sigma)\lambda^{\nu(\sigma)}.$$

We remark that ζ^φ, μ_c^φ, and $\chi^\varphi(P; \lambda)$ are generalizations of the usual zeta and Möbius functions and characteristic polynomial of P in the sense that each reduces to its classical counterpart upon the substitution $\varphi_1 = \varphi_2 = \cdots = 1$.

Simplicial Complexes and Coclosure Operators

Suppose S is a simplicial complex on a finite set V. That is, S is a collection of subsets of V such that whenever $U \in S$ and $W \subseteq U$, then $W \in S$. We can then define a map $\sigma \to \overline{\sigma}$ from $\Pi(V)$ into itself by letting $\overline{\sigma}$ be the partition of V whose nonsingleton blocks are precisely those of σ that are not contained in S. In other words, $\overline{\sigma}$ is obtained from σ by splitting into singletons all nontrivial blocks of σ that belong to S. We then have the following.

PROPOSITION 1: *Let* S *be a simplicial complex on a finite set* V; *then the associated map* $\sigma \to \overline{\sigma}$ *defined above is a* ζ^φ*-factoring coclosure operator on* Π(V).

Proof: The map $\sigma \to \overline{\sigma}$ is clearly a coclosure operator. Now for any subset B of V, let π_B be the partition of V whose only nontrivial block is B. In particular, $\pi_B = \hat{0}$ whenever $|B| = 1$. Then for all $\sigma \in \Pi(V)$, we have

$$\zeta^\varphi(\hat{0}, \sigma) = \prod_{B \in \sigma} \zeta^\varphi(\hat{0}, \pi_B).$$

Now let S' be the collection of all subsets of V that are not in the simplicial complex S. Then

$$\zeta^\varphi(\hat{0}, \overline{\sigma}) = \prod_{B \in \sigma \cap S'} \zeta^\varphi(\hat{0}, \pi_B)$$

and

$$\zeta^\varphi(\overline{\sigma}, \sigma) = \prod_{B \in \sigma \cap S} \zeta^\varphi(\hat{0}, \pi_B).$$

Thus $\zeta^\varphi(\hat{0}, \sigma) = \zeta^\varphi(\hat{0}, \overline{\sigma})\zeta^\varphi(\overline{\sigma}, \sigma)$, for all σ in $\Pi(V)$. Eq. **2** follows easily, hence the map $\sigma \to \overline{\sigma}$ is ζ^φ-factoring. □

Graphs

Let $G = (V, E)$ be a finite graph with vertex set V and edge set E. If U is a subset of V, the *induced subgraph* $G(U)$ is the graph whose vertex set is U and whose edges are those of G having both endpoints in U. A subset U of V is *independent* if the induced subgraph $G(U)$ contains no edges. The collection of all independent subsets of V is a simplicial complex on V, denoted by $S(G)$. A *k-coloring* of G is a map $f: V \to \{1, \ldots, k\}$ such that $f^{-1}(i)$ is independent for $1 \le i \le k$. A partition σ in $\Pi(V)$ is a *color partition* of G if its blocks are independent subsets of V. A *contraction* of G is a partition $\sigma \in \Pi(V)$, where $G(B)$ is a connected graph for all $B \in \sigma$. Let $\mathscr{C}(G)$ denote the lattice of contractions of G, and let $\chi(\mathscr{C}(G); \lambda)$ be the characteristic polynomial of $\mathscr{C}(G)$. The *chromatic polynomial* $\chi(G; k)$ of G is defined by the property that $\chi(G; k)$ is equal to the number of k-colorings of G, for all positive integers k. It is a classic theorem (3) that

$$\lambda^c \cdot \chi(\mathscr{C}(G); \lambda) = \chi(G; \lambda),$$

where c is the number of connected components of G. We would like to make an analogous statement about the characteristic type polynomial of $\mathscr{C}(G)$, but this is not possible for reasons that we shall explain below. Instead we must consider a larger, closely related subposet of $\Pi(V)$. Thus, we define a partition $\sigma \in \Pi(V)$ to be *admissible* if every block of σ that is not a singleton is not independent and let $\mathscr{A}(G)$ denote the poset of all admissible partitions of V. Clearly we have $\mathscr{C}(G) \subseteq \mathscr{A}(G) \subseteq \Pi(V)$. The essential property of $\mathscr{A}(G)$, which is actually just a restatement of its definition, is the following: $\mathcal{A}(G)$ is the subposet of closed elements of $\Pi(V)$ corresponding to the ζ^{φ}-factoring coclosure operator associated with $S(G)$, the complex of independent subsets of V.

According to ref. 2 we define the *umbral chromatic polynomial* $\chi^{\varphi}(G; \lambda)$ of G to be the characteristic type polynomial of $\mathcal{A}(G)$, multiplied by λ. That is

$$\chi^{\varphi}(G; \lambda) = \sum_{\sigma \in \mathcal{A}(G)} \mu_c^{\varphi}(\hat{0}, \sigma)\lambda^{|\sigma|}.$$

Now let $\rho(\lambda)$ be any polynomial in $\Phi[\lambda]$, and let k be a positive integer. The substitution of the *umbral integer* $k\varphi$ into ρ, denoted $\rho(k\varphi)$, is defined as follows: For each positive integer n, let $(k\varphi)^n$ be the result of applying the multinomial expansion to the expression $(\alpha_1 + \cdots + \alpha_k)^n$ and then setting α_i^r equal to φ_{r-1} for $1 \leq i \leq k$. Then $\rho(k\varphi)$ is obtained by substituting $(k\varphi)^n$ for λ^n in $\rho(\lambda)$, for all n.

We can now give a simplified proof of the main result of ref. 2.

THEOREM 3. *Let* G *be a finite graph with umbral chromatic polynomial* $\chi^{\varphi}(\mathrm{G}; \lambda)$. *For all positive integers* k,

$$\chi^{\varphi}(\mathrm{G}, \mathrm{k}\varphi) = \sum_{\mathrm{f}} \varphi^{\mathrm{f}}, \qquad [5]$$

where the sum is over all k-*colorings* f *of* G, *and* φ^{f} *is the monomial* $\varphi_1^{\mathrm{f}_1}\varphi_2^{\mathrm{f}_2}\cdots$, *where* $\mathrm{f_i}$ *is the number of colors* $\mathrm{j} \in \{1, 2, \ldots, \mathrm{k}\}$ *with* $|\mathrm{f}^{-1}(\mathrm{j})| = \mathrm{i} + 1$.

Proof: For all positive integers k and for $\sigma \leq \pi$ in $\Pi(V)$, let $(k)(\sigma, \pi)$ be the falling factorial $(k)_{|\sigma|}$. Then if $\sigma \in \Pi(V)$ has n blocks, we have $(k\varphi)^n = \zeta^{\varphi} * (k)(\sigma, \hat{1})$, where the convolution takes place in $\Phi(\Pi(V))$. Now define $\hat{\mu}_c^{\varphi}$ in $\Phi(\Pi(V))$ by

$$\hat{\mu}_c^{\varphi}(\sigma, \pi) = \begin{cases} \mu_c^{\varphi}(\sigma, \pi) & \text{if } \sigma, \pi \in \mathcal{A}(G) \\ 0 & \text{otherwise,} \end{cases}$$

where μ_c^{φ} is the Möbius type function of $\mathcal{A}(G)$. Then we can write

$$\chi^{\varphi}(G; k\varphi) = \sum_{\sigma \in \mathcal{A}(G)} \mu_c^{\varphi}(\hat{0}, \sigma)(k\varphi)^{|\sigma|} = \hat{\mu}_c^{\varphi} * \zeta^{\varphi} * (k)(\hat{0}, \hat{1}).$$

Now by *Proposition 1* and *Theorem 2* (Eq. **4**) we have

$$\hat{\mu}_c^{\varphi} * \zeta^{\varphi}(\hat{0}, \sigma) = \begin{cases} \zeta^{\varphi}(\hat{0}, \sigma) & \text{if } \bar{\sigma} = \hat{0} \\ 0 & \text{otherwise.} \end{cases}$$

But $\bar{\sigma} = \hat{0}$ if and only if σ is a color partition of G. Thus

$$\chi^{\varphi}(G; k\varphi) = \sum_{\substack{\sigma,\ \text{color} \\ \text{partition}}} \zeta^{\varphi}(\hat{0}, \sigma)(k)_{|\sigma|},$$

and the proof is complete. □

It follows immediately from *Theorem 3* that $\chi^{\varphi}(G; \lambda)$ reduces to the ordinary chromatic polynomial of G upon setting $\varphi_1 = \varphi_2 = \cdots = 1$.

Now suppose V is a set of n elements. Then the *Bell type number* $B_n^{\varphi} = B_n^{\varphi}(\varphi_1, \varphi_2, \ldots)$ is defined by $B_n^{\varphi} = \sum_{\sigma \in \pi(V)} \zeta^{\varphi}(\hat{0}, \sigma)$ or $B_n^{\varphi} = \zeta^{\varphi} * \zeta(\hat{0}, \hat{1})$. We will write $\chi^{\varphi}(G, B^{\varphi})$ for the image of $\chi^{\varphi}(G; \lambda)$ under the umbral substitution $\lambda^n \to B_n^{\varphi}$. Then we have the following variant of *Theorem 3*, which was not apparent from the original proof.

THEOREM 4. *If* G *is a finite graph with umbral chromatic polynomial* $\chi^{\varphi}(\mathrm{G}; \lambda)$, *then*

$$\chi^{\varphi}(\mathrm{G}, \mathrm{B}^{\varphi}) = \sum_{\substack{\sigma,\ \text{color} \\ \text{partition}}} \zeta^{\varphi}(\hat{0},\sigma).$$

The proof is essentially the same as that of *Theorem 3*.

If we further write B_n for the nth Bell number, that is, the total number of partitions of an n-element set, and $\chi(G; B)$ for the image of $\chi(G, \lambda)$ under the umbral substitution $\lambda^n \to B_n$, we obtain the following.

COROLLARY 1. *If* G *has chromatic polynomial* $\chi(\mathrm{G}, \lambda)$, *then* $\chi(\mathrm{G}; \mathrm{B})$ *is the number of color partitions of* G.

We can at last explain why the poset $\mathcal{A}(G)$ of admissible partitions forces itself upon us when defining the umbral chromatic polynomial. It is certainly true that the lattice of contractions of G is also the subposet of closed elements corresponding to a coclosure operator on $\Pi(V)$—i.e., map σ to $\tilde{\sigma}$, where $\tilde{\sigma}$ is obtained by splitting all blocks of σ into their connected components (as induced subgraphs of G). This coclosure operator has the property that $\tilde{\sigma} = \hat{0}$ if and only if σ is a color partition of G. Thus it follows from *Theorem 1* that the classical characteristic polynomial of $\mathcal{C}(G)$ is (essentially) the chromatic polynomial of G. However, the map $\sigma \to \tilde{\sigma}$ is not a ζ^{φ}-factoring coclosure operator, hence we cannot apply *Theorem 2*.

Instead, we are required to find a new coclosure operator $\sigma \to \bar{\sigma}$ that is ζ^{φ}-factoring and also has the property that $\bar{\sigma} = \hat{0}$ if and only if σ is a color partition of G. The coclosure operator associated with the simplicial complex of independent subsets of V clearly meets these conditions. Furthermore, it is easy to see that the subposet of closed elements corresponding to any other such coclosure operator must contain $\mathcal{A}(G)$. Hence $\mathcal{A}(G)$ is the unique minimal subposet of $\Pi(V)$ required for *Theorem 3*.

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1. Ray, N. (1988) *Trans. Am. Math. Soc.* **309**, 191–213.
2. Ray, N. & Wright, C. (1988) *Ars Combinatoria* **25B**, 277–286.
3. Rota, G.-C. (1964) *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **2**, 340–368.
4. Schmitt, W. (1987) *J. Comb. Theory Ser. A* **46**, 264–290.
5. Schmitt, W. & Haiman, M. (1989) *J. Comb. Theory Ser. A* **50**, 172–185.