Proc. Natl. Acad. Sci. USA
Vol. 87, pp. 9582–9584, December 1990
Mathematics

Excision in algebraic K-theory and Karoubi's conjecture

(C*-algebras/pseudodifferential operators)

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Communicated by S. S. Chern, August 31, 1990

ABSTRACT **We prove that the property of excision in algebraic K-theory is for a $\mathbb{Q}$-algebra A equivalent to the H-unitality of the latter. Our excision theorem, in particular, implies Karoubi's conjecture on the equality of algebraic and topological K-theory groups of stable C*-algebras. It also allows us to identify the algebraic K-theory of the symbol map in the theory of pseudodifferential operators.**

Recall that a ring I is said to satisfy excision in algebraic K-theory if for every ring R containing I as a two-sided ideal the K-groups of I, of R, and of R/I are related to each other by the functorial long exact sequence

$$\cdots \to K_{q+1}(R/I) \to K_q(I) \to K_q(R) \to K_q(R/I) \to \ldots.$$

Recall from ref. 1 that a $\mathbb{Q}$-algebra A is said to be homologically unital (abbreviated to "H-unital") if the following complex

$$0 \leftarrow A \overset{b'}{\leftarrow} A \otimes_{\mathbb{Q}} A \overset{b'}{\leftarrow} A \otimes_{\mathbb{Q}} A \otimes_{\mathbb{Q}} A \leftarrow \ldots$$

$(b'(a_1 \otimes \cdots \otimes a_q) = \Sigma_{i=1}^{q-1}(-1)^{i-1}a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q)$ is acyclic. It has been proven that for a $\mathbb{Q}$-algebra A the property of excision in algebraic K-theory implies the analogous property of excision in cyclic homology (2) and that the latter is equivalent to the H-unitality of A (see refs. 1 and 2). Our main objective here is to prove that H-unitality, in turn, implies excision in algebraic K-theory.

THEOREM 1. *For a $\mathbb{Q}$-algebra* A, *the following conditions are equivalent*:

(i) A *satisfies excision in algebraic* K-*theory*; *and*
(ii) A *is* H-*unital*.

As mentioned above, in view of the results of refs. 1 and 2, it remains to show that condition *ii* implies condition *i*. First, we remark that in order to prove excision in the algebraic K-theory it suffices to demonstrate that the canonical embedding $GL(A) \hookrightarrow \widetilde{GL}(A) = GL(A) \ltimes A^{\infty}$ induces an isomorphism in group homology. As for the homology with finite coefficients, that is well known and trivial (see ref. 3, for example), so the problem reduces to the similar question in rational homology. For any subgroup $G \subset GL(A)$, we will denote by $\tilde{G}$ the corresponding affine group $G \ltimes A^{\infty}$. Let E(A) denote the elementary subgroup of GL(A).

LEMMA 1. *Suppose that* $A = A^2 = \{a \in A \mid a = a'_1a''_1 + \cdots + a'_na''_n\}$ *then* $E(A) = [E(A), E(A)] = [GL(A), GL(A)]$, $\tilde{E}(A) = [\tilde{E}(A), \tilde{E}(A)] = [\widetilde{GL}(A), \widetilde{GL}(A)]$ *and* $GL(A)/E(A) = \widetilde{GL}(A)/\tilde{E}(A)$.

The Hochschild–Serre spectral sequence shows that $H_*(GL(A), \mathbb{Q}) = H_*(\widetilde{GL}(A), \mathbb{Q}) \Leftrightarrow H_*(E(A), \mathbb{Q}) = H_*(\tilde{E}(A), \mathbb{Q})$ [to show that the action of GL(A) on $H_*(E(A))$ is trivial one has to use Vaserstein's lemma (4) and the fact that $A = A^2$]. Let us consider now the system of triangular subgroups $T_n^{\sigma}(A) \subset E(A)$ [σ—a partial ordering of $\{1, \ldots, n\}$] and the associated Volodin spaces (5) $V(A) = V(E(A), \{T_n^{\sigma}(A)\})$ and $\tilde{V}(A) = V(\tilde{E}(A), \{\tilde{T}_n^{\sigma}(A)\})$. There exist spectral sequences (see ref. 5)

$$E_{pq}^2 = H_p(E(A), H_q(V(A), \mathbb{Q})) \Rightarrow H_{p+q}(\cup_{n,\sigma} BT_n^{\sigma}, \mathbb{Q})$$

and

$$\tilde{E}_{pq}^2 = H_p(\tilde{E}(A), H_q(\tilde{V}(A), \mathbb{Q})) \Rightarrow H_{p+1}(\cup_{n,\sigma} B\tilde{T}_n^{\sigma}, \mathbb{Q})$$

and the canonical morphism between them $E_{**}^* \to \tilde{E}_{**}^*$.

LEMMA 2. (i) *For any ring* A *the embedding* $V(A) \hookrightarrow \tilde{V}(A)$ *is a homotopy equivalence*; (ii) *If* $A = A^2$ *the action of* E(A) *on* $H_*(V(A))$ *is trivial.*

The standard spectral sequence comparison argument shows in conjunction with *Lemma 2* that $H_*(E(A), \mathbb{Q}) = H_*(\tilde{E}(A), \mathbb{Q}) \Leftrightarrow H_*(\cup_{n,\sigma} BT_n^{\sigma}(A), \mathbb{Q}) = H_*(\cup_{n,\sigma} B\tilde{T}_n^{\sigma}(A), \mathbb{Q})$.

Now we are going to reduce the initial problem to a problem in Lie algebra homology. Denote by $gl_n(A)$ the Lie algebra of $n \times n$ matrices over A, by $sl_n(A)$ its subalgebra of matrices whose trace in A/[A, A] vanishes, and by $t_n^{\sigma}(A)$ the corresponding triangular subalgebras. For any Lie $\mathbb{Q}$-algebra $\mathfrak{g}$, we will denote by $P_*(\mathfrak{g})$ the standard Koszul resolution of $\mathbb{Q}$ considered to be a left module over the universal enveloping algebra $U = U(\mathfrak{g})$:

$$U \leftarrow U \otimes \mathfrak{g} \leftarrow U \otimes \Lambda^2\mathfrak{g} \leftarrow \ldots$$

and by $C_*(\mathfrak{g})$ the complex $\mathbb{Q} \otimes_U P_*(\mathfrak{g}) = (\mathbb{Q} \leftarrow \mathfrak{g} \leftarrow \Lambda^2\mathfrak{g} \leftarrow \ldots)$, which computes the homology groups $H_*(\mathfrak{g}, \mathbb{Q})$. All the tools that we used before in the context of linear groups have analogs for Lie algebras; e.g., Volodin's space V(A) has a counterpart in the subcomplex $v(A) \subset P_*(gl(A))$, $v(A) = (U \leftarrow U \otimes (\Sigma_{n,\sigma} t_n^{\sigma}(A)) \leftarrow U \otimes (\Sigma_{n,\sigma} \Lambda^2 t_n^{\sigma}(A)) \leftarrow \ldots)$. So, the same arguments as before prove the following.

LEMMA 3. *If* $A = A^2$, *the following conditions are equivalent*:

(i) $H_*(gl(A), \mathbb{Q}) = H_*(\widetilde{gl}(A), \mathbb{Q})$;
(ii) $H_*(sl(A), \mathbb{Q}) = H_*(\widetilde{sl}(A), \mathbb{Q})$; *and*
(iii) $H_*(\Sigma_{n,\sigma} C_*(t_n^{\sigma}(A))) = H_*(\Sigma_{n,\sigma} C_*(\tilde{t}_n^{\sigma}(A)))$.

Now we come to the crucial point. If $\mathfrak{g}$ is a nilpotent Lie algebra and G is the corresponding nilpotent group, then as is well-known $H_*(BG, \mathbb{Q}) = H_*(\mathfrak{g}, \mathbb{Q})$ (see ref. 6). More precisely, one can produce a functorial quasi-isomorphism $C_*(BG) \to C_*(\mathfrak{g})$. Thus we get the quasi-isomorphisms $C_*(BT_n^{\sigma}(A)) \to C_*(t_n^{\sigma}(A))$ for all partial orderings σ. By functoriality, those maps patch to produce a single quasi-isomorphism $C_*(\cup_{n,\sigma} BT_n^{\sigma}(A)) = \Sigma_{n,\sigma} C_*(BT_n^{\sigma}(A)) \to \Sigma_{n,\sigma} C_*(t_n^{\sigma})$. In the same way we deduce that $C_*(\cup_{n,\sigma} B\tilde{T}_n^{\sigma}(A)) \to \Sigma_{n,\sigma} C_*(\tilde{t}_{n,\sigma}(A))$ is a quasi-isomorphism.

COROLLARY 1. *If* A *is a $\mathbb{Q}$-algebra satisfying* $A = A^2$, *the following conditions are equivalent*:

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(i) $H_*(GL(A)) = H_*(\widetilde{GL}(A))$; *and*
(ii) $H_*(gl(A)) = H_*(\widetilde{gl}(A))$.

Let A_1 be the ring $\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}$. We have natural inclusions $gl(A) \hookrightarrow \widetilde{gl}(A) \hookrightarrow gl(A_1)$. Furthermore, $\widetilde{gl}(A)$ is a retract of $gl(A_1)$. Thus, if we show that $H_*(gl(A)) = H_*(gl(A_1))$ that, in turn, would imply that $H_*(gl(A)) = H_*(\widetilde{gl}(A))$ and the proof of *Theorem 1* would be complete. If A is H-unital, the same is true for A_1 (see ref. 2) and a theorem of Hanlon (7) shows that in this case $H_*(gl(A))$ and $H_*(gl(A_1))$ are graded commutative Hopf algebras freely generated by their respective modules of primitive elements and that the latter are equal to the cyclic homology groups $HC_*(A)$ and, respectively, $HC_*(A_1)$. It has been proven that the inclusion $A \hookrightarrow A_1$ induces an isomorphism in cyclic homology under the assumption that A is H-unital (2). This brings us to the end of the proof of *Theorem 1*.

Some additional work enables us to completely compute the groups $H_*(\Sigma_{n,\sigma}C_*(t_n^\sigma(A)) = H_*(\Sigma_{n,\sigma}C_*(\tilde{t}_n^\sigma(A)))$.

THEOREM 2. *For an* H-*unital* $\mathbb{Q}$-*algebra* A, *one has*

$$\tilde{H}_*\left(\sum_{n,\sigma} C_*(t_n^\sigma(A))\right) = \tilde{H}_*\left(\sum_{n,\sigma} C_*(\tilde{t}_n^\sigma(A))\right) = 0.$$

That computation can be used to eliminate the final part of the demonstration of *Theorem 1* given above. The proof of *Theorem 2* uses in an essential way the following important result.

THEOREM 3. *The tensor product of any two* H-*unital algebras is again* H-*unital.*

This last result holds also if one replaces the ground field $\mathbb{Q}$ by an arbitrary commutative ring of coefficients. After certain modifications the arguments that prove *Theorem 1* also yield the following stronger result.

THEOREM 4. *For any ring* A, *the following conditions are equivalent:*

(i) A *satisfies excision in rational algebraic* K-*theory*; *and*
(ii) $A \otimes_{\mathbb{Z}} \mathbb{Q}$ *is* H-*unital.*

This result is conjectured in ref. 2. The other class of rings for which we can prove excision in integral algebraic K-theory is described in the following theorem.

THEOREM 5. *Suppose that for any finite set of elements* $a_1, \ldots, a_n \in A$ *there exist* $b_1, \ldots, b_n, c, d \in A$ *such that* $a_i = cdb_i$ *and the right annihilators of the elements cd and d are equal* (*see* Property Φ of ref. 9). *Then* A *satisfies excision in algebraic* K-*theory.*

Theorem 5 is demonstrated by using a method, developed in ref. 5, that allows us to show directly that $\tilde{H}_*(\cup BT_n^\sigma(A), \mathbb{Z}) = \tilde{H}_*(\cup B\tilde{T}_n^\sigma(A), \mathbb{Z}) = 0$. In all cases when we are able to prove excision we prove, in fact, a more precise statement.

COROLLARY 2. *Let* A *be either an* H-*unital* $\mathbb{Q}$-*algebra or a ring having the factorization property of* Theorem 5. *For any ring with unit* R *containing* A *as a two-sided ideal we have the homotopy fibration*

$$BGL(A)^+ \rightarrow BGL(R)^+ \rightarrow B\overline{GL}(R/A)^+$$

($\overline{GL}(R/A) = Im(GL(R) \rightarrow GL(R/A))$). *In particular*, $K_i(A) = \pi_i(BGL(A)^+)$ *and the actions of* GL(R) *by conjugation on* $K_i(A)$ *and on* $H_*(GL(A))$ *are trivial.*

The assertion of *Corollary 2* follows immediately from the results presented above (see also ref. 8). Notice that a $\mathbb{Q}$-algebra possessing the factorization property of *Theorem 5* is automatically H-unital (see ref. 9).

In the sections below we use our main result (*Theorem 1* above) to settle in the positive two conjectures, both concerned with the algebraic K-theory of certain topological rings.

Karoubi's Conjecture. The following corollary of *Theorem 1* is obtained by combining it with the results of ref. 9.

COROLLARY 3. *Every locally multiplicatively convex Fréchet algebra with a uniformly bounded left or right approximate unit satisfies excision in algebraic* K-*theory.*

The class of algebras to which *Corollary 3* applies contains all Banach algebras with bounded left or right approximate units. In particular, every C*-algebra satisfies excision in algebraic K-theory.

Recall that a functor F from the category of C*-algebras to the category of abelian groups is said to be homotopy invariant if, for every *-homomorphism $\varphi: B_1 \rightarrow B_2 \otimes C[0, 1]$, the compositions of φ with the evaluation maps at 0 and at 1 induce the same map $F(B_1) \rightarrow F(B_2)$. Cuntz and Higson (see ref. 10) have extracted from earlier results of Kasparov two simple properties of a functor that secure its homotopy invariance (in ref. 10, these are called *stability* and *split exactness*). *Corollary 3* above implies that the functor

$$B \rightsquigarrow K_*(B \otimes \mathcal{K}) \qquad [1]$$

possesses both of those properties. Here $\mathcal{K}$ stands for the C*-algebra of compact operators on the standard separable infinite-dimensional Hilbert space and $\otimes$ denotes the spatial tensor product of C*-algebras. Thus we have the following two additional corollaries of *Theorem 1*.

COROLLARY 4. *The functor given by Eq.* **1** *is homotopy invariant.*

COROLLARY 5. *If one denotes by* $\mathcal{F} \subset C[0, 1]$ *the subalgebra of functions vanishing at* $t = 0$ *then one has*

$$K_*(B \otimes \mathcal{F} \otimes \mathcal{K}) = 0$$

for any C*-*algebra* B.

The identity and zero endomorphisms of $B \otimes \mathcal{F} \otimes \mathcal{K}$ are homotopic.

Corollary 5 combined with the repeated use of the long exact sequences in algebraic K-theory associated with certain simple extensions of C*-algebras then shows that $K_q(B \otimes \mathcal{K})$ canonically identifies with $K_{q-n}(B \otimes C_0(S^n) \otimes \mathcal{K})$, $q \geq n$, where $C_0(S^n)$ denotes the algebra of continuous functions on the *n*-dimensional sphere which vanish at the "northern pole." *Theorem 6* below, which settles a long-standing Karoubi's conjecture, is an immediate corollary of the above.

THEOREM 6 (Karoubi's conjecture; see ref. 11). *The canonical comparison map connecting the algebraic and topological* K-*groups*

$$K_*(B \otimes \mathcal{K}) \rightarrow K_*^{top}(B \otimes \mathcal{K}) \simeq K_*^{top}(B)$$

is an isomorphism for every C*-*algebra* B.

Additive versions of Karoubi's conjecture in cyclic homology over an arbitrary subring k of $\mathbb{C}$ and in continuous cyclic homology have been established previously (9, 12). A special case of Karoubi's conjecture for K_2 has also been settled (10, 13).

The Algebraic K-Theory of the Symbol Map. Let CL(X, E) denote the ring of pseudodifferential operators of classical type and of integral order, acting in the space of C^∞-sections of a vector bundle E on a closed manifold X. The ring CL(X, E) is connected with the ring of complete symbols CS(X, E) by means of the symbol map $\sigma_{X,E}$: CL(X, E) $\hookrightarrow$ CS(X, E). If one replaces CL(X, E) by its subring $CL^0(X, E)$ of L^2-bounded operators and CS(X, E) by its subring $CS^0(X, E)$ of symbols of order ≤ 0, one obtains the corresponding bounded symbol map $\sigma^0_{X,E}$. *Theorem 7* below describes the algebraic

K-theory groups of $\sigma_{X,E}$ and $\sigma^0_{X,E}$, which are defined so that they enter into the following long exact sequences

$$\begin{array}{ccc} & K_*(\sigma_{X,E}) & \\ {\scriptstyle -1}\nearrow & & \searrow \\ K_*(CS(X,\, E)) & \longleftarrow & K_*(CL(X,\, E)) \end{array}$$

and

$$\begin{array}{ccc} & K_*(\sigma^0_{X,E}) & \\ {\scriptstyle -1}\nearrow & & \searrow \\ K_*(CS^0)X,\, E)) & \longleftarrow & K_*(CL^0(X,\, E)). \end{array}$$

THEOREM 7. *One has canonical isomorphisms*

$$\begin{array}{ccc} K_*(\sigma^0_{X,E}) & \xrightarrow{\;\sim\;} & K_*(\sigma_{X,E}), \\ \nwarrow & & \nearrow \\ & K_*(\mathbb{L}) & \end{array} \qquad [2]$$

where $\mathbb{L}$ *denotes the ring of infinite complex matrices of rapid decay* ($(a_{ij})_{1 \le i,j < \infty}$ *belongs to* $\mathbb{L}$ *if* $\sup_{i,j}(|a_{ij}|(i+j)^N) < \infty$ *for all real* N).

In particular, the K*-groups* $K_*(\sigma^0_{X,E})$ *and* $K_*(\sigma_{X,E})$ *are equal and do not depend on* X *or* E.

We would like to emphasize that the vertical arrows in Eq. 2 involve a *noncanonical* identification between the ring $L^{-\infty}(X, E)$ of operators with smooth Schwartz kernels and the ring $\mathbb{L}$. It follows from the main results of this article that the resulting maps on K-groups do not depend on how one chooses that identification.

The assertion of *Theorem 7* has been conjectured previously (M.W., unpublished work). *Theorem 7* suitably interpreted means that the extent to which the algebraic K-theory of pseudodifferential operators differs from the algebraic K-theory of symbols is completely described by a certain universal abelian group of values of "higher index invariants." An additive analog of *Theorem 7* in cyclic homology over an arbitrary subring k of $\mathbb{C}$ can be found in ref. 9, and a similar result for the continuous cyclic homology was established earlier (14).

The work of M.W. is supported by National Science Foundation Grant DMS-8910928.

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