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Superconnections and internal supersymmetry dynamics

(Higgs mechanism/electroweak theory/generational unification)

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ABSTRACT **In previous papers we proposed a theory of internal supersymmetry using the superalgebra su(n/1) to give rise to a unified structure that included quarks and leptons in 2^{n-5} generations. In the present paper we suggest that the notion of superconnections as introduced by Quillen provides a natural setting for the dynamics of an internally supersymmetric theory with the Higgs field occurring as the "zero-th order part" of the superconnection. The Higgs mechanism enters quadratically into the curvature of the superconnection and hence quartically into the Lagrangian. The supercovariant derivative gives a coupling of the Higgs field to the matter field similar to that put in "by hand" in the Lagrangian of the Weinberg–Salam theory.**

In ref. 1 it was noted that the number of independent assumptions required by the Weinberg–Salam SU(2) × U(1) theory can be reduced by assuming that the structure (gauge) group SU(2) × U(1) is associated to the even part of the superalgebra su(2/1) that acts internally on the matter fields (cf. also ref. 2). In ref. 3 it was shown how the basic representations of sl(2/1) occurring in ref. 1 could be extended to sl(n/1) and so give rise to a unified structure that included quarks and leptons in 2^{n-5} generations. These representations have been discussed in terms of Howe pairs and dimensional reduction in ref. 4. It was proposed in ref. 5 that for a theory of internal supersymmetry, the natural "Grassmann variables" to tensor with the internal superalgebra are the differential forms on the base manifold, and an attempt was made to construct a dynamics using a connection associated to this structure. In the present paper we suggest that the notion of superconnections as introduced by Quillen (6) provides a natural setting for the dynamics of an internally supersymmetric theory with the Higgs field occurring as the "zero-th order part" of the superconnection. The Higgs enters quadratically into the curvature of the superconnection and hence quartically into the Lagrangian. The supercovariant derivative gives a coupling of the Higgs field to the matter field similar to that put in "by hand" in the Lagrangian of the Weinberg–Salam theory. A good reference for the material on superconnections and equivariant superconnections has been given by Berlin *et al.* (7).

Section 1. Generalities

Recall (8) that if $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are superalgebras, then the superalgebra $A \otimes B$ is defined by

$$(A \otimes B)_0 = A_0 \otimes B_0 \oplus A_1 \otimes B_1,$$
$$(A \otimes B)_1 = A_1 \otimes B_0 \oplus A_0 \otimes B_1, \qquad [11]$$

with multiplication (on homogeneous elements) given by

$$(a \otimes b)(a' \otimes b') = (-1)|b||a'|aa' \otimes bb', \qquad [1.2]$$

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where $|a|$ denotes the degree of a. For example, suppose that A is the (supercommutative) superalgebra of all differential forms on a manifold M and that $B = \text{End } E$, where $E = E_0 \oplus E_1$ is a supervector space. Thus B_0 consists of all "matrices" of the form

$$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \qquad R \in \text{End}(E_0), \qquad S \in \text{End}(E_1),$$

while B_1 consists of all "matrices" of the form

$$\begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}, \qquad K \in \text{Hom}(E_1, E_0), \qquad L \in \text{Hom}(E_0, E_1).$$

If we choose bases of E_0 and E_1 then we can think of R, S, K, and L as actual matrices. We can then think of elements of $A \otimes B$ as matrices whose entries are differential-forms forms, but we must remember rules **1.1** and **1.2**. For example, if ω_0 and ω_1 are matrices of differential forms of odd exterior degree then

$$\begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix}$$

is an odd element of $A \otimes B$. Similarly, if L_{01} and L_{10} are matrices of forms of even exterior degree, then

$$\begin{pmatrix} 0 & L_{01} \\ L_{10} & 0 \end{pmatrix}$$

is an odd element of $A \otimes B$. Then rule **1.2** says that

$$\begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix}\begin{pmatrix} 0 & L_{01} \\ L_{10} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \wedge L_{01} \\ \omega_1 \wedge L_{10} & 0 \end{pmatrix} \qquad [1.3a]$$

while

$$\begin{pmatrix} 0 & L_{01} \\ L_{10} & 0 \end{pmatrix}\begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix} = \begin{pmatrix} 0 & -L_{01} \wedge \omega_1 \\ -L_{10} \wedge \omega_0 & 0 \end{pmatrix}. \qquad [1.3b]$$

The minus sign in Eq. **1.3b** arises from passing the odd elements of B through the differential forms of odd exterior degree as prescribed by rule **1.2**. In this example we can consider the supervector space $A \otimes E$ with grading as in rule **1.1**. Then $A \otimes E$ is a (left) module for $A \otimes B$ where we apply the sign rule analogous to rule **1.2**. Thus

$$A \otimes B \subset \text{End}(A \otimes E).$$

We can think of A as embedded in $A \otimes B$ as $A \otimes I$ and this makes $A \otimes E$ into an A module where the action is the obvious one. Since $A \otimes E$ is a supervector space, End($A \otimes E$) is a superalgebra. It is easy to see that an element of End($A \otimes E$) belongs to $A \otimes B$ if and only if it supercommutes with all elements of A. In other words,

$$A \otimes B \text{ is the supercentralizer of } A \text{ inside End}(A \otimes E). \qquad [1.4]$$

We can define the (odd) operator $d \in \mathrm{End}(A \otimes E)$ by

$$d(\alpha \otimes e) = d\alpha \otimes e,$$

where $d\alpha$ is the usual exterior derivative of the differential form α. We can write this definition symbolically as defining d as the "matrix"

$$d = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}.$$

If, by abuse of notation, we let α denote multiplication by α as an element of $\mathrm{End}(A \otimes E)$ then the supercommutator of d with α is given by

$$[d, \alpha] = d\alpha, \qquad [1.5]$$

where the right-hand side denotes multiplication by $d\alpha$. More generally, for any $\alpha \otimes b \in A \otimes B$ we have

$$[d, \alpha \otimes b] = d\alpha \otimes b. \qquad [1.6]$$

So if, for $\sigma \in A \otimes B$, we define $d\sigma$ by

$$d(\alpha \otimes b) = d\alpha \otimes b,$$

then we can write Eq. **1.6** as

$$[d, \sigma] = d\sigma \text{ for } \sigma \in A \otimes B. \qquad [1.7]$$

In particular, if ω is an odd element of $A \otimes B$ then

$$(d + \omega)^2 = d^2 + [d, \omega] + \omega^2 = d\omega + \omega^2. \qquad [1.8]$$

Note that the right-hand side of Eq. **1.8** is an element of $A \otimes B$. If we write ω out as a "matrix"

$$\omega = \begin{pmatrix} \omega_0 & \mathbf{L}_{01} \\ \mathbf{L}_{10} & \omega_1 \end{pmatrix}, \qquad [1.9]$$

then we can write Eq. **1.8** as

$$(d + \omega)^2 = \begin{pmatrix} d\omega_0 + \omega_0 \wedge \omega_0 + L_{01} \wedge L_{10} & dL_{01} + \omega_0 \wedge L_{01} - L_{01} \wedge \omega_1 \\ dL_{10} + \omega_1 \wedge L_{10} - L_{10} \wedge \omega_0 & d\omega_1 + \omega_1 \wedge \omega_1 + L_{10} \wedge L_{01} \end{pmatrix}. \qquad [1.10]$$

We should remember that in Eq. **1.10** the ω and L terms are matrices of even and odd forms, respectively, but not necessarily homogeneous with respect to exterior degree. Thus, if the base space is four-dimensional then

$$\omega_0 = A_0 + C_0,$$

where A_0 is a matrix of one forms and C_0 is a matrix of three forms and similarly for ω_1. Also

$$L_{01} = h_{01} + B_{01} + D_{01},$$

where h_{01} is a matrix of functions, B_{01} is a matrix of two forms, and D_{01} is a matrix of four forms and similarly for L_{10}. The $\wedge$ occurring in Eq. **1.10** denotes matrix multiplication where the matrix entries are multiplied via exterior multiplication.

Section 2. Superconnections

Now let $E \to M$ be a supervector bundle over an ordinary manifold M. So $E = E_0 \oplus E_1$, where E_0 and E_1 are ordinary vector bundles. Let $\mathbf{A}(M)$ denote the ring of smooth differential forms on M and $\mathbf{A}(M, E)$, the space of smooth E-valued forms. Then $\mathbf{A}(M, E)$ is a module for $\mathbf{A}(M)$ as before. We can consider $\mathrm{End}(\mathbf{A}(M, E))$ and $\mathbf{A}(M, \mathrm{End}(E))$ so that

$$\mathbf{A}(M, \mathrm{End}(E)) \subset \mathrm{End}(\mathbf{A}(M, E)).$$

The analogue of statement **1.4** is

$\mathbf{A}(M, \mathrm{End}(E))$ is the centralizer of

$$\mathbf{A}(M) \text{ in } \mathrm{End}(\mathbf{A}(M, E)). \qquad [2.1]$$

In fact, $\mathbf{A}(M, \mathrm{End}(E))$ is the centralizer of the ring of functions, $\mathbf{A}^0(M)$, in $\mathrm{End}(\mathbf{A}(M, E))$. An odd element, $\mathbf{D} \in \mathrm{End}(\mathbf{A}(M, E))_1$, is called a *superconnection* if

$$[\mathbf{D}, \alpha] = d\alpha \text{ for all } \alpha \in \mathbf{A}(M). \qquad [2.2]$$

In other words,

$$\mathbf{D}: \mathbf{A}(M, E)_0 \to \mathbf{A}(M, E)_1, \qquad \mathbf{D}: \mathbf{A}(M, E)_1 \to \mathbf{A}(M, E)_0$$

and

$$\mathbf{D}(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^{|\alpha|}\alpha \wedge \mathbf{D}\sigma$$

$$\text{for all } \alpha \in \mathbf{A}(M) \text{ and } \sigma \in \mathbf{A}(M, E).$$

The curvature $\mathbf{F} = \mathbf{F}(\mathbf{D})$ of the superconnection is defined as

$$\mathbf{F} = \mathbf{D}^2. \qquad [2.3]$$

Note that for any function f we have

$$[\mathbf{D}^2, f] = [\mathbf{D}, f]\mathbf{D} + \mathbf{D}[\mathbf{D}, f] = (df)\mathbf{D} + \mathbf{D}(df)$$
$$= [\mathbf{D}, df] = ddf = 0.$$

Thus

$$\mathbf{F} \in \mathbf{A}(M, \mathrm{End}(E))_0. \qquad [2.4]$$

The difference $\mathbf{D}_1 - \mathbf{D}_2$ between two superconnections is an element of $\mathbf{A}(M, \mathrm{End}(E))$ by Eq. **2.3**. Hence, in terms of a local trivialization of E, the most-general superconnection can be written locally as

$$\mathbf{D} = d + \omega, \qquad \omega \in \mathbf{A}(M, \mathrm{End}(E))_1. \qquad [2.5]$$

This means that the local expression for the curvature is given by Eq. **1.8** or, in "matrix" language, by Eq. **1.10**. If p is any polynomial (or entire function) of one variable and Str denotes the supertrace, then (cf. ref. 6 or ref. 7) $\mathrm{Str}(p(\mathbf{F}))$ is a closed form; i.e.,

$$\mathrm{Str}(p(\mathbf{F})) \in \mathbf{A}(M) \text{ and } d\mathrm{Str}(p(\mathbf{F})) = 0.$$

Furthermore, up to an exact form, $\mathrm{Str}(p(\mathbf{F}))$ is independent of the choice of the superconnection; i.e.,

$$\mathrm{Str}(p(\mathbf{F}(\mathbf{D}_1))) - \mathrm{Str}(p(\mathbf{F}(\mathbf{D}_2))) = d\alpha(\mathbf{D}_1, \mathbf{D}_2),$$

where $\alpha(\mathbf{D}_1, \mathbf{D}_2)$ is a differential form that has a simple expression in terms of $\mathbf{D}_1$ and $\mathbf{D}_2$. Thus, for example, the Chern character corresponds to $p(z) = e^{-z}$ (for all this, see refs. 6 and 7). Now $\mathrm{Str}(ab)$ is antisymmetric in a and b if a and b are both odd elements of $\mathrm{End}(E)$. Hence if α and β are odd forms, the expression $\mathrm{Str}(\alpha \otimes a)(*\beta \otimes b)$ is antisymmetric as a function of $\alpha \otimes a$ and $\beta \otimes b$. Furthermore, on the even terms, an expression such as $\mathrm{Str}(\mathbf{F}*\mathbf{F})$ in the Lagrangian will lead to negative kinetic energy terms for the dynamics (unless E_0 or E_1 is trivial). Hence we proceed as follows: choose an invariant bilinear form b on the Lie algebra $\mathrm{End}(E)_0$. Here invariant means invariant under the "even group" $\mathrm{Aut}(E_0) \times \mathrm{Aut}(E_1)$. As the adjoint representation of this group is not irreducible, there will be some choices here, beyond overall scale. In the case of eventual interest to us, this amounts to the choice of Weinberg angle. We will see how this choice is made in our theory. Then the Lagrange density for the purely Yang–Mills part of the theory is

$$\mathrm{Y\text{–}M}(\mathbf{D}) = b(\mathbf{F}, *\mathbf{F}) \qquad [2.6]$$

as usual. If we identify $\mathrm{End}(E)_1$ with the Higgs sector, then the L_{01} entering into "matrix" **1.9** contains, as components of exterior degree zero, sections of $\mathrm{End}(E)_1$, that is to say Higgs fields. From "matrix" **1.10** we see that the Higgs field enters quadratically into the curvature, and hence Eq. **2.6** is a polynomial of degree four in the Higgs. For the case of $\mathrm{su}(n/1)$ as internal superalgebra, a natural choice of b is as follows: For $\mathrm{su}(n/1)$, we have $g_0 = \mathrm{su}(n) \oplus \mathbf{R} \approx \mathrm{su}(n)$ and $g_1 = \mathbf{C}^n$. As a vector space, and also as far as the action of g_0 on g_1 is concerned, we have $g_0 \oplus g_1 \approx \mathrm{su}(n+1)$. The difference lies in the bracket of $g_1 \times g_1 \to g_0$, one bracket being symmetric and giving a Lie superalgebra and the other being antisymmetric and giving a Lie algebra. Indeed these two structures are related to one another via the notion of a Hermitian Lie algebra (see the first few pages of ref. 9 and cf. also ref. 10). So a natural choice would be to take b to be the Killing form of $\mathrm{su}(n+1)$, and this was the choice made for the case $n = 2$ in refs. 1 and 2 for determination of the Weinberg angle.

The theory of superconnections can, of course, also be formulated in terms of principal and associated bundles (cf. ref. 7): If $g = g_0 \oplus g_1$ and G is a Lie group whose Lie algebra is g_0, then a superconnection will be a g-valued form on P_G of total odd degree (subject to conditions generalizing the standard ones for connections), where P_G is a principal bundle with structure group G. If F is a supervector bundle associated to a representation of (G, g) on a supervector space V, then the superconnection form on P_G induces a superconnection $\mathbf{D}$ on F. If S is the spin bundle, then we can use $\mathbf{D}$ to modify the Dirac operator and so obtain the operator $\gamma(\mathbf{D})$: $F \otimes S \to F \otimes S$. A superinvariant bilinear form on F then gives the matter field contribution to the Lagrangian as $(\cdot, \gamma(\mathbf{D})\cdot)$ on $F \otimes S$. Notice that this involves a cubic term that is quadratic in the matter field and of first order in the Higgs field, as in the Weinberg–Salam Lagrangian.

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