Random matrices and quantum chaos

Thomas Kriecherbauer*, Jens Marklof†‡, and Alexander Soshnikov§

*Mathematisches Institut, Ludwig-Maximilians-Universität, Theresienstrasse 39, D-80333 Munich, Germany; †School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom; and §Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633

The theory of random matrices has far-reaching applications in many different areas of mathematics and physics. In this note, we briefly describe the state of the theory and two of the perhaps most surprising appearances of random matrices, namely in the theory of quantum chaos and in the theory of prime numbers.

Since the pioneering work of E. Wigner in the 1950s, it has emerged that the statistical properties of many quantum systems can be modeled by random matrices. Wigner's original work was concerned with neutron excitation spectra of heavy nuclei. These are many-particle systems whose interaction, according to Wigner, is so complex that the Hamiltonian representing the system should behave like a large random matrix. It was discovered 30 years later that even simple one-particle quantum systems exhibit random matrix statistics, if the classical limit of the system is chaotic. An example of such a system is the electron in the heart-shaped region of Fig. 1, studied in ref. 1. In Fig. 2, the distribution of energy level spacings for that system is compared with that of the Gaussian Orthogonal Ensemble of random matrices. In this situation, nearby levels seem to repel each other, because the probability of finding small spacings is small. It is, in fact, believed that all generic quantum systems follow random matrix statistics of a suitably chosen ensemble, if the underlying classical dynamics is chaotic. The choice of ensemble depends on the physical symmetries of the system, for instance, time-reversal symmetry (2). If, in contrast, the underlying dynamics is regular, i.e., nonchaotic, the energy levels will not follow random matrix statistics but rather will behave like independent random variables from a Poisson process (3). An example of such a regular system is an electron confined to a circular domain. Its level spacing distribution is shown in Fig. 3, vs. the exponential distribution of a Poisson process.

Random matrices, in fact, are not only used to describe statistical properties of physical systems (e.g., in quantum chaos, disordered mesoscopic systems, and polynuclear growth models; see ref. 4 for a recent survey), but they also appear in such distant fields as number theory (5) and combinatorics (6). The combined efforts of mathematicians and physicists have recently led to a rigorous understanding of universality for a number of different symmetry classes, and we can hope for an even more complete understanding in the near future. Moreover, the recent discovery of new applications of random matrix theory is a strong indication that the eigenvalues of random matrices provide a fundamental model for sequences of dependent random numbers with a wide range of possible applications.

Universality in Random Matrix Theory. We now briefly describe the flavor of some of the results, which have been obtained in the theory of random matrices during the past few years. We refer the reader to ref. 7 for a history of the field up to the beginning of the 1990s. All results we consider here seek to establish the universality conjecture, which—roughly speaking—claims that local statistics (i.e., properly rescaled correlation functions) of the eigenvalues of random matrices converge as the size of the matrices becomes large, and that the limit is independent of the probability measure on the matrix spaces. It is clear that this conjecture is too general to hold true. Therefore, it is more appropriate to reformulate the goal to determine different universality classes, i.e., to identify classes of probability measures that asymptotically possess the same statistical behavior. As a rule of thumb, these classes may depend on the symmetry class of the matrices. Also, the statistics in the bulk of the spectrum differ from the statistics observed near the edge of the spectrum (or more generally near points where the density of the eigenvalues vanishes). During the past 5 years, there have been three different types of matrix ensembles for which universality was established rigorously. The first type is closest to those studied by Wigner: the probability distributions on the entries of the symmetric (respectively hermitean) matrices are independent and satisfy some mild assumptions on their moments (8). The second type concerns probability measures on matrix spaces, which are invariant under certain classes of similarity transformations (see refs. 9–11). The third type concerns classical compact Lie groups (see ref. 12 and refs. therein).

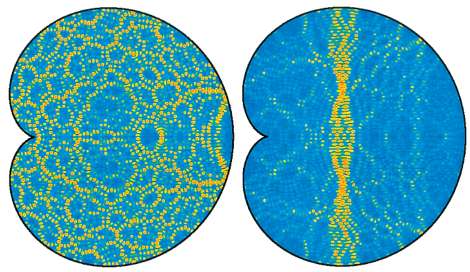

Fig. 1. Probability density of the 1,816th and 1,817th odd eigenstate of a quantum particle trapped in a chaotic heart-shaped region with Dirichlet boundary conditions. The probability of finding the particle at a given point is low in blue regions and high in red regions.

Quantum Chaos. The theory of quantum chaos is concerned with statistical properties of quantum systems that possess a classical limit. As mentioned earlier, one expects, for example, that the statistics of energy levels are typically described either by random matrix theory, when the classical limit is chaotic, or by a Poisson process, in the case when the classical dynamics is regular, i.e., completely integrable. The central tool in understanding this connection is Gutzwiller's trace formula (13), which gives a semiclassical relation between the energy eigenvalues of the quantum system and the actions of the periodic

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‡To whom reprint requests should be addressed. E-mail: j.marklof@bristol.ac.uk.

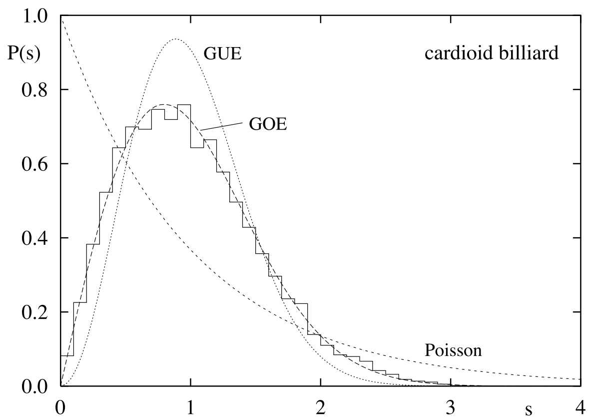

Fig. 2. Level spacing distribution for the energy spectrum of a quantum particle in the chaotic heart-shaped region of Fig. 1 vs. the level spacing distribution for Gaussian Unitary Ensemble, Gaussian Orthogonal Ensemble, and Poisson, respectively.

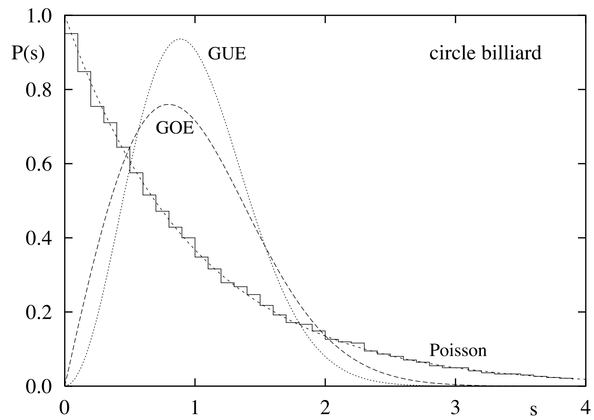

Fig. 3. Level spacing distribution for the energy spectrum of a quantum particle in a circular region vs. the level spacing distribution for Gaussian Unitary Ensemble, Gaussian Orthogonal Ensemble, and Poisson, respectively.

orbits of the classical counterpart. The hope is that the random matrix behavior of the eigenvalues can be understood in terms of the underlying classical chaotic or regular dynamics (14). There has been some recent progress in the case of simple regular systems, such as an electron confined to a rectangular box with generic ratio of sides. Here the question of spectral statistics reduces to very subtle lattice-point counting problems, which can be solved for special cases (see ref. 15 and refs. therein).

An additional aspect of quantum chaos is related to the statistical properties of the eigenstates. According to a theorem of Shnirelman, Zelditch, and Colin de Verdiere (16–18), almost all eigenfunctions of a quantum system whose classical dynamics is ergodic should become equidistributed in the classical limit on the available classical phase space. The eigenstate pictured in Fig. 1 *Left* is an example of such an "equidistributed" eigenfunction; that in Fig. 1 *Right* obviously is not. It is, in fact, localized around a classical unstable periodic orbit bouncing back and forth between opposite billiard walls. Such eigenstates have been termed scars (19), and their appearance indicates that not all eigenstates are in accordance with the random matrix prediction. Generally, it is not understood whether scars can appear at arbitrarily high energy values, or whether all (and not just almost all) highly excited eigenstates will eventually become equidistributed (see ref. 1 and refs. therein). So far, only a few systems are known where localized states can be ruled out (20). This phenomenon is called quantum unique ergodicity.

Prime Numbers. One of the biggest unsolved problems in mathematics is the Riemann Hypothesis, which asserts that the nontrivial zeros of the Riemann zeta function are lying on a straight line in the complex plane. The position of the zeros is of crucial importance in numerous problems in analytic number theory; Riemann himself investigated the zeros in connection with the number of primes below some large X.

Polya and Hilbert suggested that the Riemann Hypothesis could be solved by finding a linear self-adjoint operator whose eigenvalues are given by the Riemann zeros. This idea received a great boost when it was discovered that the Riemann zeros have the same statistical properties as eigenvalues of large random matrices from the Gaussian Unitary Ensemble (21, 22). This ensemble happens also to be responsible for the statistics of chaotic quantum systems without time-reversal symmetry, which led to speculations that the Polya–Hilbert operator might be the Hamiltonian of a quantum system with a chaotic classical limit. Riemann's explicit formula, which connects the zeros with prime numbers, could then be viewed as a special case of Gutzwiller's trace formula, where the prime numbers are interpreted as logarithms of classical actions of the (unknown) classical dynamics (23). Although the connection to random matrix theory is still rather mysterious in the case of the Riemann zeta function, there have been recent exciting developments in the case of zeta functions of curves over finite fields, where the relation with the spectral measures of the classical groups is now well established (5, 12).

We are very grateful to Arnd Bäcker for producing Figs. 1–3. More such beautiful illustrations of quantum chaotic wave functions may be found in ref. 1 and in his Ph.D. thesis (24).

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