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Mathematics

Schur maps, Young tableaux, and supersymmetric algebra

GIAN-CARLO ROTA† AND JOEL A. STEIN‡

Mathematics Department, Massachusetts Institute of Technology, Cambridge, MA 02139; and ‡Department of Mathematics and Statistics, California State University, Chico, CA 95929

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ABSTRACT **It is shown that the Hopf algebra dual of a supersymmetric Hopf algebra admits two presentations, and a natural isomorphism between them is described.**

Section 1. Introduction

We shall follow Sweedler's notation as well as the notation introduced in refs. 1–4, with the following provisos: (*i*) in dealing with supersymmetric algebras, special conventions will be introduced relative to signs (much as in our previous work) and (*ii*) the superscript $*$ to designate dual modules will not be used, since it clashes with our use of the same superscript to designate adjoint signed sets, introduced in section 3 of ref. 1 and used again below. All modules will be taken over the integers $\mathbb{Z}$.

All Hopf algebras will be assumed to be *supersymmetric algebras*, namely, Hopf algebras of the form Super $[A]$, where A is a signed alphabet.

We define a generalization of the notion of measuring in the sense of Sweedler. Let Super$[A]$, Super$[B]$, and Super$[C]$ be supersymmetric algebras over signed alphabets A, B, and C, respectively. Let η be a bilinear form from Super$[A]$ $\times$ Super$[B]$ to Super$[C]$, and let $\boldsymbol{\eta}$ be the corresponding linear form from the product Hopf algebra Super$[A] \otimes$ Super$[B]$ to Super$[C]$. We say that $\boldsymbol{\eta}$ is a *Laplace pairing* from Super$[A]$ $\otimes$ Super$[B]$ to Super$[C]$ when $\boldsymbol{\eta}$ satisfies the following identities:

1. $\boldsymbol{\eta}(1 \otimes 1) = 1$;

2. $\boldsymbol{\eta}(w \otimes w'\ w'') = \Sigma\ \pm\boldsymbol{\eta}(w_{(1)} \otimes w')\boldsymbol{\eta}(w_{(2)} \otimes w'')$

for w in Super$[A]$ and for w', w'' in Super$[B]$;

3. $\boldsymbol{\eta}(w\ w' \otimes w'') = \Sigma\ \pm\boldsymbol{\eta}(w \otimes w''_{(1)})\boldsymbol{\eta}(w' \otimes w''_{(2)})$

for w, w' in Super$[A]$ and for w'' in Super$[B]$; and

4. $\Sigma\boldsymbol{\eta}(w \otimes w')_{(1)} \otimes \boldsymbol{\eta}(w \otimes w')_{(2)}$

$$= \Sigma\ \pm\boldsymbol{\eta}(w_{(1)} \otimes w'_{(1)}) \otimes \boldsymbol{\eta}(w_{(2)} \otimes w'_{(2)}).$$

The sign $\pm$ in identity 2 is determined by the following *sign convention*: count the transpositions of pairs of negative letters that are required to transform the word $w\ w'\ w''$ in Div$(A \cup B)$ into the word $w_{(1)}\ w'\ w_{(2)}\ w''$.

The most important example of a Laplace pairing is given by the linear map $\boldsymbol{\Omega}$ from Super$[L] \otimes$ Super$[P]$ to Super$[L|P]$ that is obtained from the bilinear form Ω defined in section 2 of ref. 1 for proper signed alphabets L and P. Only identity 4 remains to be verified, namely,

$$\Sigma(w|w')_{(1)} \otimes (w|w')_{(2)} = \Sigma\ \pm(w_{(1)}|w'_{(1)}) \otimes (w_{(2)}|w'_{(2)}).$$

The proof of this identity follows familiar lines, by first establishing the identity for $w = x^{(k)}$ and $w' = \alpha^{(k)}$ and then applying polarization operators.

Let ξ be a bilinear form on Super$[A]$ $\times$ Super$[B]$, with values in Z. We say that ξ is a *pairing* of Super$[A]$ $\times$ Super$[B]$ when it satisfies the following conditions:

1. $\xi(1, 1) = 1$;

2. $\xi(w, w'\ w'') = \Sigma\ \pm\xi(w_{(1)}, w')\xi(w_{(2)}, w'')$

for w in Super$[A]$ and for w', w'' in Super$[B]$; and

3. $\xi(w\ w', w'') = \Sigma\ \pm\xi(w, w''_{(1)})\xi(w', w''_{(2)})$

for w, w' in Super$[A]$ and for w'' in Super$[B]$.

Suppose ξ is a pairing of Super$[A]$ $\times$ Super$[B]$ and ξ' is a pairing of Super$[A']$ $\times$ Super$[B']$, and suppose f: Super$[B'] \to$ Super$[B]$ and g: Super$[A] \to$ Super$[A']$ are linear maps of modules. We say that f is the *right adjoint* of g and that g is the *left adjoint* of f, relative to the pair ξ, ξ', when for all p in Super$[A]$ and for all q in Super$[B']$ the following identity holds:

$$\xi(p, f(q)) = \xi'(g(p), q).$$

Let A be a signed set. We denote by $(A^\#, \psi)$ the pair consisting of a signed set $A^\#$ and a bijection ψ: $A \to A^\#$ that maps A^+ to $(A^\#)^0$, A^- to $(A^\#)^-$, and A^0 to $(A^\#)^+$. We shall call the signed set $A^\#$ the *dual signed set*; ordinarily, the bijection ψ will be passed over in silence. If a is a letter of the signed set A, we denote the letter $\psi(a)$ of $A^\#$ by $a^\#$. We have $(A \cup B)^\# = A^\# \cup B^\#$ for disjoint sets A and B. The dual signed set $A^\#$ must be distinguished from the *adjoint* signed set A^*, which is defined in section 3 of ref. 1, when A is a proper signed set, as follows: the adjoint signed set A^* is a proper signed set A^* together with a bijection ϕ: $A \to A^*$ such that $\phi(A^+) = (A^*)^-$ and $\phi(A^-) = (A^*)^+$.

We now define some of the pairings to be considered in the present work. First, the pairing $\xi_\#$ of Super$[A^\#]$ $\times$ Super$[A]$ is defined to be the unique pairing satisfying the following condition: if a is a letter of A and b is a letter of $A^\#$, we set $\xi_\#(b, a) = 1$ if $b = a^\# = \psi(a)$, and we set $\xi_\#(b, a) = 0$ otherwise. Second, we denote by ξ_* the pairing of Super$[L^*|P^*]$ $\times$ Super$[L|P]$ defined in proposition 5 of ref. 1 and there denoted by $\langle p, q\rangle$:

$$\xi_*(p, q) = \langle p, q\rangle.$$

We can now state the motivation of the present work. The supersymmetric algebra Super$[L|P]$, like all graded Hopf algebras, has a dual Hopf algebra. However, one cannot work with the dual Hopf algebra unless a pairing is given, one of whose terms is Super$[L|P]$. Any such pairing leads to a representation of the dual of Super$[L|P]$, and different pairings lead to different representations. The pairing ξ_* between Super$[L^*|P^*]$ and Super$[L|P]$ gives one such representation. It is, however, a representation that is in some ways unusual. We introduce below another pairing $\xi_{\#\#}$, one of whose

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factors is Super$[L|P]$ and the other is a Hopf algebra that we call Symm$[L^\#, P^\#]$. This second pairing gives a representation of the dual Hopf algebra of Super$[L|P]$ that is closer to the ones previously considered. We are therefore led to conjecture that a natural map exists that implements an isomorphism between Super$[L^*|P^*]$ and Symm$[L^\#, P^\#]$. The purpose of the present work is to describe such a map.

Section 2. The Free Supersymmetric Algebra

Let L and P be proper signed sets. The *free supersymmetric algebra* Brace$\{L, P\}$ is defined as follows: As an associative algebra, Brace$\{L, P\}$ is generated by expressions $\{w|w'\}$, where w, w' are elements of Div(L) and Div(P), respectively. The *parity* $|\{w|w'\}|$ is defined to be $|w| + |w'|$, and the parity of a product (which is indicated by juxtaposition) $\{w|w'\}\{w''|w'''\}$ $\ldots$ is defined to be the sum $|\{w|w'\}| + |\{w''|w'''\}| + \ldots$. The product in Brace{L, P} is subject to the following relations:

1. If Length$(w) \neq$ Length(w'), $\quad \{w|w'\} = 0$,
2. $\{w|w'\}\{w''|w'''\} = \pm\{w''|w'''\}\{w|w'\}$, and
3. $\{1|1\} = 1$.

The coproduct in Brace$\{L, P\}$ is defined as follows:

1. Set $\Delta\{w|w'\} = \Sigma \pm\{w_{(1)}|w'_{(1)}\} \otimes \{w_{(2)}|w'_{(2)}\}$;
2. If $W = \{w|w'\}\{w''|w'''\} \ldots$, set $\Delta W = \Delta\{w|w'\}\Delta\{w''|w'''\} \ldots$.

Brace$\{L, P\}$ is a bialgebra but not in general a Hopf algebra.

PROPOSITION 1. *The map* Φ: *Brace*{L, P} $\rightarrow$ *Super*[L|P] *defined by setting* $\Phi(\{w|w'\}) = (w|w')$ *extends uniquely to a bialgebra map of Brace*{L, P} *onto Super*[L|P].

We define Brace$_\lambda\{L, P\}$ for every shape λ to be the submodule of Brace$\{L, P\}$ spanned by all elements $\{w|w'\}\{w''|w'''\}$ $\ldots$, where Length(w) = Length$(w') = \lambda_1$, Length(w'') = Length$(w''') = \lambda_2, \ldots$.

We remark in passing that Brace$\{L, P\}$ can also be defined by the following universal construction. Consider the underlying coalgebra of Super$[L] \otimes$ Super$[P]$; in this coalgebra consider the coideal J generated by all pairs $w \otimes w'$, where Length$(w) \neq$ Length(w'). Then Brace$\{L, P\}$ is the free supercommutative bialgebra generated by the quotient coalgebra Super$[L] \otimes$ Super$[P]/J$.

Section 3. The Cofree Supersymmetric Algebra

We next define another bialgebra on a pair of signed sets $L^\#$ and $P^\#$, to be called Symm$[L^\#, P^\#]$, where L and P are proper signed sets. It is defined by the following steps.

1. If λ is a shape, so that $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $D = (w_1, w_2, \ldots)$ be a Young diagram of shape λ whose entries are words in Mon$(L^\#)$ such that Length$(w_1) = \lambda_1$, Length$(w_2) = \lambda_2$, $\ldots$. Similarly, let $E = (v_1, v_2, \ldots)$ be a Young diagram whose entries are words in Mon$(P^\#)$ such that Length$(v_1) = \lambda_1$, Length$(v_2) = \lambda_2, \ldots$. We denote by Super$^\lambda[L^\#, P^\#]$ the tensor product module spanned by all elements

$$\text{Tens}(D, E) = \pm(w_1 \otimes v_1) \otimes (w_2 \otimes v_2) \otimes \ldots.$$

The *parity* $|\text{Tens}(D, E)|$ of the element Tens(D, E) equals

$$|w_1| + |w_2| + \ldots + |v_1| + |v_2| + \ldots.$$

2. Suppose that the shape λ has components $\lambda_1, \lambda_2, \ldots, \lambda_k$, and recall that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. Let Σ_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. We consider the subgroup $\Sigma(\lambda)$ of Σ_k consisting of all permutations such that $(\lambda_{\sigma 1}, \lambda_{\sigma 2}, \ldots, \lambda_{\sigma k}) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.
3. We define an action of the group $\Sigma(\lambda)$ on Super$^\lambda[L^\#, P^\#]$ by setting

$$\sigma\text{Tens}(D, E) = \pm(w_{\sigma 1} \otimes v_{\sigma 1}) \otimes (w_{\sigma 2} \otimes v_{\sigma 2}) \otimes \ldots \otimes (w_{\sigma k} \otimes v_{\sigma k}),$$

and extend to all of Super$^\lambda[L^\#, P^\#]$ by linearity.
4. An element p of Super$^\lambda[L^\#, P^\#]$ is said to be a *supersymmetric element* when $\sigma(p) = p$ for all σ in $\Sigma(\lambda)$. We stress the fact that, in general, a supersymmetric element is a linear combination of several elements of the form Tens(D, E). The submodule of Super$^\lambda[L^\#, P^\#]$ consisting of all supersymmetric elements is denoted by Symm$_\lambda[L^\#, P^\#]$.

Example: Let $x^\#$ be a negative element of $L^\#$, let $y^\#$ be a neutral element of $L^\#$, and let $\alpha^\#$ and $\beta^\#$ be negative elements of $P^\#$. Then the element

$$(x^\# \otimes \alpha^\#) \otimes (y^\# \otimes \beta^\#) + (y^\# \otimes \beta^\#) \otimes (x^\# \otimes \alpha^\#)$$

is supersymmetric, even though neither the element $(x^\# \otimes \alpha^\#) \otimes (y^\# \otimes \beta^\#)$ nor the element $(y^\# \otimes \beta^\#) \otimes (x^\# \otimes \alpha^\#)$ is supersymmetric.

5. We are at last in a position to define the underlying module of the yet-to-be-defined bialgebra Symm$[L^\#, P^\#]$. It is the direct sum

$$\sum_\lambda \text{Symm}_\lambda[L^\#, P^\#],$$

where the sum ranges over all shapes λ. Every element of Symm$[L^\#, P^\#]$ is a linear combination of elements of well-defined parities.

6. We next define an integer-valued bilinear form $\xi_{\#\#}$ on Symm$[L^\#, P^\#] \times$ Brace$\{L, P\}$ as follows. Suppose W is an element of Symm$_\mu[L^\#, P^\#]$ and V is an element of Brace$_\lambda\{L, P\}$. Then, if $\mu \neq \lambda$, define $\xi_{\#\#}(W, V) = 0$. If $\mu = \lambda$, suppose

$$W = \Sigma(w_1 \otimes v_1) \otimes (w_2 \otimes v_2) \ldots \otimes (w_r \otimes v_r)$$

is a supersymmetric element, and $V = \{w'_1|v'_1\}\{w'_2|v'_2\} \ldots \{w'_r|v'_r\}$, with Length$(w'_1)$ = Length$(v'_1) = \lambda_1$, Length(w'_2) = Length$(v'_2) = \lambda_2 \ldots$. Define $\xi_{\#\#}(W, V)$ to be

$$\Sigma \pm\xi_\#(w_1, w'_1)\, \xi_\#(w_2, w'_2) \ldots \xi_\#(w_r, w'_r)\, \xi_\#(v_1, v'_1)\, \xi_\#(v_2, v'_2) \ldots \xi_\#(v_r, v'_r).$$

In view of the definition of supersymmetry of W, this expression is well-defined!

7. We now define the structure of a graded bialgebra on Symm$[L^\#, P^\#]$ to be the adjoint bialgebra of the bialgebra Brace$\{L, P\}$ relative to the bilinear form $\xi_{\#\#}$. Specifically, we define product and coproduct in Symm$[L^\#, P^\#]$ so as to satisfy the following identities. Let

$$W = \Sigma(w_1 \otimes v_1) \otimes (w_2 \otimes v_2) \ldots \otimes (w_r \otimes v_r),$$

$$U = \Sigma(u_1 \otimes t_1) \otimes (u_2 \otimes t_2) \ldots \otimes (u_s \otimes t_s)$$

be elements of Symm$[L^\#, P^\#]$ having parities $|W|$ and $|U|$. Let

$$V = \{w'_1|v'_1\}\{w'_2|v'_2\} \ldots \{w'_n|v'_n\}$$

be an element of Brace$\{L, P\}$ having parity $|V|$.

The product WU in Symm$[L^\#, P^\#]$ is uniquely defined by the identity

$$\xi_{\#\#}(WU, V) = \Sigma \pm\xi_{\#\#}(W, V_{(1)})\xi_{\#\#}(U, V_{(2)}), \qquad [1]$$

where the sign $\pm$ equals sign($|U||V_{(1)}|$). The coproduct in Symm$[L^\#, P^\#]$ is defined to satisfy the similar identity

$$\xi_{\#\#}(W, VV') = \Sigma \pm \xi_{\#\#}(W_{(1)}, V)\xi_{\#\#}(W_{(2)}, V') \qquad [2]$$

with a similar sign convention. Identities **1** and **2** are similar to identities 2 and 3 of *Section 1* in the definition of a pairing of supersymmetric algebras. By an abuse of language, we shall also refer to the bilinear form $\xi_{\#\#}$ as a *pairing* of Symm$[L^\#, P^\#] \times$ Brace$\{L, P\}$.

Recall that the left adjoint map Ψ to the map Φ, relative to the pairing $\xi_{\#\#}$ of Symm$[L^\#, P^\#] \times$ Brace$\{L, P\}$ and the pairing ξ_* of Super$\{L^*|P^*\] \times$ Super$[L|P]$, is defined to satisfy

$$\xi_{\#\#}(\Psi(p), q) = \xi_*(p, \Phi(q)).$$

Thus we have that the map

$$\Psi\colon \text{Super}[L^*|P^*] \to \text{Symm}[L^\#, P^\#]$$

is injective.

We remark in passing that Symm$[L^\#, P^\#]$ can also be defined by the following universal construction. Consider the underlying algebra of Super$[L^\#] \otimes$ Super$[P^\#]$; in this algebra consider the subalgebra A generated by all pairs $w \otimes w'$, where Length(w) = Length(w'). Then Symm$[L^\#, P^\#]$ is the cofree supercocommutative bialgebra generated by A. The universal constructions leading to Brace$\{L, P\}$ and Symm$[L^\#, P^\#]$ are dual to each other.

Section 4. Definition of the Schur Map

We now consider the left adjoint ω of the linear map $\mathbf{\Omega}$, relative to the pairings ξ_* and $\xi_{\#\#}$. The operator ω is uniquely defined by the identity

$$\xi_*(p, \mathbf{\Omega}(w \otimes w')) = \xi_{\#\#}(\omega(p), w \otimes w')$$

holding for all p in Super$[L^*|P^*]$, w in Super$[L]$, and w' in Super$[P]$.

Since $\mathbf{\Omega}$ is a map of coalgebras, it follows that ω: Super$[L^*|P^*] \to$ Super$[L^\#] \otimes$ Super$[P^\#]$ is a map of algebras. It is therefore uniquely determined by the following rules:

1. If x^* is positive and α^* is positive, then, for $k > 1$, $\omega((x^*|\alpha^*)^{(k)}) = 0$ and $\omega((x^*|\alpha^*)^{(1)}) = -x^\# \otimes \alpha^\#$;
2. If x^* is positive and α^* is negative, then $\omega((x^*|\alpha^*)) = -x^\# \otimes \alpha^\#$;
3. If x^* is negative and α^* is positive, then $\omega((x^*|\alpha^*)) = +x^\# \otimes \alpha^\#$;
4. If x^* is negative and α^* is negative, then $\omega((x^*|\alpha^*)) = +x^\# \otimes \alpha^\#$; and
5. If p and q are elements of Super$[L^*|P^*]$, then $\omega(pq) = \omega(p)\omega(q)$; and
6. $\omega(1) = 1$.

The most important property of the map ω is specified by the following:

PROPOSITION 2. *Let Length*(w^*) = *Length*$(u^*) > 1$, *where* $w^* \in$ *Super*$[L^*]$ *and* $u^* \in$ *Super*$[P^*]$. *Then*

$$\omega((w^*|u^*)) = 0.$$

Proof. By proposition 6 of ref. 1 we have for all $(w'|u')$ in Super$[L|P]$ with Length(w') = Length$(u') > 1$ that

$$0 = \xi_*((w^*|u^*), (w'|u')) = \xi_*((w^*|u^*), \mathbf{\Omega}(w' \otimes u'))$$
$$= \xi_{\#\#}(\omega((w^*|u^*)), w' \otimes u'),$$

as desired. Using the map ω, we can now give an explicit computation of the map Ψ, as follows:

PROPOSITION 3. *The map* Ψ *is uniquely determined by specifying* $\Psi(1) = 1$ *and for biproducts of positive lengths that*

$$\Psi((\mathrm{w}|\mathrm{u})) = \sum_{\mathrm{r}} \pm\, \omega((\mathrm{w}_{(1)}|\mathrm{u}_{(1)}))\otimes\omega((\mathrm{w}_{(2)}|\mathrm{u}_{(2)}))\otimes\ldots\otimes\omega((\mathrm{w}_{(\mathrm{r})}|\mathrm{u}_{(\mathrm{r})})),$$

where the sum ranges over all nonnegative integers r, *and with the restriction that the Length*$(\mathrm{w}_{(1)})$ = *Length* $(\mathrm{u}_{(1)}) \geq$ *Length*$(\mathrm{w}_{(2)})$ = *Length*$(\mathrm{u}_{(2)}) \geq \ldots \geq$ *Length*$(\mathrm{w}_{(\mathrm{r})})$ = *Length*$(\mathrm{u}_{(\mathrm{r})}) > 0$.

Since ω maps to zero all biproducts of length greater than one, we have the following:

PROPOSITION 4. *The image of* $\Psi((\mathrm{w}|\mathrm{u}))$ *lies in Symm*$_\lambda[\mathrm{L}^\#, \mathrm{P}^\#]$, *where* $\lambda = (1, 1, 1, \ldots, 1)$.

Let D and E be Young diagrams of the same shape λ on L^* and P^*, respectively. Then $\Psi(\text{Tab}(D|E))$ is a linear combination of supersymmetric elements in Symm$[L^\#, P^\#]$, whose shapes are all $\leq \lambda^*$ in the dominance order of shapes.

We are now ready to present our main definition, namely, the definition of the Schur map.

Definition: The Schur map

$$\text{Schur: Brace}\{L^*, P^*\} \to \text{Symm}[L^\#, P^\#] \qquad [3]$$

is the map

$$\text{Schur} = \Psi\, \Phi.$$

Since both Φ and Ψ are maps of bialgebras, the Schur map is a map of bialgebras. Furthermore, since Φ is a surjective map onto the Hopf algebra Super$[L^*, P^*]$, and since Ψ is an injective map, the image of the Schur map is a sub-Hopf algebra of the bialgebra Symm$[L^\#, P^\#]$. In this way, we obtain two new presentations of the supersymmetric algebra Super$[L^*|P^*]$: the first, as a quotient Hopf algebra of the bialgebra Brace$\{L^*, P^*\}$, and the second, as a sub-Hopf algebra of the bialgebra Symm$[L^\#, P^\#]$. We see from definition 3 that the Schur map changes signs of all letters, from positive to negative and from negative to neutral.

Our objective is to show that the Schur map is selfadjoint. To this end, we make a new start, leading to the definition of the adjoint Schur map Schur*. Starting with the identity

$$\xi_*(\mathbf{\Omega}(w \otimes w'), p) = \xi_{\#\#}(w \otimes w', \omega^*(p)), \qquad p \in \text{Super}[L|P],$$

which defines the *right* adjoint ω^* of

$$\mathbf{\Omega}\colon \text{Super}[L^*] \otimes \text{Super}[P^*] \to \text{Super}[L^*|P^*].$$

Note that the maps ω and ω^* can be succinctly defined by the formulas

$$\omega((x^*|\alpha^*)) = (-1)^{|\alpha^*|}x^\# \otimes \alpha^\#$$

and

$$\omega^*((x|\alpha)) = (-1)^{|a^*|}(x^*)^\# \otimes (a^*)^\#.$$

Second, we replace the map Ψ by the map

$$\Psi^*\colon \text{Super}[L|P] \to \text{Symm}[(L^*)^\#, \quad (P^*)^\#]$$

defined to satisfy $\xi_*(\Phi(p), q) = \xi_{\#\#}(p, \Psi^*(q))$. This map is the unique algebra map satisfying $\Psi^*(1) = 1$ and

$$\Psi^*((w|u)) = \sum_r \pm\omega^*((w_{(1)}|u_{(1)})) \otimes \omega^*((w_{(2)}|u_{(2)})) \otimes \ldots \otimes \omega^*((w_{(r)}|u_{(r)}))$$

with $\mathrm{Length}(w_{(1)}) = \mathrm{Length}(u_{(1)}) \geq \mathrm{Length}(w_{(2)}) = \mathrm{Length}(u_{(2)}) \geq \ldots \geq \mathrm{Length}(w_{(r)}) = \mathrm{Length}(u_{(r)}) > 0$. For Φ: Brace$\{L^*, P^*\} \to$ Super$[L^*|P^*]$, we obtain a map

$$\mathrm{Schur}^* = \Psi^*\Phi$$

and we can state the adjointness relations

$$\xi_{\#\#}(\mathrm{Schur}(p), q) = \xi_*(\Phi(p), \Phi(q)) = \xi_{\#\#}(p, \mathrm{Schur}^*(q)).$$

Thus we see that the map Schur can be viewed as selfadjoint.

Section 5. Computation of the Schur Map

Since Schur maps the graded bialgebra Brace$\{L^*, P^*\}$ to the graded bialgebra Symm$[L^\#, P^\#]$, and since both these algebras are graded by shapes, the map Schur decomposes into a direct sum of maps of modules

$$\mathrm{Schur}(\mu, \lambda)\colon \mathrm{Brace}_\mu\{L^*, P^*\} \to \mathrm{Symm}_\lambda[L^\#, P^\#].$$

Our present objective is to obtain explicit formulas for Schur(μ, λ) in some special cases. We shall do this by analyzing the identity

$$\xi_{\#\#}(\mathrm{Schur}(p), q) = \xi_{\#\#}(\Psi\Phi(p), q) = \xi_*(\Phi(p), \Phi(q)),$$

which follows from the definitions of Schur, Φ, and Ψ. Let $\{w_1'|u_1'\}\{w_2'|u_2'\} \ldots \{w_s'|u_s'\}$ be an element of Brace$_\mu\{L^*, P^*\}$. Since the Schur map is a map of algebras, we have that

$$\mathrm{Schur}(\mu, \lambda)(\{w_1'|u_1'\}\{w_2'|u_2'\} \ldots \{w_s'|u_s'\})$$

equals the component in degree λ of the expression

$$(\omega \otimes \omega \otimes \ldots \otimes \omega)(\Delta^{\mu_1}(w_1'|u_1'))(\omega \otimes \omega \otimes \ldots \otimes \omega)$$
$$(\Delta^{\mu_2}(w_2'|\mu_2')) \ldots (\omega \otimes \omega \otimes \ldots \otimes \omega)\Delta^{\mu_s}(w_s'|u_s'),$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$. Thus, Schur$(\mu, \lambda) = 0$ unless $\lambda \leq \mu^*$ in the dominance order.

Example 1: Let $x^* \in (L^*)^+$ and $\alpha^* \in (P^*)^+$. The image under the Schur map of the element $\{x^{*(n)}|\alpha^{*(n)}\}$ of Brace$\{L^*, P^*\}$ is the element of Symm$[L^\#, P^\#]$ given by

$$\mathrm{Schur}(\{x^{*(n)}|\alpha^{*(n)}\}) = (-1)^n(x^\# \otimes \alpha^\#) \otimes \ldots \otimes (x^\# \otimes \alpha^\#).$$

Example 2: Let $x_1^*, x_2^*, \ldots, x_n^* \in (L^*)^-$, and let $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^* \in (P^*)^-$. The image under the Schur map of the element $\{x_1^*x_2^* \ldots x_n^*|\alpha_1^*\alpha_2^* \ldots \alpha_n^*\}$ of Brace$\{L^*, P^*\}$ is the element of Symm$[L^\#, P^\#]$ given by

$$\mathrm{Schur}(\{x_1^*x_2^* \ldots x_n^*|\alpha_1^*\alpha_2^* \ldots \alpha_n^*\})$$
$$= (-1)^{n(n-1)/2}\sum_{\sigma,\tau}(-1)^\sigma(-1)^\tau(x_{\sigma 1}^\# \otimes \alpha_{\tau 1}^\#) \otimes \ldots \otimes (x_{\sigma n}^\# \otimes \alpha_{\tau n}^\#).$$

Example 3: Choose a shape λ such that $\lambda = (p, q, \ldots, r)$ (a total of k nonzero components) and $\lambda^* = (k, m, \ldots, n)$ (a total of p nonzero components). When the letters x_i^* and α_i^* are all positive, we have

$$\mathrm{Schur}(\lambda, \lambda^*)(\{(x_1^*)^{(p)}|(\alpha_1^*)^{(p)}\}\{(x_2^*)^{(q)}|(\alpha_2^*)^{(q)}\} \ldots$$
$$\{(x_k^*)^{(r)}|(\alpha_k^*)^{(r)}\}) = (-1)^{k(k+1)/2}(-1)^{m(m+1)/2} \ldots$$
$$(-1)^{n(n+1)/2}(x_1^\# x_2^\# \ldots x_k^\# \otimes \alpha_1^\#\alpha_2^\# \ldots \alpha_k^\#) \otimes$$
$$(x_1^\# x_2^\# \ldots x_m^\# \otimes \alpha_1^\#\alpha_2^\# \ldots \alpha_m^\#) \otimes \ldots$$
$$\otimes (x_1^\# x_2^\# \ldots x_n^\# \otimes \alpha_1^\#\alpha_2^\# \ldots \alpha_n^\#).$$

Section 6. Polarizations

We introduce polarization operators $\mathbf{D}(\mathbf{b}^*, \mathbf{a}^*)$ and $\mathbf{D}(\boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$ on Brace$\{L^*, P^*\}$ and polarization operators $\mathbf{D}(\mathbf{b}^\#, \mathbf{a}^\#)$ and $\mathbf{D}(\boldsymbol{\beta}^\#, \boldsymbol{\alpha}^\#)$ on Symm$[L^\#, P^\#]$. They will commute with the Schur map, in the sense that

$$\mathrm{Schur}\ \mathbf{D}(\mathbf{b}^*, \mathbf{a}^*) = \pm\mathbf{D}(\mathbf{b}^\#, \mathbf{a}^\#)\mathrm{Schur}$$

and

$$\mathrm{Schur}\ \mathbf{D}(\boldsymbol{\beta}^*, \boldsymbol{\alpha}^*) = \pm\mathbf{D}(\boldsymbol{\beta}^\#, \boldsymbol{\alpha}^\#)\mathrm{Schur}.$$

The operators $\mathbf{D}(\mathbf{b}^*, \mathbf{a}^*)$ are defined to satisfy the following.
1. $\mathbf{D}(\mathbf{b}^*, \mathbf{a}^*)\{w|w'\} = \{D(b^*, a^*)\, w|w'\}$, where $D(b^*, a^*)$ is the polarization on Super$[L^*]$ introduced in refs. 2 and 3.
2. When the letters a^* and b^* are of the same sign, then the polarization $\mathbf{D}(\mathbf{b}^*, \mathbf{a}^*)$ is positive, as noted in ref. 3. We shall presently need the divided powers $D^{(k)}(b^*, a^*)$ and $\mathbf{D}^{(\mathbf{k})}(\mathbf{b}^*, \mathbf{a}^*)$ of $D(b^*, a^*)$ and $\mathbf{D}(\mathbf{b}^*, \mathbf{a}^*)$, which are defined to satisfy

$$\mathbf{D}^{(\mathbf{k})}(\mathbf{b}^*, \mathbf{a}^*)\{w|w'\} = \{D^{(k)}(b^*, a^*)w|w'\};$$
$$\mathbf{D}^{(\mathbf{k})}(\mathbf{b}^*, \mathbf{a}^*)(pq) = \sum_{i+j=k} (\mathbf{D}^{(\mathbf{i})}(\mathbf{b}^*, \mathbf{a}^*)p)\mathbf{D}^{(\mathbf{j})}(\mathbf{b}^*, \mathbf{a}^*)q$$

and similarly for $\mathbf{D}(\boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$.

The polarizations $D(b^\#, a^\#)$ and $\mathbf{D}(\mathbf{b}^\#, \mathbf{a}^\#)$ are positive when both letters are of the same sign, and negative otherwise. The polarization $D(b^\#, a^\#)$ is defined on Super$[L^\#]$; similarly, $D(\beta^\#, \alpha^\#)$ is defined on Super$[P^\#]$.

PROPOSITION 5. *The polarization operators thus defined satisfy the following commutation identities:*
1. *when* a* *and* b* *are of the same sign, then*

$$\mathit{Schur}\ \mathbf{D}(\mathbf{b}^*, \mathbf{a}^*) = \mathbf{D}(\mathbf{b}^\#, \mathbf{a}^\#)\mathit{Schur};$$

2. *when* a* *and* b* *are of different signs, then*

$$\mathit{Schur}\ \mathbf{D}(\mathbf{b}^*, \mathbf{a}^*) = -\mathbf{D}(\mathbf{b}^\#, \mathbf{a}^\#)\mathit{Schur},$$

and similarly for $\mathrm{D}(\beta^*, \alpha^*)$ *and* $\mathrm{D}(\beta^\#, \alpha^\#)$.

By judicious use of polarization operators, as in refs. 2 and 3, one can compute the Schur map in more general situations than the ones given above. For example, to compute

$$\mathrm{Schur}(\lambda, \lambda^*)W,$$

where

$$W = \{x_{11}^*x_{12}^* \ldots x_{1p}^*|\alpha_{11}^*\alpha_{12}^* \ldots \alpha_{1p}^*\}\{x_{21}^*x_{22}^* \ldots x_{2q}^*|\alpha_{21}^*\alpha_{22}^*$$
$$\ldots \alpha_{2q}^*\} \ldots \{x_{k1}^*x_{k2}^* \ldots x_{kr}^*|\alpha_{k1}^*\alpha_{k2}^* \ldots \alpha_{kr}^*\},$$

where all letters are negative, we find that

$$W = \mathbf{T}^*(\{(x_1^*)^{(p)}|(\alpha_1^*)^{(p)}\}\{(x_2^*)^{(q)}|(\alpha_2^*)^{(q)}\} \ldots \{(x_k^*)^{(r)}|(\alpha_k^*)^{(r)}\})$$

for a suitable operator $\mathbf{T}^*$ that is a product of polarization operators. Thus,

$$\mathrm{Schur}(\lambda, \lambda^*)W = \mathrm{Schur}(\lambda, \lambda^*)\mathbf{T}^*(\{(x_1^*)^{(p)}|(\alpha_1^*)^{(p)}\}$$
$$\{(x_2^*)^{(q)}|(\alpha_2^*)^{(q)}\} \ldots \{(x_k^*)^{(r)}|(\alpha_k^*)^{(r)}\}) =$$
$$\mathbf{T}^\#\ \mathrm{Schur}(\lambda, \lambda^*)(\{(x_1^*)^{(p)}|(\alpha_1^*)^{(p)}\}\{(x_2^*)^{(q)}|(\alpha_2^*)^{(q)}\} \ldots$$
$$\{(x_k^*)^{(r)}|(\alpha_k^*)^{(r)}\}),$$

where $\mathbf{T}^\#$ is the corresponding product of polarization operators on Symm$[L^\#, P^\#]$.

Section 7. Time-Ordering

We extend the definition of a time-ordered supersymmetric algebra, given in section 6 of ref. 3, to the more general case of an arbitrary proper signed alphabet, rather than positively signed letters only, as we did in ref. 3.

Let Q be a proper alphabet (that is, a linearly ordered signed set). Proceeding as in proposition 6.1 of ref. 3, we define an isomorphism of Tens(Super$[L^* \cup L^\#]$) with Super$[L^*|Q]$. Such an isomorphism is implemented by the *Feynman entangling operator*

$$\mathcal{F}: \text{Tens}(\text{Super}[L^* \cup L^\#]) \rightarrow \text{Super}[L^*|Q]$$

and the *Feynman disentangling operator*

$$\mathfrak{D}: \text{Super}[L^*|Q] \rightarrow \text{Tens}(\text{Super}[L^* \cup L^\#]),$$

which we proceed to define. Let p be a monomial in Tens (Super$[L^* \cup L^\#]$) of the form

$$p = w_1 x^\#_{11} x^\#_{12} \ldots x^\#_{1k} \otimes w_2 x^\#_{21} x^\#_{22} \ldots x^\#_{2m} \otimes \ldots .$$

Here, $w_1, w_2, \ldots$ are words in Super$[L^*]$, and $x^\#_{ij}$ are letters in $L^\#$, $x^\#_{11} \leq x^\#_{12} \leq \ldots \leq x^\#_{1k}$, and $x^\#_{21} \leq x^\#_{22} \leq \ldots \leq x^\#_{2m}$, etc. We entangle the monomial p as

$$\mathcal{F}(p) = (w_1|q_1^{(i_1)})(x^\#_{11}|n_1)(x^\#_{12}|n_1) \ldots$$
$$(x^\#_{1k}|n_1)(w_2|q_2^{(i_2)})(x^\#_{21}|n_2)(x^\#_{22}|n_2) \ldots (x^\#_{2m}|n_2) \ldots ,$$

where $q_1, q_2, \ldots$ are the first, second, ... letters of Q^+ and $n_1, n_2, \ldots$ are the first, second, ... letters of Q^-.

To define the Feynman disentangling operator, let p' be a monomial in Super$[L^*|Q]$. Such a monomial can be uniquely written in the form

$$p' = (w_1|q_1^{(i_1)})(x^\#_{11}|n_1)(x^\#_{12}|n_1) \ldots$$
$$(x^\#_{1k}|n_1)(w_2|q_2^{(i_2)})(x^\#_{21}|n_2)(x^\#_{22}|n_2) \ldots (x^\#_{2m}|n_2) \ldots ,$$

with the same assumptions as to ordering of the letters, and we set

$$\mathfrak{D}(p') = w_1 x^\#_{11} x^\#_{12} \ldots x^\#_{1k} \otimes w_2 x^\#_{21} x^\#_{22} \ldots x^\#_{2m} \otimes \ldots .$$

Clearly, both $\mathcal{F}\mathfrak{D}$ and $\mathfrak{D}\mathcal{F}$ are identity operators.

Denote by Super$^\lambda[L]$ the tensor product module spanned by all elements

$$\text{Tens}(D) = w_1 \otimes w_2 \otimes \ldots ,$$

where $D = (w_1, w_2, \ldots)$ is a Young diagram of shape λ whose entries are works in Mon(L).

Using the Feynman entangling and disentangling operators, we now define the *Buchsbaum maps* (see refs. 6 and 7),

$$B_\lambda: \text{Super}^\lambda[L^*] \rightarrow \text{Super}^{\lambda^*}[L^\#],$$

as follows. Let $D = (w_1, w_2, \ldots)$ be a Young diagram of shape λ in Super$[L^*]$. Let F be the diagram of shape λ in Super$[Q]$ defined as $F = (q_1^{(\lambda 1)}, q_2^{(\lambda 2)}, \ldots)$, and let G be the diagram of shape λ defined as

$$G = (n_1 n_2 \ldots n_{\lambda_1}, n_1 n_2 \ldots n_{\lambda_2}, \ldots).$$

The Buchsbaum map B_λ is defined as

$$B_\lambda(w_1 \otimes w_2 \otimes \ldots \otimes w_r) = B_\lambda\, \mathfrak{D}\mathcal{F}(w_1 \otimes w_2 \otimes \ldots \otimes w_r)$$
$$= B_\lambda\, \mathfrak{D}\, \text{Tab}(D|F) = \mathfrak{D}\mathbf{D}(n_1, q_1)\mathbf{D}(n_2, q_1) \ldots$$
$$\mathbf{D}(n_1, q_2)\mathbf{D}(n_2, q_2) \ldots \mathbf{D}(n_1, q_r)\mathbf{D}(n_2, q_r) \ldots$$
$$\mathbf{D}(n_{\lambda r}, q_r)\text{Tab}(D|F).$$

Thus,

$$B_\lambda(w_1 \otimes w_2 \otimes \ldots \otimes w_r) = \pm\mathfrak{D}\ \text{Tab}(D|G).$$

Thus, we see that the Buchsbaum map B_λ can be defined by the commutation identity

$$B_\lambda\mathfrak{D} = \mathfrak{D}\mathbf{D}(n_1, q_1)\mathbf{D}(n_2, q_1) \ldots \mathbf{D}(n_1, q_2)\mathbf{D}(n_2, q_2) \ldots$$
$$\mathbf{D}(n_1, q_r)\mathbf{D}(n_2, q_r) \ldots \mathbf{D}(n_{\lambda_r}, q_r) = \mathfrak{D}\mathbf{D}_\lambda,$$

where

$$\mathbf{D}_\lambda = \mathbf{D}(n_1, q_1)\mathbf{D}(n_2, q_1) \ldots \mathbf{D}(n_1, q_2)\mathbf{D}(n_2, q_2) \ldots$$
$$\mathbf{D}(n_1, q_r)\mathbf{D}(n_2, q_r) \ldots \mathbf{D}(n_{\lambda_r}, q_r).$$

Multiplying on the right by the operator $\mathcal{F}$, we obtain

$$B_\lambda = \mathfrak{D}\mathbf{D}_\lambda\mathcal{F}.$$

Using a more general sum of products of polarization operators $D_{\lambda\mu}$ (which we shall not describe here) one can define a more general Buchsbaum map

$$B_{\lambda\mu}: \text{Super}^\lambda[L^*] \rightarrow \text{Super}^\mu[L^\#]$$

by a similar commutation identity

$$B_{\lambda\mu} = \mathfrak{D}\mathbf{D}_{\lambda\mu}\mathcal{F}.$$

Let $D = (w_1, w_2, \ldots)$ be a diagram of shape λ in Super$[L^*]$ and let $E = (v_1, v_2, \ldots)$ be a diagram of shape λ in Super$[P^*]$. Set

$$\text{Tab}\{D|E\} = \pm\{w_1|v_1\}\{w_2|v_2\} \ldots . \qquad [4]$$

We are now ready to state our main result.

THEOREM 1. *Let λ be a shape and assume that the dual shape λ^* has* k *parts. Suppose that*

$$B_\lambda(\mathfrak{D}Tab(D|F)) \otimes B_\lambda(\mathfrak{D}Tab(E|F))$$
$$= \Sigma\, s_1 \otimes s_2 \otimes \ldots \otimes s_k \otimes t_1 \otimes t_2 \otimes \ldots \otimes t_k,$$

where $s_1 \otimes s_2 \otimes \ldots \otimes s_k \in Super^{\lambda^*}[L^\#]$ *and* $t_1 \otimes t_2 \otimes \ldots\ t_k \in Super^{\lambda^*}[P^\#]$. *Then,*

$$Schur(\lambda, \lambda^*)Tab\{D|E\}$$
$$= \Sigma \pm (s_1 \otimes t_1) \otimes (s_2 \otimes t_2) \otimes \ldots \otimes (s_k \otimes t_k).$$

The preceding result can be extended to Schur(λ, μ).

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