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Evidence Combination From an Evolutionary Game Theory Perspective

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Abstract

Dempster-Shafer evidence theory is a primary methodology for multi-source information fusion because it is good at dealing with uncertain information. This theory provides a Dempster's rule of combination to synthesize multiple evidences from various information sources. However, in some cases, counter-intuitive results may be obtained based on that combination rule. Numerous new or improved methods have been proposed to suppress these counter-intuitive results based on perspectives, such as minimizing the information loss or deviation. Inspired by evolutionary game theory, this paper considers a biological and evolutionary perspective to study the combination of evidences. An evolutionary combination rule (ECR) is proposed to help find the most biologically supported proposition in a multi-evidence system. Within the proposed ECR, we develop a Jaccard matrix game (JMG) to formalize the interaction between propositions in evidences, and utilize the replicator dynamics to mimick the evolution of propositions. Experimental results show that the proposed ECR can effectively suppress the counter-intuitive behaviors appeared in typical paradoxes of evidence theory, compared with many existing methods. Properties of the ECR, such as solution's stability and convergence, have been mathematically proved as well.

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Index Terms

Dempster-Shafer evidence theory; Belief function; Evolutionary game theory; Evolutionarily stable strategy; Replicator equation; Multi-source information fusion

I. Introduction

Multi-source information fusion aims to integrate multiple data and knowledge from various information sources into a consistent, comprehensive, and useful estimation for objects. Many issues in a variety of disciplines can be translated into canonical multi-source information fusion problems. Because of that, many fields extensively use multi-source information fusion [1], [2]. Yet, a crucial issue remains in multi-source information fusion regarding how to represent and dispose the imprecise, fuzzy, ambiguous, inconsistent and even incomplete information [3]–[5]. As a tool of reasoning in an uncertain environment, Dempter-Shafer evidence theory [6], [7] established a rounded system for uncertainty management [8]–[12]. In this theory, the data from each information source is represented as a mass function, also referred to as evidence, and Dempster's rule of combination is provided to combine multiple evidences for the fusion of multi-source information. However, the combination rule is controversial. Even though it has many desirable characteristics, such as commutativity, associativity, and fast and clear convergence towards a solution, in some cases, especially when the evidences to be combined are highly conflicting, Dempster's rule of combination may produce counter-intuitive results illustrated by Zadeh's paradox [13], evidence shifting [14], [15], dictatorial power of Dempster's rule [16], [17], etc.

Since Dempster's rule can produce counter-intuitive results, much debate and effort has gone into developing alternatives or new methods for evidence combination, such as conjunctive rule [18], disjunctive rule [19], cautious conjunctive rule and bold disjunctive rule [20], etc [15], [21]–[27]. In the controversy of Dempster's rule [28]–[33], some believe the rule is inadequate and advocate modification of the original rule or propose new rules entirely, whereas others defend Dempster's rule and advocate the modification or pre-process of original evidences [16]. Corporately, both sides are expected to draw a synthetic evidence that can best represent the system consisting of all original evidences, which is a typical information fusion process. During the process, a hidden criterion is to minimize the information loss or information deviation between the obtained synthetic evidence and the original evidences. This criterion is typically based on physical or engineering perspective, such as principle of conservation of information.

Inspired by evolutionary game theory, this paper uses a biological and evolutionary perspective to study information fusion. In order to obtain the most supported propositions, the process of evidence combination is compared to the evolution of species to find individuals with the highest fitness and survival rate in a population. During this process, evolutionary game theory [34], [35] provides a theoretical framework. Specifically, the supported propositions in evidences are treated as strategies adopted by individuals in a given population. First, by interacting with others, individuals receive payoff which determines their fitness within the population. Then, individuals with high fitness have a higher probability to reproduce, thereby increasing their rate within the population. Finally, through the evolution of population, individuals with the highest fitness survive, and the strategy adopted by them succeeds in becoming the most supported or acceptable proposition. The proposed method is called as evolutionary combination rule (ECR). Although the method is referred to as a rule, the ECR is not a traditional evidence combination rule, which would be expected to obtain a synthetic evidence. Rather, the ECR aims to find the most supported proposition in a multi-evidence system. And because ECR can directly find the proposition(s) with the highest fitness, the proposed ECR provides fast decision-making support for evidence-based multi-source information fusion without relying on a transformation function between evidence and probability distribution.

The rest of the paper is organized as follows. Section II provides a brief introduction to Dempster-Shafer evidence theory and evolutionary game theory. Section III presents the proposed ECR, which primarily contains five parts, including evidence weighted averaging, construction of Jaccard matrix game, evolutionary dynamics, evolution of averaging evidence, and two-dimensional measure output. After that, illustrative examples are given to show the effectiveness of the proposed method in Section IV. Finally, Section V concludes this paper.

II. Preliminaries

A. Dempster-Shafer evidence theory

Dempster-Shafer evidence theory [6], [7], also called Dempster-Shafer theory (DST) or evidence theory, is used to handle uncertain information. This theory requires weaker conditions than that of Bayesian theory of probability, so DST is often regarded as an extension of Bayesian theory – see discussion in [17]. Recently, different fields have expressed concerns over the applications of DST, such as parameter estimation [36], [37], classification and clustering [38]–[40], decision-making [41]–[44], etc [45]–[51]. And, as an extension of this theory, Dezert-Smarandache Theory (DSmT) of plausible and paradoxical reasoning [52]–[54] has also been given much attention [27], [55]. A few basic concepts on DST are introduced below.

Let Ω be a set of mutually exclusive and collectively exhaustive events, indicated by $\Omega = \{E_1, E_2, \cdots, E_i, \cdots, E_N\}$, where set Ω is called a frame of discernment (FOD). The power set of Ω is indicated by 2^Ω, namely $2^\Omega = \{\varnothing, \{E_1\}, \cdots, \{E_N\}, \{E_1, E_2\}, \cdots, \{E_1, E_2, \cdots, E_i\}, \cdots, \Omega\}$. The elements of 2^Ω or subset of Ω are called propositions. For example, if $A \in 2^\Omega$, then A is a proposition.

For a FOD $\Omega = \{E_1, E_2, \cdots, E_N\}$, a mass function is a mapping of m from 2^Ω to $[0, 1]$ which satisfies the following condition:

$$m(\varnothing)=0 \quad \text{and} \quad \sum_{A \in 2^\Omega} m(A)=1. \tag{1}$$

In DST, a mass function is also called a basic probability assignment (BPA), a belief function, or a piece of evidence. The assigned probability $m(A)$ measures the belief exactly assigned to A and represents how strongly the evidence supports A. If $m(A) > 0$, A is called a focal element.

In DST, two independent evidences m_1 and m_2 are combined with Dempster's rule of combination, denoted by $m = m_1 \oplus m_2$, and defined as follows:

$$m(A)=\begin{cases} \frac{1}{1-K}\sum_{B\cap C=A} m_1(B)m_2(C), & A \neq \varnothing; \\ 0, & A=\varnothing. \end{cases} \quad (2)$$

with

$$K=\sum_{B\cap C=\varnothing} m_1(B)m_2(C). \quad (3)$$

B. Evolutionary game theory

Developed by John Maynard Smith, evolutionary game theory (EGT) [34], [35] studies the interaction among different players or populations. In recent years, EGT has become a paradigmatic framework to understand the emergence and evolution of cooperation among unrelated individuals [56]–[63], and inspired wide applications in myriad disciplines [64]–[71]. Nowak and May [72] first investigated a spatial prisoner's dilemma to demonstrate that spatial structure can promote cooperation. Axelrod's computer tournament [73] provided a platform to observe how cooperation evolves in a competitive environment. Santos and Pacheco [74] found that the cooperation can be greatly promoted in a scale-free network. Recent advances about the EGT can be found in [75], [76]. The main idea of the EGT is to track the change of each strategy's frequency in the population during the evolutionary process. Two key concepts of EGT are evolutionary stable strategy (ESS) [34] and replicator equation (RE) [77], [78].

1) Evolutionarily stable strategy (ESS)—In a given environment, ESS is a strategy adopted by a population that can not be invaded by any other alternative strategy which is initially rare. The conditions required by an ESS are formulated as [34], [35]:

$$E(S,S)>E(T,S) \quad (4)$$

or

$$E(S,S)=E(T,S) \quad \text{and} \quad E(S,T)>E(T,T) \quad (5)$$

for all $T \neq S$, where strategy S is an ESS, T is an alternative strategy, and $E(T,S)$ denotes the payoff of strategy T playing against strategy S.

Biologically, an evolutionarily stable strategy can not be successfully invaded by any mutant strategies. In game theory, there are two kinds of strategies: pure strategies and mixed strategies. A pure strategy defines an absolutely certain action or move that a player will play in every possible attainable situation within a game. In contrast, a mixed strategy is an assignment of a probability to pure strategies, allowing a player to randomly select a pure strategy. In other words, in a mixed strategy, there are two or more pure strategies that can be selected by chance. If the evolutionarily stable strategy S is a pure strategy, then S is called a pure ESS. Once S is a mixed strategy, S is called mixed ESS.

2) Replicator equation (RE)—In EGT, the replicator equation (RE) [77], [78] plays a key role in determining the evolutionary process of a population, by providing a frequency-dependent evolutionary dynamics.

Assume there exists n strategies in a well-mixed population. A game payoff matrix $A = [a_{ij}]$ determines the payoff of a player with strategy i if he meets another player who carries out strategy j. The relative frequency of strategy i, denoted as x_i, changes with time by the following differential equation:

$$\frac{dx_i}{dt}=x_i(f_i-\phi), \quad i=1,\cdots,n, \tag{6}$$

where f_i is the fitness of strategy i which is defined by $f_i=\sum_{j=1}^{n}x_j a_{ij}$, and ϕ is the average fitness of all strategies which is defined by $\phi=\sum_{i=1}^{n}x_i f_i$.

Eq.(6) is the so called replicator equation, which implies that the change of x_i depends on the fitness of strategy i and x_i itself. By solving $\frac{dx_i}{dt}=0$, $i = 1, \cdots, n$, the fixed points of this evolutionary system, denoted as ($x_1^*, \cdots, x_n^*$), can be found. Regarding the stability of each fixed point ($x_1^*, \cdots, x_n^*$), a theorem is usually used to verify whether or not the fixed point is stable, which is given below.

<u>**Theorem 1:**</u> [78] Given a set of replicator equations $\frac{dx_i}{dt}=x_i(f_i-\phi)$, $i= 1, \cdots, n$, the fixed point $p^*=(x_1^*, \cdots, x_n^*)$ is asymptotically stable if all eigenvalues associated with p^* are negative numbers or have negative real parts.

For more details on the theorem, please refer to [78]. As shown in Theorem 1, the stability of a solution is based on the resulting sign of the real part of the eigenvalues. However, when the eigenvalues are pure imaginary numbers, this theorem is not capable for the stability analysis. In that case, the vector field method [79] can be used for the stability analysis of replicator equations.

III. Evolutionary combination rule (ECR)

Traditional approaches of evidence combination aim to obtain a new evidence that best synthesizes all information. Then, decisions can be made based on the obtained synthetic evidence. In this paper, instead of seeking the best synthetic evidence, our purpose is to find the best supported proposition (i.e., a nonempty subset of the FOD) - analogous to finding the most probable element given a probability distribution. In order to achieve that purpose, an intuitive approach is to directly compare the mass of belief of each proposition among all evidences. However, the drawback of the intuitive approach is obvious because it ignores the interactions between propositions. Natural selection, which provides an excellent mechanism to find individuals with higher fitness, has inspired us to consider this problem from a biological and evolutionary standpoint. It is interesting to transplant the mechanism of natural selection to information fusion.

In this paper, we propose a new method, called evolutionary combination rule (ECR), for evidence combination based on evolutionary game theory. In the ECR, propositions in the FOD are naturally seen as strategies that can be adopted by a population in a given environment. Through the evolutionary process of population, the propositions with the higher fitness are found out to facilitate the subsequent analysis and decision. Fig. 1 gives the framework of proposed ECR. There are five parts, including evidence weighted averaging, construction of the Jaccard matrix game, evolutionary dynamics, evolution of averaging evidence, and two-dimensional measure output, all of which will be detailed further in this section. Most notably, in the proposed ECR, there are two basic problems: (i) what are the interactive relationships between propositions (strategies) ? (ii) how do propositions evolve in a population? As shown in the following content, the two questions are answered detailedly in part B and C, respectively.

A. Evidence weighted averaging

Given multiple evidences from different information sources, in accord with the traditional assumption, these evidences are mutually independent. For these independent evidences, by considering the difference of importance among information sources, the weighted averaging approach is used to integrate multiple evidences.

Given a FOD Ω, assume there are n evidences or BPAs indicated by $m_1, \cdots, m_n$, with each evidence having a weighting factor indicated by w_i, $\sum_{i=1}^{n} w_i = 1$. The averaging evidence is denoted as m, which is obtained by

$$m(A)=\sum_{i=1}^{n} w_i m_i(A), \qquad A \subseteq \Omega. \tag{7}$$

B. Jaccard matrix game (JMG)

In the ECR, propositions (i.e., nonempty subsets of the FOD) are treated as strategies. A key problem is to define the interaction rule between these propositions. Given a FOD Ω, suppose proposition A meets proposition B, how many payoffs will be obtained for A, and then for B as well? In biology, there is a so-called greenbeard effect that shows cooperative behaviors are more likely to appear between individuals with similar phenotypes [80]. Inspired by that, an individual with a proposition, say A, should obtain more payoffs if he interacts with another individual with a proposition similar to A. Specially, in an interaction, each individual obtains 1 if the two propositions are identical and 0 if the propositions are totally different. Here, since propositions are represented in the forms of sets, the measure of similarity degree between sets is considered to solve this problem. Mathematically, the Jaccard similarity coefficient measures the similarity degree between two sets [81]. Therefore, a game model, named as Jaccard matrix game (JMG), is proposed to formalize the interaction relationship between propositions.

Definition 1—Given a FOD Ω, a Jaccard matrix game (JMG) on Ω is defined as

$$\Gamma=(\Omega, J_{\Omega}), \quad (8)$$

where the set of strategies is composed by propositions of FOD Ω. The payoff matrix is $J_{\Omega} = [J_{\Omega}(A,B)]_{A,B\subseteq\Omega}$, in which $J_{\Omega}(A,B)$ represents the payoff of proposition A playing against proposition B, and it is defined by the Jaccard similarity coefficient between sets A and B, i.e., $J_{\Omega}(A,B)=\frac{|A\cap B|}{|A\cup B|}$.

Example 1—Given a FOD with two elements $\Omega = \{a, b\}$, a JMG on Ω, denoted as (Ω, J_{Ω}), can be constructed. In this game, the set of all potential strategies is $\{a, b, ab\}$ (for the sake of presentation, set $\{x, y\}$ is abbreviated as xy), and the payoff matrix is easily calculated

$$J_{\Omega}=\begin{bmatrix} & a & b & ab \\ a & 1 & 0 & 1/2 \\ b & 0 & 1 & 1/2 \\ ab & 1/2 & 1/2 & 1 \end{bmatrix}. \quad (9)$$

Each proposition $A \subseteq \Omega$ in FOD is equivalent to a pure strategy in the JMG, all propositions consist of the set of pure strategies adopted by a population. As shown in the above example, a JMG can be constructed if a FOD is given. Two important corollaries about the JMG are shown below.

Corollary 1—A JMG (Ω, J_{Ω}) is a partnership game.

Proof: Assume a two-person game is in normal form $\Gamma(S_1, S_2, E_1, E_2)$ where S_1, S_2 are players' sets of strategies, and their payoff functions are $E_i : S_1 \times S_2 \to R$, $i = 1, 2$. The game

$\Gamma(S_1, S_2, E_1, E_2)$ is called a partnership game if $S_1 = S_2$, and $E_1(s_1, s_2) = E_2(s_1, s_2)$ for all $s_1 \in S_1$, $s_2 \in S_2$. Due to the symmetry of strategy spaces and payoff matrices, the partnership game can also be written as $\Gamma(S,E)$ where E is the shared payoff matrix by two players and $E = E^T$. According to Definition 1, in a JMG (Ω, J_Ω), the strategy sets of all players are the same, i.e., $\{A|A \subseteq \Omega, A \neq \varnothing\}$. And $J_\Omega(A,B)=\frac{|A\cap B|}{|A\cup B|}$, $J_\Omega(B,A)=\frac{|B\cap A|}{|B\cup A|}$, then we have $J_\Omega(A,B) = J_\Omega(B,A)$. So the JMG is a partnership game.

Corollary 2—Let Δ_Ω^{ESS} be the set of all ESSs (evolutionarily stable strategies) in the JMG (Ω, J_Ω), then

$$\Delta_\Omega^{ESS}=\{A|A\subseteq\Omega, A\neq\varnothing\}.$$

Proof: First, let us prove that each pure strategy in the JMG is definitely an ESS.

Assume all propositions in Ω are indexed by $1, 2, \cdots, 2^{|\Omega|}-1$. A strategy $A = (p_1, \cdots, p_k, \cdots, p_{2^{|\Omega|}-1})$ is a pure strategy associated with proposition k if $p_k = 1$ and $\forall l \neq k, p_l = 0$. The payoff of pure strategy A playing against itself is

$$E(A,A)=\sum_{i}^{2^{|\Omega|}-1}\sum_{j}^{2^{|\Omega|}-1} p_i J_\Omega(i,j)p_j=p_k J_\Omega(k,k)p_k=1.$$

For any other strategy $B = (q_1, q_2, \cdots, q_{2^{|\Omega|}-1})$ where $q_i \geq 0$ and $\sum_{i=1}^{2^{|\Omega|}-1} q_i=1$, and $B \neq A$ (i.e., $\exists j$ such that $q_j \neq p_j$), the payoff of B playing against A is

$$\begin{aligned}E(B,A)&=\sum_{i}^{2^{|\Omega|}-1}\sum_{j}^{2^{|\Omega|}-1} q_i J_\Omega(i,j)p_j=\sum_{i}^{2^{|\Omega|}-1} q_i J_\Omega(i,k)p_k\\&=q_k J_\Omega(k,k)p_k+\sum_{i,i\neq k}^{2^{|\Omega|}-1} q_i J_\Omega(i,k)p_k\\&<q_k+\sum_{i,i\neq k}^{2^{|\Omega|}-1} q_i=1.\end{aligned}$$

So, $E(A,A) > E(B,A)$. According to Eq.(4), A is an ESS.

Second, let us prove that all ESSs are pure strategies, which just needs to prove that any mixed strategy in the JMG is not an ESS. Assume there are two strategies, $P = (p_1, p_2, \cdots, p_{2^{|\Omega|}-1})$ and $Q = (q_1, q_2, \cdots, q_{2^{|\Omega|}-1})$, where P is a mixed strategy and Q can be either a mixed or pure strategy, and $P \neq Q$. Then,

$$E(P,P)=\sum_{i}^{2^{|\Omega|}-1}\sum_{j}^{2^{|\Omega|}-1}p_i J_\Omega(i,j)p_j$$
$$=\sum_{i}^{2^{|\Omega|}-1}\left[p_i\sum_{j}^{2^{|\Omega|}-1}(J_\Omega(i,j)p_j)\right].$$

Let $T_i=\sum_{j}^{2^{|\Omega|}-1}(J_\Omega(i,j)p_j)$, we have $E(P,P)=\sum_{i}^{2^{|\Omega|}-1}p_iT_i$. Since P can only be a mixed strategy, i.e., $\sum_{i}^{2^{|\Omega|}-1}p_i=1$ and $\forall i, p_i \in [0, 1)$, it is found that: (i) $E(P, P) < \max[T_i]$, if $\exists i, j$, such that $T_i \neq T_j$; (ii) $E(P, P) = T_i$, if $\forall i, j$, such that $T_i = T_j$.

Similarly, it is found that $E(Q,P)=\sum_{i}^{2^{|\Omega|}-1}q_iT_i$. Because Q can be a pure or mixed strategy, let $q_k = 1$, where $T_k = \max[T_i]$, and we can obtain that $E(Q, P) = \max[T_i]$.

So, a Q always exists such that $E(Q, P) \geq E(P, P)$. The first alternative condition of mixed strategy P as an ESS, shown in Eq.(4), does not hold.

Let us check the second alternative condition shown in Eq.(5). Assume $U_i=\sum_{j}^{2^{|\Omega|}-1}(J_\Omega(i,j)q_j)$, we can obtain that $E(P,Q)=\sum_{i}^{2^{|\Omega|}-1}p_iU_i$ and $E(Q,Q)=\sum_{i}^{2^{|\Omega|}-1}q_iU_i$. Similarly, $E(P,Q) \leq \max[U_i]$ where equality holds if $\forall i, j, U_i = U_j$, and there always exists a Q such that $E(Q,Q) = \max[U_i]$. Thus, it is found that $\exists Q$ where $Q \neq P$, such that $E(Q,Q) \geq E(P,Q)$, for each P. Hence, for mixed strategy P, the condition shown in Eq.(5) also does not hold. That means mixed strategies can not be ESSs in the JMG, which finishes the proof.

Corollary 2 shows a one-to-one correspondence between pure strategies and ESSs in the JMG. Since propositions and pure strategies are equivalent in the JMG, we also refer to the evolutionary stable strategy (ESS) as the evolutionary stable proposition (ESP). Therefore, assuming the set of ESPs in the JMG is indicated by Δ_Ω^{ESP}, we have $\Delta_\Omega^{ESP} \iff \Delta_\Omega^{ESS}$.

C. Replicator dynamics on the JMG

Once the interaction relationship between propositions has been determined, the next step is examining how the propositions evolve. In other words, what types of evolutionary dynamics is adopted by the population? To answer this problem, the replicator equation is considered to mimick the evolutionary process of the population because of two reasons. First, it accurately simulates the evolution of the population in a well-mixed environment where individuals can randomly interact with other members of the population. Second, the replicator equation is mathematically equivalent to the Lotka-Volterra equation of ecology [78] which describes the dynamics of species in an interactive biological system. The Lotka-Volterra equation closely models reality because it does not rely upon the rationality assumption which is fundamental but challenged in classical game theory. Therefore, the equivalent replicator equation eliminates the controversy of the rationality assumption. The replicator dynamics on a JMG is defined as follows.

Definition 2—Given evidence m on a FOD Ω, the belief or basic probability for proposition A, denoted as $m(A)$, evolves in time according to the equation

$$\frac{dm(A)}{dt}=m(A)\left(f_A-\phi\right), \quad A\subseteq\Omega, A\neq\varnothing, \tag{10}$$

where f_A is the fitness of proposition A that

$$f_A=\sum_{B\subseteq\Omega,B\neq\varnothing}m(B)J_\Omega(A,B), \tag{11}$$

and ϕ is the average fitness of population

$$\phi=\sum_A\sum_B m(A)J_\Omega(A,B)m(B), \quad A,B\subseteq\Omega, and\ A,B\neq\varnothing. \tag{12}$$

For simplification, the differential dynamic system shown in Eqs.(10)–(12) can be rewritten as

$$\dot{m}_i=m_i\left(\left(J_\Omega\mathbf{m}\right)_i-\mathbf{m}^T\cdot J_\Omega\mathbf{m}\right), \tag{13}$$

where $m_i = \mathbf{m}(i)$ and $i \subseteq \Omega$, $i \neq \varnothing$. In regards to the convergence and stability of the system Eq.(13), we have two corollaries shown as follows.

Corollary 3: Convergence—Every solution trajectory of replicator dynamics system Eq.(13) converges to a fixed point.

Proof: We define a function V by the average payoff of population

$$V(\mathbf{m})=\mathbf{m}^T\cdot J_\Omega\mathbf{m}.$$

Clearly, $\dot{V}(\mathbf{m}) = \dot{\mathbf{m}}^T\cdot J_\Omega\mathbf{m} + \mathbf{m}^T\cdot J_\Omega\dot{\mathbf{m}}$. Because J_Ω is symmetric, $\dot{\mathbf{m}}_T\cdot J_\Omega\mathbf{m} = \mathbf{m}^T\cdot J_\Omega\dot{\mathbf{m}}$. Thus, we have

$$\dot{V}(\mathbf{m}) = 2\sum_i \dot{m}_i (J_\Omega \mathbf{m})_i$$
$$= 2\sum_i m_i \left[(J_\Omega \mathbf{m})_i - \mathbf{m}^T \cdot J_\Omega \mathbf{m}\right](J_\Omega \mathbf{m})_i$$

and therefore, since $\sum_i m_i = 1$.

$$\dot{V}(\mathbf{m}) = 2\sum_i m_i \left[(J_\Omega \mathbf{m})_i - \mathbf{m}^T \cdot J_\Omega \mathbf{m}\right]^2 \geq 0.$$

Equality holds if and only if all terms $(J_\Omega \mathbf{m})_i$ for which $m_i > 0$ take the same value, i.e.,

$$(J_\Omega \mathbf{m})_i = (J_\Omega \mathbf{m})_j = \mathbf{m}^T \cdot J_\Omega \mathbf{m}, \forall i, j, m_i > 0, m_j > 0.$$

Clearly, the above condition characterizes the fixed points of replicator dynamics system Eq.(13) where all pure strategies that are present in the population get precisely the same payoff. It shows that the average payoff of population $V(\mathbf{m})$ increases over time first and then converges until the system reaches a fixed point, which implies that every solution trajectory converges to a fixed point.

Corollary 4: Stability—Let Δ_Ω^{ASP} be the set of asymptotically stable points (ASPs) of replicator dynamics system Eq.(13), and Δ_Ω^{ESP} be the set of evolutionarily stable propositions (ESPs) of the corresponding JMG (Ω, J_Ω), then $\mathbf{m}^* \in \Delta_\Omega^{ESP} \iff \mathbf{m}^* \in \Delta_\Omega^{ASP}$.

Proof: Let us first prove "⇒". Suppose $\mathbf{m}^* \in \Delta_\Omega^{ESP}$, $\mathbf{m}$ is a solution of Eq.(13) in a neighborhood U of $\mathbf{m}^*$, and 0 log0 = 0 log∞ = 0. We define a Lyapunov function by the relative entropy:

$$H_{\mathbf{m}^*}(\mathbf{m}) = -\sum_i m_i^* \log\left(\frac{m_i}{m_i^*}\right).$$

Because logarithm is concave function and in terms of Jensen's inequality,

$$H_{\mathbf{m}^*}(\mathbf{m}) = -\sum_{m_i^*>0} m_i^* \log\left(\frac{m_i}{m_i^*}\right)$$
$$\geq -\log\left(\sum_{m_i^*>0} \frac{m_i^* m_i}{m_i^*}\right)$$
$$\geq -\log\left(\sum_i m_i\right) = -\log(1) = 0.$$

Then, we calculate the derivative of H with respect to time:

$$\dot{H}_{\mathbf{m}^*}(\mathbf{m}) = -\sum_{m_i^*>0} m_i^* \frac{\dot{m}_i}{m_i}$$
$$= -\sum_{m_i^*>0} m_i^* \frac{1}{m_i} m_i \left[(J_\Omega \mathbf{m})_i - \mathbf{m}^T \cdot J_\Omega \mathbf{m}\right]$$
$$= -\left[(\mathbf{m}^*)^T \cdot J_\Omega \mathbf{m} - \mathbf{m}^T \cdot J_\Omega \mathbf{m}\right].$$

According to [78] (Theorem 6.4.1), given a payoff matrix E, a strategy $\mathbf{p}^*$ is an ESS if and only if $(\mathbf{p}^*)^T \cdot E\mathbf{q} > \mathbf{q}^T \cdot E\mathbf{q}$ for any $\mathbf{q}$ in some neighborhood of $\mathbf{p}^*$ and $\mathbf{q} \neq \mathbf{p}^*$. Here, since $\mathbf{m}^* \in \Delta_\Omega^{ESP}$ and $\Delta_\Omega^{ESP} \iff \Delta_\Omega^{ESS}$ (Corollary 2), we have $(\mathbf{m}^*)^T \cdot J_\Omega \mathbf{m} > \mathbf{m}^T \cdot J_\Omega \mathbf{m}$. Thus, $\dot{H}_{\mathbf{m}^*}(\mathbf{m}) < 0$. Based on Lyapunov's stability theory, $\mathbf{m}^*$ is asymptotically stable, i.e., $\mathbf{m}^* \in \Delta_\Omega^{ASP}$.

Second, let us prove "⇐". Assume $\mathbf{m}^* \in \Delta_\Omega^{ASP}$. According to Corollary 3, for the replicator dynamics system Eq.(13), the average payoff of population increases over time unless it reaches a fixed point. Because $\mathbf{m}^*$ is a asymptotically stable fixed point, then

$$(\mathbf{m}^*)^T \cdot J_\Omega \mathbf{m}^* > \mathbf{m}^T \cdot J_\Omega \mathbf{m}$$

for all $\mathbf{m} \neq \mathbf{m}^*$ in some neighbourhood of $\mathbf{m}^*$. We replace $\mathbf{m}$ by $2\mathbf{m}-\mathbf{m}^*$(which is also near $\mathbf{m}^*$), and can get

$$(\mathbf{m}^*)^T \cdot J_\Omega \mathbf{m}^* > (2\mathbf{m}-\mathbf{m}^*)^T \cdot J_\Omega (2\mathbf{m}-\mathbf{m}^*).$$

By expanding the right hand side and transposing, it is found

$$(\mathbf{m}^*)^T \cdot J_\Omega \mathbf{m} > \mathbf{m}^T \cdot J_\Omega \mathbf{m},$$

which is the condition for an ESS (Theorem 6.4.1 in [78]). Hence, we have proved $\mathbf{m}^* \in \Delta_\Omega^{ESP}$.

Corollary 3 ensures that a unique and convergent solution will be absolutely obtained by using the proposed ECR. Corollary 4 shows a one-to-one correspondence between ASPs of the replicator dynamics and ESPs of the JMG. According to Corollary 4, we can directly judge the stability of solution obtained by using the ECR, without extra calculating the eigenvalues of solution shown in Theorem 1.

Example 2—Given a FOD $\Omega = \{a, b\}$, there is a JMG $\Gamma = (\Omega, J_\Omega)$ where its payoff matrix J_Ω is shown in Eq.(9). In this JMG, there are three propositions as pure strategies: a, b, and ab. Assume $m(a) = x$, $m(b) = y$, and $m(ab) = z$, where $x + y + z = 1$. According to the replicator dynamics, we can write (replacing, as usual, the time derivatives of x, y, z with $\dot{x}$, $\dot{y}$, $\dot{z}$),

$$\begin{cases} \dot{x} = x(f_x - \phi) \\ \dot{y} = y(f_y - \phi) \\ \dot{z} = z(f_z - \phi) \end{cases} \quad (14)$$

where

$$\begin{cases} f_x = x + z/2 \\ f_y = y + z/2 \\ f_z = x/2 + y/2 + z \end{cases} \quad (15)$$

and

$$\begin{aligned} \phi &= (x, y, z) J_\Omega (x, y, z)^T \\ &= x(x + z/2) + y(y + z/2) + z(x/2 + y/2 + z). \end{aligned} \quad (16)$$

The fixed points of replicator dynamic system Eqs. (14)–(16) can be obtained by $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$. They are $P_1^* = (1, 0, 0)$, $P_2^* = (0, 1, 0)$, $P_3^* = (0, 0, 1)$, $P_4^* = (0, 0.5, 0.5)$, $P_5^* = (0.5, 0.5, 0)$, and $P_6^* = (0.5, 0, 0.5)$. According to Corollaries 2 and 4, fixed points P_1^*, P_2^* and P_3^* are asymptotically stable, and P_4^*, P_5^*, P_6^* are unstable.

The stability of each fixed point can also be checked through Theorem 1. First, we need construct a Jacobian matrix:

$$JM = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix}. \quad (17)$$

For example, for fixed point P_1^*,

$$A{=}JM|_{P_1^*}=\begin{pmatrix} -1 & 0 & -0.5 \\ 0 & -1 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}. \quad (18)$$

Then, the stability of P_1^* is determined by the eigenvalues of the following characteristic equation

$$\det(A-\lambda I)=0, \quad (19)$$

where det is the determinant of a matrix, and I is the identity matrix. So, the eigenvalue λ can be calculated as $\lambda_1 = -1$, $\lambda_2 = -1$, and $\lambda_3 = -0.5$. According to Theorem 1, P_1^* is an asymptotically stable fixed point since all eigenvalues are negative numbers. Through this approach, the stability of other fixed points can be found, as shown in TABLE I. They are identical with that obtained by using Corollaries 2 and 4.

Graphically, a two-dimensional space known as simplex can clearly represent the evolutionary dynamics of propositions *a*, *b* and *ab* in this example, as shown in Fig. 2. In the simplex, every vertex of means that there only exists a sole proposition (i.e., strategy) in the population, edges represent that at least a proposition is missing in the population. The interior of the simplex corresponds to the condition of all propositions in coexistence. At each point of the simplex, the sum of the belief of all propositions is 1. In addition, the arrows in Fig. 2 represent the directions of evolution, where black circles indicate ASPs, and white circles are unstable fixed points. Fig. 2 clearly shows that there are three ASPs that are precisely ESPs of Γ, and that there are three unstable fixed points located on the midpoints of each edge.

D. Evolution of averaging evidence

As previously stated, given an evidence, which is usually the average of multiple evidences in multi-source information fusion, the replicator dynamics determines into which proposition this evidence will evolve. The following example shows the evolutionary process.

Example 3—Given an evidence m on a FOD Ω = {*a, b, c*}:

$$m(a){=}0.25,\ m(b){=}0.25,\ m(c){=}0.25,\ m(ab){=}0.05,$$
$$m(ac){=}0.1,\ m(bc){=}0.05,\ m(abc){=}0.05.$$

We want to find which is the most possible object among *a*, *b*, *c*, in terms of m. However, as shown above, the support degrees for objects *a*, *b* and *c* are very similar in m. In order to solve the problem, let us use the proposed ECR. Let m be the initial configuration of this population at time $t = 0$, and $m_t(A)$ be the mass value of $m(A)$ at time t, where $A \subseteq \Omega$. The evolutionary process is illustrated by

$$m_{t+\Delta t}(A) = m_t(A) + \frac{dm_t(A)}{dt}\Delta t. \quad (20)$$

In this paper, we simulate the evolutionary process of each proposition by using the fourth-order Runge-Kutta method, as shown in Fig. 3, where the horizontal axis "Time" indicates time points in the Runge-Kutta method. According to Fig. 3, given an initial configuration determined by evidence m, proposition ac survives and finally occupies the population at the end of evolution, while other propositions become extinct. For evidence m, the corresponding ESP is ac, denoted as $ESP_m = ac$. Therefore, in m, the most supported object is either a or c. As shown in Fig. 3, by using the Runge-Kutta method, the evolutionary curve of each proposition can be tracked, and the time required for reaching equilibrium can be obtained. In this paper, we assume the replicator dynamics equation reaches equilibrium when the maximum increment or decrement of propositions' mass values or beliefs between two adjacent time points is less than 10^{-3}.

E. Two-dimensional measure

Within the ECR, Corollary 3 shows that an equilibrium state (ES), which is an either stable or unstable fixed point, will be obtained given any evidence. As proved in Corollary 4, the obtained ES is asymptotically stable if it is an ESP, otherwise it is an unstable ES. Yet, there is still another problem: If two different evidences evolve into the same ES, how can we distinguish them? By considering that the ECR contains a dynamic evolutionary process, we suggest to reflect such difference by using the time of evolution from initial state to the ES. Two examples are given as follows.

Example 4—Given two evidences, indicated by m_1 and m_2, on FOD $\Omega = \{a, b\}$,

$$m_1(a) = 0.7,\ m_1(b) = 0.1,\ m_1(ab) = 0.2;$$
$$m_2(a) = 0.5,\ m_2(b) = 0.3,\ m_2(ab) = 0.2.$$

The evolutionary process of propositions in m_1 and m_2 are shown in Fig. 4. Evidences m_1 and m_2 evolve to a same ES which is ESP a, namely $ES_{m_1} = a$, and $ES_{m_2} = a$. But the time of evolving to the ES are different for the two evidences. For m_1, $t_{ES} = 13.0459$. For m_2, $t_{ES} = 16.8553$. From the viewpoint of evolution, less time of reaching the ES implies that m_1 is much closer to ESP a, hence proposition a is better supported in m_1.

Example 5—Given a FOD $\Omega = \{a, b, c\}$, there is an evidence m, shown as below:

$$m(a) = x,\ m(b) = 0.9 - x,\ m(bc) = 0.05,\ m(abc) = 0.05.$$

where $x \in [0, 0.9]$. Now let us investigate the ES associated with m and the time evolving to that ES, with the change of x from 0 to 0.9 with an increment of 0.01.

Fig. 5 illustrates the results. From the figure, we can see that the ES of m is ESP b if $x \leq 0.45$, and ESP a if $x \geq 0.46$. It is found that t_{ES} changes with the initial configuration of evidence m. When $x \leq 0.45$, as the population evolves to ESP b, t_{ES} increases with the decrease of $m(b)$; when $x \geq 0.46$, as $ES_m = a$, t_{ES} decreases with the increase of $m(a)$. These results means that, if two evidences evolve to the same ES, say A, a higher $m(A)$ leads to a lower t_{ES}.

Therefore, we say that t_{ES} measures the time cost of an evidence evolving to its associated ES. From a biology perspective, t_{ES} reflects the evolutionary distance from the given evidence m to its associated ES ES_M. Based on this consideration, we use t_{ES} to further depict the relationship between m and ES_M. Then, a two-dimensional measure is defined as the outcomes of the ECR.

Definition 3—Assume there are n evidences indicated by $m_1, \cdots, m_n$, where m is the average of these n evidences, the evolutionary output of the ECR is represented as

$$<ES_m, t_{ES}>=f_{ECR}(m_1, \cdots, m_n), \quad (21)$$

where ES_m is the equilibrium state associated with m based on replicator dynamics, and t_{ES} is the time of m evolving to the equilibrium state.

The time of evolving to an ES provides a reference for the evolutionary distance between the averaging evidence m and its associated ES ES_m. It is worthy to notice that the final equilibrium state is asymptotically stable if ES_m is an ESP, otherwise the equilibrium state is an unstable state where two or more propositions coexist and a slight disturbance will cause the output deviates from such unstable equilibrium state. In this sense, the stability of ES_m implies the robustness of the outcomes of the ECR to some degrees. If ES_m is stable, the multi-evidence system $m_1, \cdots, m_n$ has a good consistency. Conversely, if ES_m is unstable, it means there may be highly discordant or conflicting information among evidences $m_1, \cdots, m_n$. In addition, the proposed ECR is with two other properties:

- The ECR is commutative, i.e., $f_{ECR}(m_1, m_2) = f_{ECR}(m_2, m_1)$.
- The ECR is idempotent, i.e., $f_{ECR}(m) = f_{ECR}(m, \cdots, m)$.

Proof: Within the ECR, mainly, the first is to calculate the averaging evidence m by Eq.(7), the second is the evolution of averaging evidence m based on replicator dynamics. Since,

$$average(m_1, m_2)=average(m_2, m_1),$$

and

$$average(m)=average(m, \cdots, m),$$

the ECR is commutative and idempotent.

IV. Illustrative examples and analysis

A. Combination of highly conflicting evidences

Conflicting evidence combination [82], [83] is a main concern to verify the effectiveness of combination rules in multi-source information fusion.

Example 6—Zadeh's paradox [13]. Two doctors diagnose a patient, and they agree that the patient suffers from one of three diseases: meningitis (M), brain tumor (T), and concussion (C). A FOD is determined as $\Omega = \{M, T, C\}$. Both of the doctors believe a tumor is unlikely, but they hold different opinions about the likely cause. Two diagnosis are given as follows.

$$m_1(M)=0.9,\ m_1(T)=0.1,\ m_1(C)=0.0.$$
$$m_2(M)=0.0,\ m_2(T)=0.1,\ m_2(C)=0.9.$$

It is found that these two evidences are highly conflicting. If using the classical Dempter's rule of combination to combine them, as shown in Eqs.(2) and (3), the combination result is

$$m_{\oplus}(M)=0,\ m_{\oplus}(T)=1,\ m_{\oplus}(T)=0$$

and the conflict coefficient $K = 0.99$. This is an apparently counter-intuitive result. In each doctor's diagnosis, the patient most likely does not suffer from a tumor, but the synthesizing result shows that the patient 100% suffering from a tumor. This example was first given by Zadeh to show the doubts on the validity of Dempster's rule when information is highly conflicting [13], [84].

Now, let's use the proposed ECR to integrate these two evidences. Assume each doctor is weighted the same, the averaging evidence is calculated as $m(M) = 0.45$, $m(T) = 0.1$, and $m(C) = 0.45$. Fig. 6 shows the evolutionary process of each proposition in m. In this figure, as evolutionary time increases, the belief of T goes to 0, and propositions M and C collectively average a total belief of 1. At the end of evolution, the equilibrium state is $\{(M, 0.5), (C, 0.5)\}$ where M and C coexist. Moreover, we know that the equilibrium state is not stable resulting from highly conflicting information in m_1 and m_2. Through the ECR, we obtain a reasonable result, and Zadeh's paradox is explained by the instability of an equilibrium state.

B. More illustrative examples

In the following content, we will examine the proposed ECR through more classical paradoxes in DST.

Example 7—In [16], [17], the authors have presented the *dictatorial power* (DP) of Dempster's rule. A simple version [15] of that paradox is shown below.

Given a FOD $\Theta = \{\theta_1, \theta_2, \theta_3\}$, there are four evidences:

$$m_1(\theta_1)=a,\ m_1(\theta_1\theta_2)=1-a;$$
$$m_i(\theta_3)=b,\ m_i(\Theta)=1-b,\ \text{where}\, i=2,3,4.$$

When using Dempster's rule of combination, one gets:

$$m_{\oplus}(\theta_1)=a=m_1(\theta_1), m_{\oplus}(\theta_1\theta_2)=1-a=m_1(\theta_1\theta_2).$$

It clearly shows that Dempster's rule does not respond to the combination of different evidences. It seems that evidence m_1 dominates other evidences since the combination result is always m_1, which does not accord with people's expectation.

Now let us reconsider this example by using the proposed ECR. Moreover, several improved methods for evidence combination, including Murphy's simple average [22], Deng's weighted average [23], Han's sequential weighted combination [15], proportional conflict redistribution (PCR6) rule [53], have also been adopted to test the results. Because PCR6 is not associative, to get optimal results, the PCR6 rule is implemented by combining all evidences altogether at the same time. Here, assume $a = 0.7$, $b = 0.6$. The results are listed in TABLE II. As illustrated in that table, apart from Dempster's rule, the counter-intuitive behaviors have been suppressed in all other methods. And in every case of combination, the most supported proposition obtained by the ECR totally accords with the results obtained by other improved methods. In addition, the decrease of t_{ES} shows that the evolutionary distance to the ES_m reduces with the accumulation of evidences, which further coincides with people's expectation.

Example 8—In [15], the authors constructed a fictitious example to verify the rationality of new evidence combination rule. In a multisensor-based automatic target recognition system, assume the FOD is $\Theta = \{\theta_1, \theta_2, \theta_3\}$. In order to recognize an unknown target, the system has collected five evidences shown as follows.

$$m_1(\theta_1)=0.60, m_1(\theta_2)=0.10, m_1(\theta_2,\theta_3)=0.30.$$
$$m_2(\theta_1)=0.65, m_2(\theta_2)=0.10, m_2(\theta_3)=0.25.$$
$$m_3(\theta_1)=0.00, m_3(\theta_2)=0.90, m_3(\theta_2,\theta_3)=0.10.$$
$$m_4(\theta_1)=0.55, m_4(\theta_2)=0.10, m_4(\theta_2,\theta_3)=0.35.$$
$$m_5(\theta_1)=0.55, m_5(\theta_2)=0.10, m_5(\theta_2,\theta_3)=0.35.$$

Based on different methods, the combination results are obtained in TABLE III. From TABLE III, in the results of Dempter's rule, $m(\theta_1)$ always equals 0 after combining m_3, yet regardless of m_4 and m_5. In contrast, if using other five methods, the counter-intuitive results are suppressed. In Deng's weighted average and Han's sequential weighted combination, the most supported target is always θ_1 for any cases, and $m(\theta_1)$ has an increasing trend when m_4 and m_5 arrives. For Murphy's simple average, PCR6 rule, and the proposed ECR, although $m(\theta_2)$ ever became the biggest one when combining m_1, m_2 and m_3, the mistake is corrected as soon as m_4 arrives. Hence, the counter-intuitive behavior is also suppressed in the three methods. Our proposed ECR remains effective in this example.

Example 9—Evidence shifting paradox [14], [15] describes another counter-intuitive behavior in Demspter-Shafer theory. Consider a case in which a target is evaluated by l different experts, each with the same importance. The FOD is $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Each expert provides an identical assessment as given below:

$$m_i(\theta_1)=0.06, m_i(\Theta)=0.94, \text{ where } i=1, \cdots, l.$$

By using Dempster's rule, the combination result is $m_\oplus(\theta_1) = 1 - 0.94^l$, $m_\oplus(\Theta) = 0.94^l$. If l is a big number, such as 100, we have $m_\oplus(\theta_1) = 1 - 0.94^{100} = 0.9979$, so the combination result strongly supports θ_1. However, in each expert's assessment, $m_l(\theta_1) = 0.06$, which is very small. The example show that the aggregation of a crowd's wisdom may generate counter-intuitive results when using Dempster's rule of combination.

Now, let's study this paradox by using the proposed ECR and other improved methods. The results are shown in TABLE IV. Here, the PCR6 results in Table TABLE IV have been obtained by sequential (suboptimal) implementation of PCR6. As illustrated in TABLE IV, when the number of combined evidences increases, Dempster's rule, Murphy's simple average, Deng's weighted average, and PCR6 rule all generate counter-intuitive results. For Han's sequential weighted combination, although it generates reasonable results, yet $m(\theta_1)$ has a rising trend as l increases. Han's method will eventually produce a counter-intuitive result where $m(\theta_1) > 0.5$ when l becomes large enough. Conversely, only the ECR always gets the reasonable result that $ES_m = \Theta$, and the evolutionary time remains constant as the increase of l. The idempotent property of the ECR inhibits the evidence shifting paradox.

V. Conclusion

In this paper, we have proposed an evolutionary combination rule (ECR) for the evidence-based multi-source information fusion from an evolutionary game theory perspective. Within the framework of ECR, original evidences are averaged by their weights, and a game model, called Jaccard matrix game, is proposed to formalize the interaction relationship between propositions. Then, we utilize the replicator dynamics equation to mimick the evolution of a population. And finally, an equilibrium state is obtained to express the combination results. According to the obtained equilibrium state, we can find the most biologically supported proposition for the decision-making and other purposes. Experimental results show that the proposed ECR has suppressed the counter-intuitive results in many paradoxes of Dempster-Shafer theory, thereby demonstrating the rationality and effectiveness of the proposed method. Some mathematical properties of the ECR, such as solution's stability and convergence, have also been analyzed and proved. Of course, there still remains some problems to be solved in future research. Below are summaries of several noticeable issues.

First, the ECR is not associative, all evidences must be combined together at the same time. Two main reasons lead to the non-associativity of the ECR: average mechanism and the replicator dynamics equation. Arguments about the necessity of associativity were presented in many literatures [24], [85], [86]. For the case of requiring the associativity, as an alternative, we are trying to build a modified ECR which meets the quasi-associativity [86]. Second, the ECR does not preserve the neutrality for a vacuous evidence $m_V(\text{FOD}) = 1$, i.e., $m \oplus_{ECR} m_V \neq m$. Third, in some cases the ECR is sensitive to the discord of evidences but fails to produce a reasonable result. For example, in [28] the author presented an example that combines $m_1(A) = m_1(B, C) = 0.5$ and $m_2(C) = m_2(A, B) = 0.5$. Dempster's rule gives $m(A) = m(B) = m(C) = 1/3$. As reported by Voorbraak [28], this result is counter-intuitive because B intuitively seems to share a probability mass of 0.5 twice, whereas both A and C share a probability mass of 0.5 with B only once and are once assigned 0.5 individually. Intuitively, B is less confirmed than A and C, but they are equally confirmed by Dempster's rule. By using the proposed ECR, we obtain an unstable equilibrium state $\{(AB, 0.5), (BC, 0.5)\}$. The ECR has detected the discord of evidences which results in an unstable solution, but does not give a reasonable combination result.

In summary, although there are some drawbacks at present, the proposed ECR is still a useful method for evidence combination. First, it imports the idea of population evolution into evidence combination, which is not presented in previous studies. Second, the ECR has a mathematical basis to ensure that the solution is with some desirable properties, such as convergence, stability, and idempotence. Third, compared with other methods, the ECR can effectively restrain the counter-intuitive behaviors appeared in many typical paradoxes of DST. In future research, we will continue to improve and perfect the framework of ECR.

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Biographies

Xinyang Deng received the bachelor's and master's degrees from Southwest University, Chongqing, China, in 2010 and 2013, respectively, and is currently pursuing the Ph.D. degree from Southwest University.

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Dr. Dezert has edited four books and published more than 150 papers in conferences and journals on tracking and information fusion. He has given seminars and workshops in IF and tracking in North America, Europe, Australia and China, and he is reviewer for several international journals and Associate Editor of ISIF Journal of Advances in Information Fusion. Jean Dezert is the co-founder with Prof. Smarandache of DSmT (Dezert-Smarandache Theory) of information fusion based on belief functions. More details are available at http://www.onera.fr/staff/jean-dezert.

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Dr. Deng won SMC Excellent Scholarship of Shanghai Jiao Tong University and National New Century Excellent Talents in University in 2008, Shanghai Rising-Star Program in 2009 and Chongqing Distinguished Young Scientists in 2010, China Highly Cited Scientist in 2014 by Elsevier. He is the PI for National High Technology (863) project and three National Natural Science projects in China. He is the editorial board member of PLOS ONE and more than 20 peer reviewed journals' reviewer. He has published more than 100 papers in peer reviewed journals such as Scientific Reports, PLOS ONE, IEEE TRANSACTION, European Journal of Operation Research, Decision Support Systems and so on. He presented Generalized Evidence Theory, D Numbers theory and Deng entropy. His current research interests include intelligent computing and intelligent information processing, information fusion, uncertain information modeling and decision making, etc.

Yu Shyr received his bachelor's degree from Tamkang University, Taiwan, in 1985 and his master's degree from Michigan State University, USA, in 1989. Yu Shyr received his PhD in biostatistics from the University of Michigan (Ann Arbor) in 1994 and subsequently joined the faculty at Vanderbilt University School of Medicine. He currently serves as the Director of both the Vanderbilt Center for Quantitative Sciences (CQS) and the Vanderbilt Technologies for Advanced Genomics Analysis and Research Design (VANGARD), while serving as the Harold L. Moses Chair in Cancer Research and as a professor of biostatistics, biomedical informatics, cancer biology, and health policy.

Dr. Shyr is a Fellow of the American Statistical Association and an FDA advisory committee voting member. He has delivered more than 200 abstracts at professional meetings and has published more than 375 peer-reviewed papers. Dr. Shyr has served on numerous NIH/NCI SPORE, P01, and CCSG review panels/committees and has been a member of the invited faculty at the AACR/ASCO Methods in Clinical Cancer Research Vail Workshop since 2004. He currently serves on the external advisory board for a dozen national cancer centers, and directs the biostatistics and bioinformatics cores for the NCI-funded Vanderbilt University Breast Cancer SPORE, GI Cancer SPORE, and other program projects. In addition, Dr. Shyr is the Principle Investigator of a UO1 grant for the Barrett's esophagus translational research network coordinating center (BETRNetCC). Dr. Shyr's current research interests focus on developing statistical bioinformatic methods for analyzing nextgeneration sequencing data, including a series of papers on estimating the sample size requirements for studies conducting RNA sequencing analysis.

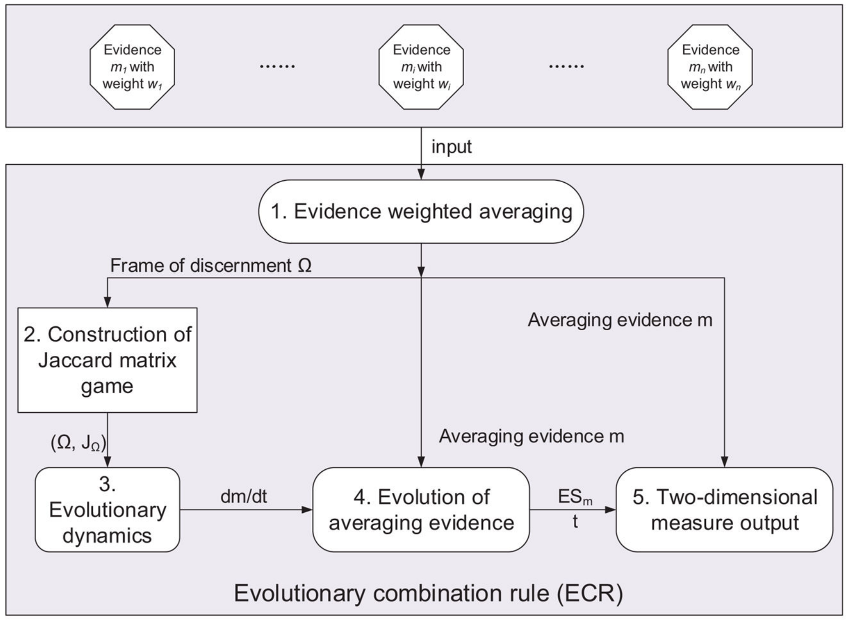

Fig. 1.

Framework of the proposed evolutionary combination rule

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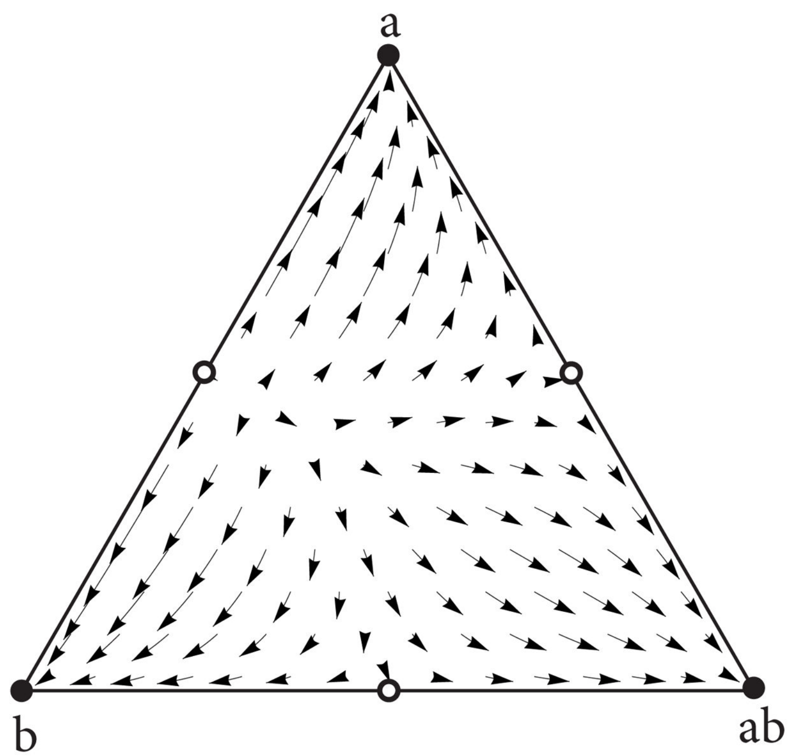

Fig. 2.
Evolutionary dynamics of propositions a, b and ab in Example 2

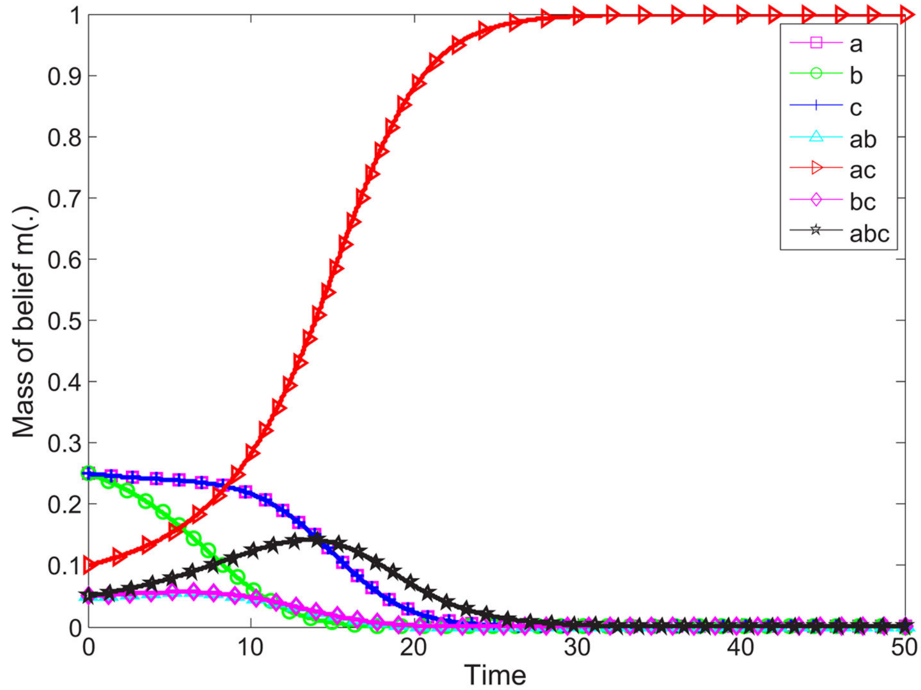

Fig. 3.
The evolutionary process of propositions in Example 3

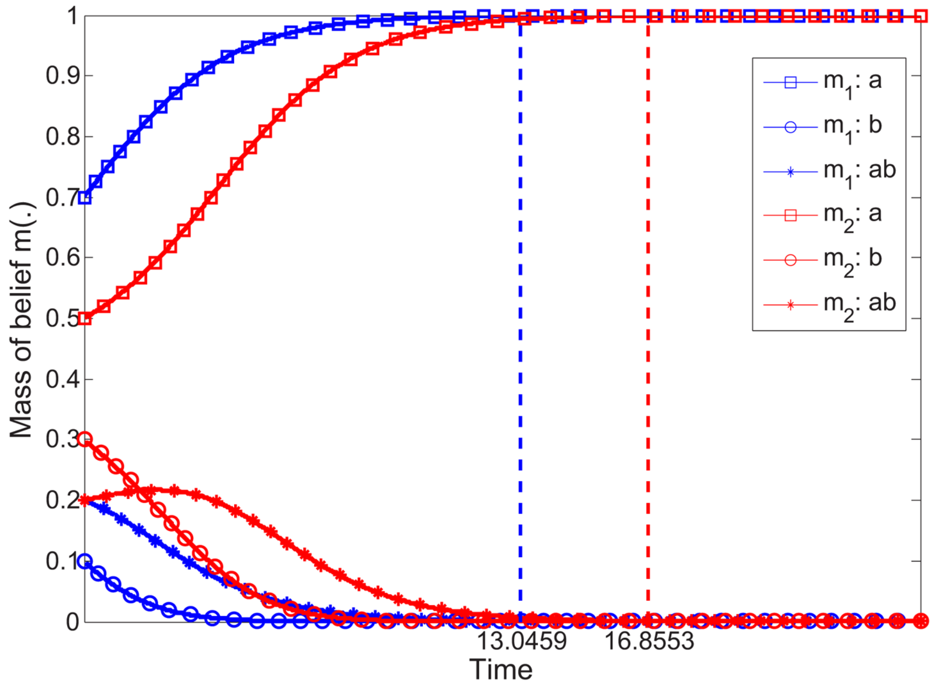

Fig. 4.

The evolutionary process of propositions in m_1 and m_2 of Example

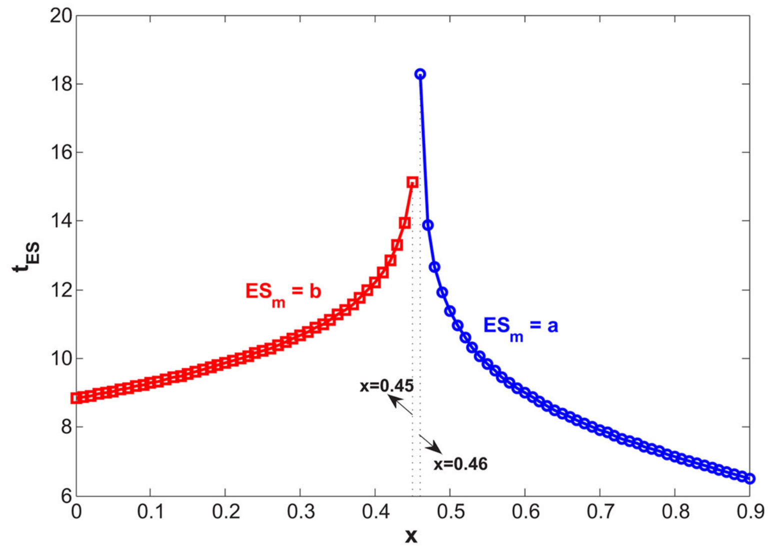

Fig. 5.

Evolutionary results in Example 5

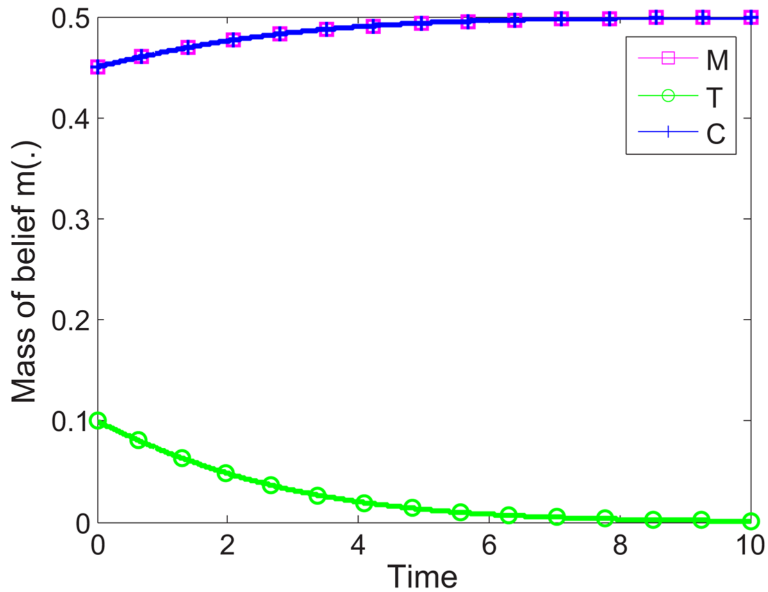

Fig. 6.
Evolutionary result of Zadeh's paradox

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TABLE I

Fixed points and their stability in Example 2

Fixed points	Associated eigenvalues	Stablity
(1, 0, 0)	−1, −1, −0.5	asymptotically stable
(0, 1, 0)	−1, −1, −0.5	asymptotically stable
(0, 0, 1)	−1, −0.5, −0.5	asymptotically stable
(0, 0.5, 0.5)	−0.75, −0.5, 0.25	unstable
(0.5, 0.5, 0)	−0.5, 0, 0.5	unstable
(0.5, 0, 0.5)	−0.75, −0.5, 0.25	unstable

TABLE II

Results for Example 7

Evidences	m_1, m_2	m_1, m_2, m_3	m_1, m_2, m_3, m_4
Dempster's rule of combination	$m(\theta_1) = 0.7000$, $m(\theta_1\theta_2) = 0.3000$.	$m(\theta_1) = 0.7000$, $m(\theta_1\theta_2) = 0.3000$.	$m(\theta_1) = 0.7000$, $m(\theta_1\theta_2) = 0.3000$.
Murphy's simple average	$m(\theta_1) = 0.5250$, $m(\theta_1\theta_2) = 0.1179$, $m(\theta_3) = 0.3000$, $m(\Theta) = 0.0571$.	$m(\theta_1) = 0.3379$, $m(\theta_1\theta_2) = 0.0615$, $m(\theta_3) = 0.5622$, $m(\Theta) = 0.0384$.	$m(\theta_1) = 0.1794$, $m(\theta_1\theta_2) = 0.0292$, $m(\theta_3) = 0.7711$, $m(\Theta) = 0.0203$.
Deng's weighted average	$m(\theta_1) = 0.5250$, $m(\theta_1\theta_2) = 0.1179$, $m(\theta_3) = 0.3000$, $m(\Theta) = 0.0571$.	$m(\theta_1) = 0.1032$, $m(\theta_1\theta_2) = 0.0290$, $m(\theta_3) = 0.8122$, $m(\Theta) = 0.0555$.	$m(\theta_1) = 0.1032$, $m(\theta_1\theta_2) = 0.0093$, $m(\theta_3) = 0.9344$, $m(\Theta) = 0.0246$.
Han's sequential weighted combination	$m(\theta_1) = 0.5250$, $m(\theta_1\theta_2) = 0.1179$, $m(\theta_3) = 0.3000$, $m(\Theta) = 0.0571$.	$m(\theta_1) = 0.2362$, $m(\theta_1\theta_2) = 0.0363$, $m(\theta_3) = 0.6369$, $m(\Theta) = 0.0906$.	$m(\theta_1) = 0.0676$, $m(\theta_1\theta_2) = 0.0089$, $m(\theta_3) = 0.8298$, $m(\Theta) = 0.0937$.
PCR6 rule	$m(\theta_1) = 0.5062$, $m(\theta_1\theta_2) = 0.1800$, $m(\theta_3) = 0.3138$.	$m(\theta_1) = 0.3432$, $m(\theta_1\theta_2) = 0.1028$, $m(\theta_3) = 0.4306$, $m(\Theta) = 0.1234$.	$m(\theta_1) = 0.2464$, $m(\theta_1\theta_2) = 0.0642$, $m(\theta_3) = 0.4921$, $m(\Theta) = 0.1973$.
The proposed ECR	$< ES_m = \theta_1, t_{ES} = 21.4913 >$	$< ES_m = \theta_3, t_{ES} = 21.6188 >$	$< ES_m = \theta_3, t_{ES} = 16.0809 >$

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TABLE III

Results for Example 8

Evidences	$\mathbf{m_1, m_2}$	$\mathbf{m_1, m_2, m_3}$	$\mathbf{m_1, m_2, m_3, m_4}$	$\mathbf{m_1, m_2, m_3, m_4, m_5}$
Dempster's rule of combination	$m(\theta_1) = 0.7723$, $m(\theta_2) = 0.0792$, $m(\theta_3) = 0.1485$.	$m(\theta_1) = 0.0000$, $m(\theta_2) = 0.8421$, $m(\theta_3) = 0.1579$.	$m(\theta_1) = 0.0000$, $m(\theta_2) = 0.8727$, $m(\theta_3) = 0.1273$.	$m(\theta_1) = 0.0000$, $m(\theta_2) = 0.8981$, $m(\theta_3) = 0.1019$.
Murphy's simple average	$m(\theta_1) = 0.7716$, $m(\theta_2) = 0.0790$, $m(\theta_3) = 0.1049$, $m(\theta_2\theta_3) = 0.0444$.	$m(\theta_1) = 0.3526$, $m(\theta_2) = 0.5978$, $m(\theta_3) = 0.0380$, $m(\theta_2\theta_3) = 0.0116$.	$m(\theta_1) = 0.5167$, $m(\theta_2) = 0.4573$, $m(\theta_3) = 0.0189$, $m(\theta_2\theta_3) = 0.0071$.	$m(\theta_1) = 0.6706$, $m(\theta_2) = 0.3169$, $m(\theta_3) = 0.0088$, $m(\theta_2\theta_3) = 0.0037$.
Deng's weighted average	$m(\theta_1) = 0.7716$, $m(\theta_2) = 0.0790$, $m(\theta_3) = 0.1049$, $m(\theta_2\theta_3) = 0.0444$.	$m(\theta_1) = 0.6013$, $m(\theta_2) = 0.3302$, $m(\theta_3) = 0.0532$, $m(\theta_2\theta_3) = 0.0153$.	$m(\theta_1) = 0.7987$, $m(\theta_2) = 0.1698$, $m(\theta_3) = 0.0227$, $m(\theta_2\theta_3) = 0.0088$.	$m(\theta_1) = 0.8975$, $m(\theta_2) = 0.0897$, $m(\theta_3) = 0.0088$, $m(\theta_2\theta_3) = 0.0040$.
Han's sequential weighted combination	$m(\theta_1) = 0.7716$, $m(\theta_2) = 0.0790$, $m(\theta_3) = 0.1049$, $m(\theta_2\theta_3) = 0.0444$.	$m(\theta_1) = 0.5591$, $m(\theta_2) = 0.4021$, $m(\theta_3) = 0.0297$, $m(\theta_2\theta_3) = 0.0091$.	$m(\theta_1) = 0.6781$, $m(\theta_2) = 0.3054$, $m(\theta_3) = 0.0071$, $m(\theta_2\theta_3) = 0.0093$.	$m(\theta_1) = 0.8103$, $m(\theta_2) = 0.1797$, $m(\theta_3) = 0.0014$, $m(\theta_2\theta_3) = 0.0086$.
PCR6 rule	$m(\theta_1) = 0.7371$, $m(\theta_2) = 0.0644$, $m(\theta_3) = 0.1370$, $m(\theta_2\theta_3) = 0.0615$.	$m(\theta_1) = 0.4224$, $m(\theta_2) = 0.4729$, $m(\theta_3) = 0.0483$, $m(\theta_2\theta_3) = 0.0564$.	$m(\theta_1) = 0.4755$, $m(\theta_2) = 0.3849$, $m(\theta_3) = 0.0351$, $m(\theta_2\theta_3) = 0.1045$.	$m(\theta_1) = 0.5111$, $m(\theta_2) = 0.3244$, $m(\theta_3) = 0.0276$, $m(\theta_2\theta_3) = 0.1369$.
The proposed ECR	$< ES_m = \theta_1, t_{ES} = 7.5708 >$	$< ES_m = \theta_2, t_{ES} = 19.3900 >$	$< ES_m = \theta_1, t_{ES} = 11.5065 >$	$< ES_m = \theta_1, t_{ES} = 10.6493 >$

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TABLE IV

Results for Example 9

The number of evidences *l*	***l* = 10**	***l* = 25**	***l* = 50**
Dempster's rule of combination	$m(\theta_1) = 0.4614$, $m(\Theta) = 0.5386$.	$m(\theta_1) = 0.7871$, $m(\Theta) = 0.2129$.	$m(\theta_1) = 0.9547$, $m(\Theta) = 0.0453$.
Murphy's simple average	$m(\theta_1) = 0.4614$, $m(\Theta) = 0.5386$.	$m(\theta_1) = 0.7871$, $m(\Theta) = 0.2129$.	$m(\theta_1) = 0.9547$, $m(\Theta) = 0.0453$.
Deng's weighted average	$m(\theta_1) = 0.4614$, $m(\Theta) = 0.5386$.	$m(\theta_1) = 0.7871$, $m(\Theta) = 0.2129$.	$m(\theta_1) = 0.9547$, $m(\Theta) = 0.0453$.
Han's sequential weighted combination	$m(\theta_1) = 0.3195$, $m(\Theta) = 0.6805$.	$m(\theta_1) = 0.3684$, $m(\Theta) = 0.6316$.	$m(\theta_1) = 0.3924$, $m(\Theta) = 0.6076$.
PCR6 rule	$m(\theta_1) = 0.4614$, $m(\Theta) = 0.5386$.	$m(\theta_1) = 0.7871$, $m(\Theta) = 0.2129$.	$m(\theta_1) = 0.9547$, $m(\Theta) = 0.0453$.
The proposed ECR	$< ES_m = \Theta, t_{ES} = 6.4296 >$	$< ES_m = \Theta, t_{ES} = 6.4296 >$	$< ES_m = \Theta, t_{ES} = 6.4296 >$