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Homological properties of rings of functional-analytic type

(C*-algebras/cyclic homology/algebraic K-theory)

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ABSTRACT **Strong flatness properties are established for a large class of functional-analytic rings including all C*-algebras. This is later used to prove that all those rings satisfy excision in Hochschild and in cyclic homology over almost arbitrary rings of coefficients and that, for stable C*-algebras, the Hochschild and cyclic homology groups defined over an arbitrary coefficient ring $k \subset \mathbf{C}$ of complex numbers (e.g., $k = \mathbf{Z}$ or $\overline{\mathbf{Q}}$) vanish in all dimensions.**

In this note we study homological properties of rings of functional-analytic type. Rings and algebras are not assumed to have unit unless stated.

Definition: A ring A is said to be *right universally flat* if, for every unital ring R containing A as a right ideal, A is a flat right R-module.

Left universally flat rings are defined in a similar way. A ring will be referred to as *universally flat* if it is both left and right universally flat. Since A is left universally flat if and only if the opposite ring A^{op} is right universally flat, we shall usually consider only one of the two properties.

With every k-algebra structure $k \to \mathrm{Hom}_{A\text{-}A}(A, A)$ on A, where k denotes a commutative unital ring, one can associate the Hochschild homology groups $HH_*(A/k)$ and the cyclic homology groups $HC_*(A/k)$ of the k-algebra A, cf., e.g., section 4 of ref. 1.

Our first theorem establishes a connection between the universal flatness and the excision in Hochschild and in cyclic homology (cf. ref. 2).

THEOREM 1. *Any left or right universally flat ring* A *satisfies excision in* $HH_*(\ /k)$ *and* $HC_*(\ /k)$ *for all flat* k*-algebra structures on* A *and arbitrary commutative unital rings* k.

Warning: No nonzero ring has the property that every k-algebra structure on it is flat, not even a universally flat ring. A model example: $k_0 = \mathbf{Z}[\varepsilon]/(\varepsilon^2)$ and A is a $\mathbf{Z}[\varepsilon]/(\varepsilon^2)$-algebra via the augmentation map $\mathbf{Z}[\varepsilon]/(\varepsilon^2) \to \mathbf{Z}$. Then $\mathrm{Tor}_q^{k_0}(A, \mathbf{Z}) \simeq A$ for any $q \geq 0$; here A is an arbitrary ring. Thus the flatness of the k-algebra structure in the formulation of *Theorem 1* is an essential restriction.

Theorem 1 says that for any flat k-algebra structure on A all *pure* extensions [cf. ref. 2 (p. 591)] in the category of k-algebras

$$A \overset{i}{\rightarrowtail} R \overset{p}{\twoheadrightarrow} S$$

induce natural long exact sequences in the cyclic homology of k-algebras

$$\ldots HC_{q+1}(S/k) \xrightarrow{\partial_q} HC_q(A/k) \xrightarrow{i_q} HC_q(R/k) \xrightarrow{p_q} HC_q(S/k) \xrightarrow{\partial_{q-1}} \ldots$$

and similar long exact sequences in the Hochschild homology $HH_*(\ /k)$ of k-algebras.

Because of the connection between the universal flatness and excision, it is important to determine how large is the class of right universally flat rings. *Propositions 1* and *2* below demonstrate that it contains, e.g., all rings having a left unit or, more generally, a "local" left unit (see below).

PROPOSITION 1. *For any ring* A, *the following conditions are equivalent*: (a) *for every unital ring* R *containing* A *as a right ideal, the cyclic right* R*-module* R/A *is projective*; (b) *there exists a unital ring* R_0 *containing* A *as a right ideal such that the right* R_0*-module* R_0/A *is projective*; (c) A *has a left unit, i.e.,* ea = a *for some* e ∈ A *and all* a ∈ A.

PROPOSITION 2. *For any ring* A, *the following conditions are equivalent*: (a) *for every unital ring* R *containing* A *as a right ideal, the cyclic right* R*-module* R/A *is flat*; (b) *there exists a unital ring* R_0 *containing* A *as a right ideal such that the right* R_0*-module* R_0/A *is flat*; (c) A *has a "local" left unit, i.e., for every finite collection* $\{a_1, \ldots, a_n\} \subset A$, *there exists* $\varepsilon \in A$ *such that* $\varepsilon a_i = a_i$, $i = 1, \ldots, n$.

COROLLARY 1. *A ring having a "local" left unit is right universally flat. Similarly, a ring having a "local" right unit is left universally flat.*

One also has the following.

PROPOSITION 3. *A ring* A *that can be embedded into a unital simple ring as a left ideal is right universally flat.*

COROLLARY 2. *Every left ideal in the ring of differential operators on a smooth affine algebraic variety over a perfect field* k *is right universally flat.*

The ring of differential operators is simple.

Our next proposition provides a very useful general criterion of right universal flatness.

Notation: The right annihilator in A of an element $x \in A$ will be denoted $r(x) := \{a \in A \mid xa = 0\}$.

PROPOSITION 4. *Let us assume that* A *satisfies the following condition*:

for every finite collection $\{a_1, \ldots, a_n\} \subset A$ *there exists a unital ring* B *containing* A *as a left ideal, a finite partition of unity* $\psi_1 + \ldots + \psi_l = 1$, *where*

(Φ) $\{\psi_1, \ldots, \psi_l\} \subset B$, *and elements* $\rho_j, \sigma_j, b_{ji} \in A$, *such that*

$$\psi_j a_i = \rho_j \sigma_j b_{ji} \ (i = 1, \ldots, n;\ j = 1, \ldots, l)$$

and $\quad r(\rho_j \sigma_j) = r(\sigma_j),\ j = 1, \ldots, l.$

Then A *is right universally flat.*

The proof uses a criterion of flatness due to H. Cartan and S. Eilenberg [example 6 in chapter VI of ref. 3 (p. 123)]. By considering the opposite ring A^{op}, it is easy to formulate the corresponding right factorization property (Φ_R) that would similarly imply the left universal flatness of A.

Proposition 7 is then used to establish the universal flatness (i.e., both left and right universal flatness) of a number of important operator and function rings.

THEOREM 2. *The class of universally flat rings includes:* (a) *every* C*-*algebra (real or complex),* (b) *the convolution algebra* $L^1(G)$, *for any locally compact group* G; (c) *the ring*

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$C_K(X,Z)$ *of continuous* K*-valued functions on an arbitrary topological space* X *which vanish on a subset* $Z \subset X$; *here* $K = \mathbb{R}, \mathbb{C}$, *or* $\mathbb{C}_p$ (*the complex* p*-adic numbers* [*see* section 3.4.3 of ref. 4 (p. 150)]); (d) *the ring* $C^\infty(M, L)$ *of* C^∞ *functions on an arbitrary* C^∞ *manifold* M *which vanish with all derivatives on a closed subset* $L \subset M$; (e) *the ring* $L^{-\infty}(M, E)$ *of continuous linear operators* $C^\infty(M; E) \to C^\infty(M; E)$ *whose Schwartz kernels are* C^∞ (*"smoothing" operators*); M *is an arbitrary closed* C^∞ *manifold and* E *is a vector bundle on* M; (f) *the ring of compact operators* $\mathcal{K}(V)$ *on an arbitrary Banach space* V *whose strong dual* V′ *has a basis; for* V *having the Bounded Approximation Property,* $\mathcal{K}(V)$ *is always right totally flat.*

In the corollaries below, A can be any one of the rings listed in *Theorem 2*.

COROLLARY 3. A *satisfies excision in* $HH_*(/k)$ *and* $HC_*(/k)$ *for all flat* k*-algebra structures on* A *over an arbitrary unital commutative ring* k.

COROLLARY 4. *For every unital ring* R *of cardinality* $\aleph_m$ *containing* A *as a right ideal, the projective dimension* $dp_R(A)$ *of the right* R*-module* A *is* $\leq m + 1$.

The last corollary follows from the well-known correlation between the projective and flat dimensions [see corollary 1.4 of ref. 5, proposition 3 of ref. 6, and exercise 19 in section 8 of ref. 7 (p. 204)]. In particular, assuming the Continuum Hypothesis, one has $dp_R(A) \leq 2$ for all rings R of cardinality continuum. Thus the assertion: "$dp_R(A) \leq 2$" is in this case either true or undecidable.

Theorem 2 finds an application in a large number of situations. Here are two examples: (a) The ring of smoothing operators $L^{-\infty}(X, E)$ on a closed C^∞ manifold X, E being a coefficient vector bundle, is flat both as a left and as a right module over each of the following rings: (*i*) the ring $CL^0(X, E)$ of L_2-bounded pseudodifferential operators and (*ii*) the ring CL(X, E) of pseudodifferential operators of unbounded order. (b) Every closed right ideal I in an arbitrary unital C*-algebra C is flat over C; a similar assertion also holds for all closed left ideals.

In particular, $\mathrm{Tor}_q^C(C/I, M) = 0$ for $q \geq 2$ and all left C-modules M. This last statement can be amplified as follows.

THEOREM 3. *For every closed right ideal* I *in a unital* C**-algebra* C *and every topological Hausdorff left* C*-module* N, *one has* $\mathrm{Tor}_q^C(C/I, N) = 0$, $q \geq 1$.

Remark: The assertion of *Theorem 3* remains true for any left C-module N with the property that $\mathrm{Ann}_I(x) := \{a \in I | ax = 0\}$ is closed in I for all $x \in N$. If I does not have a left "local" unit, however, one can always find a right C-module M such that $\mathrm{Tor}_1^C(C/I, M) \neq 0$ (cf. *Proposition 2* above).

Stable C**-algebras*. For given C*-algebras C_1 and C_2, let us denote their *spatial*, also called *minimal*, tensor product by $C_1 \hat{\otimes} C_2$ [cf. definition IV.4.8 of ref. 8 (p. 207)]. Recall that a C*-algebra B is called *stable* if B is C*-isomorphic to $C \hat{\otimes} \mathcal{K}$ for some C*-algebra C, where $\mathcal{K} = \mathcal{K}(H)$ denotes the C*-algebra of compact operators on a separable Hilbert space *H*.

THEOREM 4. *For any unital subring* $k \subset \mathbb{C}$ *and any stable* C**-algebra* B, *one has:*

$$HH_*(B/k) = HC_*(B/k) = 0.$$

In other words, for all subrings k of $\mathbb{C}$, the Hochschild and cyclic homology groups over k of an arbitrary C*-algebra vanish in all dimensions.

A special case of *Theorem 4* corresponding to $k = \mathbb{C}$ has already been proved in ref. 9 where it was also proved that the continuous Hochschild and cyclic homology groups of stable C*-algebras vanish too:

$$HH_*^{\mathrm{cont}}(B) = HC_*^{\mathrm{cont}}(B) = 0.$$

The key steps in the proof of *Theorem 4* are *Theorems 1* and *2* above.

Theorem 4 has, e.g., the following application. Let $C^q_{\mathrm{add}}(B)$ denote the space of mappings $\phi: B \times \ldots \times B \to \mathbb{C}$ ($q+1$ times), which are assumed to be only additive in each variable (in particular, they are not assumed to be continuous or $\mathbb{C}$-multilinear). The standard Hochschild coboundary homomorphism operates on $C^*_{\mathrm{add}}(B)$ according to the formula

$$(\delta\phi)(b_0, \ldots, b_{q+1}) = \sum_{i=0}^{q}(-1)^i\phi(b_0, \ldots, b_ib_{i+1}, \ldots, b_{q+1}) + (-1)^{q+1}\phi(b_{q+1}b_0, b_1, \ldots, b_q),$$

[$\phi \in C^q_{\mathrm{add}}(B)$, $\delta\phi \in C^{q+1}_{\mathrm{add}}(B)$, $\delta \circ \delta = 0$]. Let $HH^*_{\mathrm{add}}(B)$ denote the cohomology groups of $(C^*_{\mathrm{add}}(B), \delta)$. Mappings $\phi: B \times \ldots \times B \to \mathbb{C}$ that possess the cyclic symmetry

$$\phi(b_q, b_0, \ldots, b_{q-1}) = (-1)^q\phi(b_0, b_1, \ldots, b_q)$$

form a subcomplex; let us denote its cohomology by $HC^*_{\mathrm{add}}(B)$.

COROLLARY 5. *For every stable* C**-algebra* B, *one has*

$$HH^*_{\mathrm{add}}(B) = HC^*_{\mathrm{add}}(B) = 0.$$

Karoubi's conjecture. In the 1970s Karoubi (10) conjectured that the canonical comparison map between the algebraic and topological K-groups $\iota_*: K_*^{\mathrm{alg}}(B) \to K_*^{\mathrm{top}}(B)$ is an isomorphism for any stable C*-algebra. This conjecture since then has been partially established in dimensions $* \leq 2$ and it is known that the conjecture would follow if someone would prove that C*-algebras satisfy excision in algebraic K-theory (cf., e.g., refs. 11 and 12).

In ref. 2 the following implication was proved:

A ring A *satisfies excision in* K_*^{alg} $\Rightarrow$ *the* $\mathbb{Q}$*-algebra* $A \otimes_{\mathbb{Z}} \mathbb{Q}$ *satisfies excision in* $HC_*(/\mathbb{Q})$ **[E]**

and it was later conjectured that for rings $A = A \otimes_{\mathbb{Z}} \mathbb{Q}$ the two excision properties are, in fact, equivalent. In the present note we establish the right-hand side of implication **E** for a large number of functional-analytic rings, including all C*-algebras. Thus Karoubi's conjecture is now reduced to the above-mentioned purely algebraic conjecture on the reverse implication in **E**.

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