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Global dynamics for switching systems and their extensions by linear differential equations

Zane Huttinga[†], **Bree Cummins**[†], **Tomáš Gedeon**[†], and **Konstantin Mischaikow**[‡]

[†]Department of Mathematical Sciences, Montana State University, Bozeman, MT 59715

[‡]Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghusen Rd, Piscataway, New Jersey 08854-8019, USA

Abstract

Switching systems use piecewise constant nonlinearities to model gene regulatory networks. This choice provides advantages in the analysis of behavior and allows the global description of dynamics in terms of Morse graphs associated to nodes of a parameter graph. The parameter graph captures spatial characteristics of a decomposition of parameter space into domains with identical Morse graphs. However, there are many cellular processes that do not exhibit threshold-like behavior and thus are not well described by a switching system. We consider a class of extensions of switching systems formed by a mixture of switching interactions and chains of variables governed by linear differential equations. We show that the parameter graphs associated to the switching system and any of its extensions are identical. For each parameter graph node, there is an order-preserving map from the Morse graph of the switching system to the Morse graph of any of its extensions. We provide counterexamples that show why possible stronger relationships between the Morse graphs are not valid.

Keywords

switching systems; gene regulation; transcription/translation model; Morse graphs

1 Introduction

While the last twenty years have brought unprecedented advances in experimental techniques allowing us to gather data on many cellular processes, the development of methods to combine this data into informative models is lagging behind [2, 33, 32]. Models of dynamic cellular processes based on ordinary differential equations require a choice of nonlinearities, parameters and initial conditions, and most of these are difficult to measure experimentally. Moreover, since the nonlinearities of multi-scale cellular processes are by necessity phenomenological, the measurement of the parameters depends on the model with which the data is interpreted. Changing the model leads to re-interpretation of the measurements, and often the need to remeasure all the parameters in the new model.

In this situation it is important to develop methods that compare the outcomes of models of different acuity and at different scales. It also calls for new approaches to modeling, which de-emphasizes comparison of individual trajectories of dynamical models with data. Instead, we should look for global coarse structures in dynamical systems, that are robust under perturbations, can be refined when more information becomes available, and that are computable.

Switching systems, also known as Glass systems, were developed [23, 24, 12, 6, 2] as an attempt to combine the advantages of ordinary differential equation models with Boolean rules, that are often used to represent the logic of gene regulation [3, 9, 8]. These models have the form

$$\dot{y}_j = -\gamma_j y_j + M_j(\sigma_i(y_i)) \quad (1)$$

where $\sigma_i(y_i)$ are piecewise constant functions whose values represent expression of the gene y_i in its “on” state and its “off” state, and M_j is a multi-linear function that prescribes how different inputs into gene j are combined.

The piecewise constant character of σ_i has its advantages and disadvantages. The disadvantages are mostly mathematical, and involve difficulties with interpreting the interaction of trajectories with thresholds, which are places where variables switch between the on state and the off state. Many papers have been devoted to making sense of continuation of solutions in switching systems [36, 31, 13, 1, 7]. A key biological objection to switching models is that in the majority of the cellular networks it is not reasonable to assume that all variables are switch-like in their behavior. For instance, while (1) can be a reasonable model for a network of mutually regulating proteins, including mRNA concentration for one of the proteins as an additional variable brings up a modeling difficulty since the rate of protein production usually depends linearly on the concentration of its mRNA. Such interaction cannot be modeled in a switching system. To address this challenge, in this paper we consider systems of the form

$$\begin{aligned} \dot{x}^1 &= -a^1 x^1 + b^1 y_i \\ \dot{x}^k &= -a^k x^k + b^k x^{k-1}, \quad k = 2, 3, \ldots, n, \\ \dot{y}_j &= -\gamma_j y_j + M_j(\sigma_i(x^n)) \end{aligned} \quad (2)$$

and compare their dynamics to that of system (1). Here a direct influence from variable y_i on variable y_j in (1) is mediated in (2) by a chain of variables $y_i \to x^1 \to x^2 \to \ldots \to x^n \to y_j$ that depend linearly on the previous variable in the chain. The nonlinear interaction is in the dependence of y_j on x^n which is mediated by a piecewise constant (i.e. switching) function.

The advantages of model (1) have been recognized since its inception [24, 25]. In each domain bounded by thresholds in the phase space the equations (1) are decoupled and linear. Their readily computable solutions can be organized in a *state transition diagram* (STD) which is a graph whose vertices are these domains and whose directed edges unambiguously represent flow between the domains [25, 18, 35]. The state transition diagram combinatorializes the dynamics of (1), and there are several results on the correspondence between paths in STD and invariant sets of (1) [25, 14, 15]. Another interesting approach that uses decomposition of phase space into rectangles in the context of multi-affine vector fields to address questions of control of trajectories into specific region is discussed in [5, 26, 27, 28].

The recent results in [11] have gone a step further. A *Morse graph* collects strongly connected components of the STD as vertices, and edges represent reachability between these components. This is a coarse version of a fundamental concept of a Morse decomposition of a dynamical system [10] and represents a summary of global dynamics of (1). In [21] it is shown that the Morse graph of (1) provides information about a Morse decomposition of nearby smooth perturbations of (1). Since Morse graphs are readily computable, the system (1) can now be viewed as a computational tool to compute coarse descriptions of the global dynamics of smooth systems.

The second insight that has come from changing the perspective from considering trajectories to looking at Morse graphs, is that Morse graphs are locally constant in the parameter space of (1). Furthermore, the boundaries of the constancy regions are given by readily computable inequalities in parameters [11, 31]. We represent each region as a node of a *parameter graph* and connect two nodes by an edge when they share a codimension 1 boundary in the parameter space. The parameter graph, together with a Morse graph attached to each of its nodes, form a *DSGRN database* [29, 11]. This database encodes global dynamics over the parameter space of the system (1). As such, it can be readily searched for particular dynamic signatures (stable equilibria, periodic patterns), and their robustness in the parameter space can be readily assessed [4].

In this paper we study the relationship between the Morse graphs and the parameter graphs of the switching system (1) and the corresponding extended system (2). The motivation comes from several sources. In [20], the correspondence between the dynamics of a network of proteins and a network with included mRNA for each protein is studied. The additional equations for mRNA are linear. The authors assumed arbitrary monotone nonlinearities and showed that while there is a correspondence between the equilibria in the two systems, their stability can differ.

In a series of papers, Edwards and collaborators [16, 30] studied the same question in the context of switching system (1) with appended linear equations, or appended switching equations [17]. The key insight that we take from [16, 30] is that there is a natural way to define domain boundaries in the new linear variables and use these new domains to define a state transition diagram for the extended system. In a generalization of the approach in [16, 30], each interaction in the switching system (1), corresponds in the extended system (2) to a chain of new variables, each depending on the previous one in a linear fashion. We call any such system an *chain linear extension* (or simply an *extension*) of (1).

Our central result is that the parameter graph of the original system (1) and any of its extensions are the same. Since the parameter graph tracks the changes in the Morse graphs and is therefore an analog of a bifurcation diagram in classical theory of dynamical systems, this shows that the structure of changes in dynamics in parameter space is the same. While the parameter graphs of the two systems are the same, this does not necessarily imply that the Morse graphs describing the phase space dynamics at the same parameter node are the same. In fact, since the dimension of the phase space of the switching system and its extension can be vastly different, we would expect the dynamics of the extension to be richer, and hence the Morse graphs larger.

We prove two positive results about the relationship between the Morse graphs of the two systems. We first show that each fixed point attractor in the Morse graph of the switching system corresponds to a unique fixed point attractor in the Morse graph of the extended system. Second, for every node of the parameter graph, we construct an order preserving map φ that maps the Morse graph (viewed as a partially ordered set) of the switching system into the Morse graph of its extension.

We close our results by three examples that show that φ is not injective in general and that in general there is no surjection from the Morse graph of the switching system onto the Morse graph of its extension. While these examples rule out certain natural stronger relationships between the corresponding Morse graphs of the two systems, there is a possibility that there exists a stronger relationship between a Morse graph computed from a properly formulated extension of the state transition graph of the switching system and the Morse graph of its chain linear extension. Switching systems with variables that affect their own dynamics, i.e. self-feedback, always contain pairs of neighboring domains where flows in both either point toward, or away, from their common boundary. These boundaries are known as black (white) walls and present mathematical challenges to extending the solutions entering these walls. In our perspective of global dynamics, the Morse graphs of the switching systems do not reflect the potential existence of invariant sets in these walls. However, there will be Morse nodes in a Morse graph for any smooth system that is a small perturbation of a switching system, that correspond to these invariant sets. We conjecture that there is a surjection from any Morse graph that captures invariant sets of nearby smooth approximations of the switching system to a coarsened Morse graph of the extended system (2).

2 Chain linear extension of the switching system

This paper relies on concepts developed in [11] for switching systems. In this section, as we recall these concepts, we generalize them to the extended system (2).

2.1 Network and equations

Definition 2.1: A *regulatory network* **RN** = (V,E) is a directed graph with N distinct network nodes $V = \{\beta_1, \beta_2, \dots, \beta_N\}$ and signed edges $E \subset V \times V \times \{\rightarrow, \dashv\}$. Edges of sign $\rightarrow$ represent *positive regulation* or *activation* and those of sign $\dashv$ represent *negative regulation* or *repression*. We will use the notation $(\beta_i, \beta_j) \in E$ to describe a directed edge from β_i to β_j of either sign, and $\beta_i \rightarrow \beta_j$ or $\beta_i \dashv \beta_j$ to denote signed directed edges. We require that at most one edge exists from β_i to β_j.

Let

$$S(\beta_i) := \{\beta_j \mid (\beta_j, \beta_i) \in E\}$$
$$T(\beta_i) := \{\beta_j \mid (\beta_i, \beta_j) \in E\}$$

be the *sources* and *targets* of β_i, respectively.

Definition 2.2: Let **RN** = (V,E) be a regulatory network. Then an *extension* of **RN** is a directed annotated graph **EX** = ($V \cup X$, E_X). The vertex set X and the edge set E_X are constructed in the following way. To edge $(\beta_i, \beta_j) \in E$ we associate a set of nodes $X_{j,i}$, disjoint from V ($X_{j,i} = \varnothing$ is permitted). Each nonempty set $X_{j,i}$ admits a linear ordering of its vertices $X_{j,i} := \{\alpha^1_{j,i}, \dots, \alpha^{n_{j,i}}_{j,i}\}$. Then a *chain* from β_i to β_j is a subgraph $G_{j,i} = (X_{j,i} \cup \{\beta_i, \beta_j\}, C_{j,i})$ where the set of edges $C_{j,i} \subset (X_{j,i} \cup \{\beta_i\}) \times (X_{j,i} \cup \{\beta_j\}) \times \{\rightarrow, \dashv\}$ is

$$C_{j,i} := \{(\beta_i, \alpha^1_{j,i}), (\alpha^1_{j,i}, \alpha^2_{j,i}), \dots, (\alpha^k_{j,i}, \alpha^{k+1}_{j,i}), \dots, (\alpha^{n_{j,i}}_{j,i}, \beta_j)\} \text{ if } X_{j,i} \neq \varnothing, \text{ or}$$
$$C_{j,i} := \{(\beta_i, \beta_j)\} \text{ if } X_{j,i} = \varnothing,$$

and the signs of the edges may take either value in $\{\rightarrow, \dashv\}$. We require that $X_{j,i}$ and $X_{t,s}$ are disjoint for each $(\beta_i, \beta_j) \neq (\beta_s, \beta_t)$ in E. Then we define

$$X := \bigcup_{(\beta_i, \beta_j) \in E} X_{j,i}, \quad E_X := \bigcup_{(\beta_i, \beta_j) \in E} C_{j,i}.$$

We refer to $\alpha^k_{j,i} \in X_{j,i}$ as the k^{th} *intermediary* between β_i and β_j. We define a *canonical extension* of **RN** to be an extension that fulfills the additional conditions:

1. The vertex set X is nonempty.
2. $X_{j,i}$ is nonempty whenever $(\beta_i, \beta_i) \in E$,
3. if $X_{j,i}$ is nonempty with $n_{j,i} > 0$ elements, then
   a. $\beta_i \rightarrow \alpha^1_{j,i} \in C_{j,i}$ and

b. for $k = 1, 2, \ldots, n_{j,i} - 1$, $\alpha_{j,i}^{k} \rightarrow \alpha_{j,i}^{k+1} \in C_{j,i}$,

c. if $\beta_i \rightarrow \beta_j \in E$, then $\alpha_{j,i}^{n_{j,i}} \rightarrow \beta_j \in C_{j,i}$

d. if $\beta_i \dashv \beta_j \in E$, then $\alpha_{j,i}^{n_{j,i}} \dashv \beta_j \in C_{j,i}$.

Whenever we will be considering a single chain and the context is clear, we will simplify the notation from $\alpha_{j,i}^{k}$ to α^k to improve readability.

As we introduce new concepts, we will illustrate them on a pair of simple networks in Figure 1.

Definition 2.3: A *switching system* associated to **RN** is a set of ordinary differential equations in variables $V' := \{y_j \mid j = 1, \ldots, N\}$, with piecewise constant nonlinearities of the form

$$\begin{aligned} \dot{y}_j &= -\gamma_j y_j + \widetilde{\Lambda}_j(y) \\ \widetilde{\Lambda}_j &= M_j \circ \widetilde{\sigma}_j \\ \widetilde{\sigma}_j &= (\widetilde{\sigma}(y_j, y_1; \theta_{j,1}), \widetilde{\sigma}(y_j, y_2; \theta_{j,2}), \ldots, \widetilde{\sigma}(y_j, y_N; \theta_{j,N})) . \end{aligned} \tag{3}$$

Here γ_j is a decay rate, M_j is a multilinear function, and $\theta_{j,i}$ are *regulatory thresholds*. The functions $\widetilde{\sigma}(y_j, y_i; \theta_{j,i})$ are step functions of the form

$$\widetilde{\sigma}(y_j, y_i; \theta_{j,i}) = \begin{cases} l_{j,i} & \text{if } \beta_i \rightarrow \beta_j \in E \text{ and } y_i < \theta_{j,i} \\ & \text{or } \beta_i \dashv \beta_j \in E \text{ and } y_i > \theta_{j,i} \\ u_{j,i} & \text{if } \beta_i \rightarrow \beta_j \in E \text{ and } y_i > \theta_{j,i} \\ & \text{or } \beta_i \dashv \beta_j \in E \text{ and } y_i < \theta_{j,i} \\ \text{undefined} & \text{if } (\beta_i, \beta_j) \notin E \end{cases} \tag{4}$$

We will say that $\theta_{j,i}$ exists if and only if $(\beta_i, \beta_j) \in E$. In order for the step function $\widetilde{\sigma}$ to correspond to positive or negative regulation, we require that $l_{j,i} \le u_{j,i}$ for every $(\beta_i, \beta_j) \in E$, with $l_{j,i} = u_{j,i}$ being the degenerate case. We also assume strictly positive parameters, so that $0 < \gamma_i, l_{j,i}, u_{j,i}, \theta_{j,i}$ for all i, j.

In an abuse of notation we define

$$\begin{aligned} S(y_i) &:= \{y_j \mid (\beta_j, \beta_i) \in E\} \\ T(y_i) &:= \{y_j \mid (\beta_i, \beta_j) \in E\} \end{aligned}$$

to be the *sources* and *targets* of variable y_i, respectively. It should be clear from the context whether we work with nodes of the graph, or the corresponding variables.

We now define an analog of the switching system (3), in which we introduce intermediate variables x that are governed by linear equations.

Definition 2.4: The *extended system* associated to **EX** is a set of ordinary differential equations in variables $V' \cup X'$, where $X' := \{x_{j,i}^k \mid \alpha_{j,i}^k \in X\}$. The equations take the form

$$\begin{aligned} \dot{y}_i &= -\gamma_i y_i + \Lambda_i(y, x), \quad i = 1, 2, \ldots, N \\ \dot{\vec{x}}_i &= A_i \vec{x}_i + \vec{b}_i, \quad i \in \mathcal{I} \subseteq \{1, 2, \ldots, N\} \end{aligned} \quad (5)$$

where $\mathcal{I} := \{i \,/\, X_{j,i} \neq = \varnothing \text{ for some } y_j \in \mathcal{T}(y_i)\}$. The matrix A_i is block diagonal and $\vec{x}_i$ and $\vec{b}_i$ are block vectors, each containing a unique block $A_{j,i}$, $\vec{x}_{j,i}$, and $\vec{b}_{j,i}$, respectively, for each $y_j \in \mathcal{T}(y_i)$. In particular, if $C_{j,i}$ is a chain $\{(\beta_i, \alpha^1), (\alpha^1, \alpha^2), \ldots, (\alpha^n, \beta_j)\} \subset E_X$, these blocks take the form

$$A_{j,i} = \begin{bmatrix} -a^1 & 0 & 0 & \cdots & 0 \\ b^2 & -a^2 & 0 & \cdots & 0 \\ 0 & b^3 & -a^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & b^n & -a^n \end{bmatrix}$$

$$\vec{x}_{j,i} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \quad \vec{b}_{j,i} = \begin{bmatrix} b^1 y_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where we have written x^k, a^k, and b^k in lieu of $x_{j,i}^k$, $a_{j,i}^k$, and $b_{j,i}^k$ for brevity. In non-matrix form, this becomes

$$\dot{y}_i = -\gamma_i y_i + \Lambda_i(y, x) \quad (6)$$

$$\dot{x}^1 = -a^1 x^1 + b^1 y_i \quad (7)$$

$$\dot{x}^k = -a^k x^k + b^k x^{k-1}, \quad k = 2, 3, \ldots, n \quad (8)$$

The function $\Lambda_i := M_i \circ \sigma_i$, where M_i is the same multilinear function as in the switching system in (3), and σ_i, defined in Definition 2.7, is a step function similar to $\tilde{\sigma}_i$ in (4). The new parameters $a^k\,(=a^k_{j,i})$ and $b^k\,(=b^k_{j,i})$ denote a decay rate and a production rate for $x^k\,(=x^k_{j,i})$ and are strictly positive. We take

$$a_{j,i} = \{a^k_{j,i} \mid k = 1, \dots, n_{j,i}\}, \quad b_{j,i} = \{b^k_{j,i} \mid k = 1, \dots, n_{j,i}\}. \quad (9)$$

Note that $a_{j,i} = b_{j,i} = \emptyset$ whenever $X_{j,i} = \emptyset$.

For the examples in Figure 1 both single loop **RN** networks give rise to a single differential equation

$$\dot{y}_1 = -\gamma_1 y_1 + \tilde{\Lambda}_1(y_1),$$

the only difference between the positive and negative self-loop is reflected in the function $\tilde{\Lambda}^{\pm}_1$:

$$\tilde{\Lambda}^{+}_1(y_1) = \begin{cases} l_{1,1} & \text{if } y_1 < \theta_{1,1} \\ u_{1,1} & \text{if } y_1 > \theta_{1,1} \end{cases} \quad \tilde{\Lambda}^{-}_1(y_1) = \begin{cases} u_{1,1} & \text{if } y_1 < \theta_{1,1} \\ l_{1,1} & \text{if } y_1 > \theta_{1,1} \end{cases}$$

The extended system for **EX** in both cases has the form

$$\begin{aligned} \dot{y}_1 &= -\gamma_1 y_1 + \Lambda^{\pm}_1(y_1, x^1) \\ \dot{x}^1 &= -a^1 x^1 + b^1 y_1 \end{aligned}$$

We describe functions $\Lambda^{\pm}_1(y_1, x^1)$ below in Definition 2.7.

2.2 Parameters

A *parameter* of the switching system (3) is $z := (l, u, \theta, \gamma)$, where $l = \{l_{j,i}\}$, $u = \{u_{j,i}\}$, and $\theta = \{\theta_{j,i}\}$ are indexed by edges $(\beta_i, \beta_j) \in E$ in **RN**, and $\gamma = \{\gamma_i\}$ is indexed by nodes $\beta_i \in V$, or equivalently $y_i \in V'$. The parameter z is an element of $(0,\infty)^D$ where $D = |V| + 3|E| = N + 3|E|$.

An *extended parameter* or a *parameter* for **EX** is a tuple $h = (z, a, b) = (l, u, \theta, \gamma, a, b) \in (0,\infty)^{D'}$, where a and b are vectors of length $|X|$ (a is the union of sets $a_{j,i}$ and b is the union of sets $b_{j,i}$ in (9)). Moreover $z = \pi(h)$, where π is the projection from $\mathbb{R}^{D'}$ to $\mathbb{R}^D$. The new parameter length D' is then

$$D' = D + 2|X| = N + 3|E| + 2|X|.$$

2.3 Phase space

The phase space of a switching system admits a natural decomposition into domains.

Definition 2.5: Fix a parameter $z = (l, u, \theta, \gamma)$ for the switching system (3). We will use the convention that for each $y_s \in V'$, $\theta_{-\infty,s} = 0$ and $\theta_{\infty,s} = \infty$. Let $\theta_{v_s,s}$ and $\theta_{w_s,s}$ be consecutive threshold values. Then a *domain* is a subset of the phase space $(0,\infty)^N$ of the switching system at z, given by

$$\tilde{\kappa} := \prod_{s=1}^{N} (\theta_{v_s,s}, \theta_{w_s,s}).$$

We denote the collection of domains in $(0, \infty)^N$ by $\mathscr{K}(z)$.

A *face* of $\tilde{\kappa}$ is a subset of the boundary of $\tilde{\kappa}$ given by

$$\tilde{\tau} := \prod_{s=1}^{i-1} (\theta_{v_s,s}, \theta_{w_s,s}) \times \{\theta_{m_i,i}\} \times \prod_{s=i+1}^{N} (\theta_{v_s,s}, \theta_{w_s,s}), \quad m_i \notin \{\pm\infty\}$$

on which $y_i = \theta_{m_i,i}$ is constant.

Let $\tilde{\tau}$ be a face of $\tilde{\kappa}$ on which $y_i = \theta_{m_i,i}$. The pair $(\tilde{\tau}, \tilde{\kappa})$ is called a *wall*. We say that $\tilde{\tau}$ is a *left face* (*right face*) of $\tilde{\kappa}$ if $y_i > \theta_{m_i,i}$ ($y_i < \theta_{m_i,i}$) everywhere in $\tilde{\kappa}$. We say the sign of the wall $(\tilde{\tau}, \tilde{\kappa})$ is 1, $\operatorname{sgn}(\tilde{\tau}, \tilde{\kappa}) = 1$, if $\tilde{\tau}$ is a left face of $\tilde{\kappa}$. Similarly, $\operatorname{sgn}(\tilde{\tau}, \tilde{\kappa}) = -1$ when $\tilde{\tau}$ is a right face of $\tilde{\kappa}$.

We would like to define domains for the extended system. A naive solution would be to define them as products $\tilde{\kappa} \times (0, \infty)^{|X|}$ for each $\tilde{\kappa} \in \mathscr{K}(z)$. However, the key ingredient in describing the dynamics in terms of a partition of phase space has been the fact that the solutions to (3) cross each face $\tilde{\tau}$ in one direction. This is not true under extended flow for extended walls $(\tilde{\tau} \times (0, \infty)^{|X|}, \tilde{\kappa} \times (0, \infty)^{|X|})$. To resolve this problem we follow [16, 30] and define new domain boundaries in the x variables to further partition phase space. The choice of these values is not arbitrary, as shall be demonstrated later in Theorem 2.15. We give a geometric reason for their definition in Appendix A.

Definition 2.6: Consider a chain $C_{j,i} = \{(\beta_i, \alpha^1), (\alpha^1, \alpha^2), \ldots, (\alpha^n, \beta_j)\} \in E_X$, where $X_{j,i} \neq \varnothing$. Fix a parameter $h = (l, u, \theta, \gamma, a, b) \in (0,\infty)^{D'}$. Define values ϑ^k associated to edges in $C_{j,i}$ as follows.

1. Let $\vartheta^0 := \theta_{j,i}$ be associated to the edge $(\beta_i, \alpha^1) \in C_{j,i}$.

2. For $k = 1, 2, \ldots, n$, let $\vartheta^k = \frac{b^k}{a^k}\vartheta^{k-1}$ be associated to edge (α^k, α^{k+1}) or (α^n, β_j) when $k = n$.

Because the values ϑ^k are defined recursively, we can rewrite ϑ^k associated to edges in $C_{j,i}$ in terms of the threshold $\theta_{j,i}$:

$$\vartheta^k = \left(\prod_{\ell=1}^{k} \frac{b^\ell}{a^\ell}\right)\theta_{j,i}, \quad k = 1, 2, \ldots, n. \quad (10)$$

When dealing with multiple chains, we denote the value associated to the edge $(\alpha_{j,i}^k, \alpha_{j,i}^{k+1})$ by the triple indexing scheme $\vartheta_{j,i}^k$. If $X_{j,i} = \varnothing$, then we say these thresholds $\vartheta_{j,i}^k$ do not exist.

We now explicitly define the vector of step functions $\sigma_j = (\sigma_{j,1}, \ldots, \sigma_{j,N})$ associated to variable $y_j \in V'$, which is based on Definition 2.3 for the step function in the switching system.

Definition 2.7: Let $C_{j,i} \subset E_X$ be a chain from β_i to β_j and y_i, y_j the associated variables. If $X_{j,i} = \varnothing$, then

$$\sigma_{j,i} := \tilde{\sigma}(y_j, y_i; \theta_{j,i}),$$

referring back to the step function in (4). If $X_{j,i}$ is nonempty, denote the last intermediate variable by x^n with associated value ϑ^n. Then

$$\sigma_{j,i} := \tilde{\sigma}(y_j, x^n; \vartheta^n).$$

In other words, the number and form of the step functions in the extended system is the same as the original switching system, except that the inputs into the step function will change in each chain.

Since the value ϑ^n plays a role of a threshold in the function $\sigma_{j,i}$ we will refer to the values ϑ^n also as thresholds. However, we want to emphasize that they are constructed with the sole purpose of defining domains for the extended system, and do not have the same role in the dynamics of system (5) as thresholds $\theta_{j,i}$ have in the dynamics of (3).

According to Definition 2.7 the extended systems for the positive (left) and negative (right) self loop have the form

$$\dot{y}_1 = -\gamma_1 y_1 + \begin{cases} l_{1,1} & \text{if } x^1 < \vartheta^1 \\ u_{1,1} & \text{if } x^1 > \vartheta^1 \end{cases} \qquad \dot{y}_1 = -\gamma_1 y_1 + \begin{cases} u_{1,1} & \text{if } x^1 < \vartheta^1 \\ l_{1,1} & \text{if } x^1 > \vartheta^1 \end{cases}$$
$$\dot{x}^1 = -a^1 x^1 + b^1 y_1 \qquad \dot{x}^1 = -a^1 x^1 + b^1 y_1$$

where $\vartheta^1 = \frac{b^1}{a^1}\theta_{1,1}$. The phase space of these extended systems is depicted in Figures 2b and 2d.

Definition 2.8: Let **RN** be a regulatory network and **EX** an extension of **RN**. Let the associated switching and extended systems be parameterized with $z = (l, u, \theta, \gamma)$ and $h = (z, a, b)$, respectively. For a domain $\tilde{\kappa} \in \mathscr{K}(z)$ of the switching system, we define an *extended domain* $\kappa \in \mathscr{K}(h)$ to be a choice

$$\kappa := \tilde{\kappa} \times \prod_{s \in \mathscr{I}} \Xi(y_s)$$

where recall from Definition 2.4 that $\mathscr{I} = \{i \mid X_{j,i} \neq \varnothing \text{ for some } y_j \in T(y_i)\}$. Here each $\Xi(y_s)$ is a product

$$\Xi(y_s) = \prod_{t \in \mathscr{J}(s)} \prod_{\ell = 1}^{n_{t,s}} (\vartheta^{\ell}_{v_t, s}, \vartheta^{\ell}_{w_t, s})$$

where $\mathscr{J}(s) := \{t \,/\, y_t \in T(y_s) \text{ and } X_{t,s} \neq \varnothing\}$ and for each $t \in \mathscr{J}(s)$ either $(v_t, w_t) = (-\infty, t)$ or (t, ∞). Note that $\Xi(y_s)$ represents the values of all of the chain variables $x^{\ell}_{t,s}$ in each chain with initial vertex β_s. As in the switching system, we will use the convention that for each $x^{\ell}_{t,s} \in X'$, $\vartheta^{\ell}_{-\infty,s} = 0$ and $\vartheta^{\ell}_{\infty,s} = \infty$. We illustrate this on Figure 2. In both cases the phase space of the switching system contains two domains $(0, \theta_{1,1})$ and $(\theta_{1,1}, \infty)$, which share the face $\{\theta_{1,1}\}$ on which y_1 is constant. The phase spaces of both extensions contain four domains, which are Cartesian products of both domain of the switching system with both $(0, \vartheta^1)$ and (ϑ^1, ∞). Both of $(0, \vartheta^1)$ and (ϑ^1, ∞) are themselves products of the form $\Pi_{s \in \mathscr{I}} \Xi(y_s)$ with $\mathscr{I} = \{1\}$, $\mathscr{J}(1) = \{1\}$, and $n_{1,1} = 1$.

Let $\tilde{\tau}$ be a face of the domain $\tilde{\kappa}$. A *y-face* of an extended domain κ is given by

$$\tau_y := \tilde{\tau} \times \prod_{s \in \mathscr{I}} \Xi(y_s)$$

Clearly this is a subset of one of the y-hyperplanes bounding κ. In Figure 2 each extended domain has a y-face equal to either $\{\theta_{1,1}\} \times (0, \vartheta^1)$ or $\{\theta_{1,1}\} \times (\vartheta^1, \infty)$. An *x-face* of the extended domain κ is

$$\tau_x := \tilde{\kappa} \prod_{\substack{s \in \mathcal{I}, \\ s \neq i}} \Xi(y_s) \times \prod_{\substack{t \in \mathcal{J}(s), \\ t \neq j}} \prod_{\ell = 1}^{n_{t,i}} (\vartheta^{\ell}_{v_t, i}, \vartheta^{\ell}_{w_t, i}) \times \Upsilon(i, j, k)$$

where

$$\Upsilon(i, j, k) = \prod_{\ell = 1}^{k-1} (\vartheta^{\ell}_{v_j, i}, \vartheta^{\ell}_{w_j, i}) \times \{\vartheta^{k}_{j, i}\} \times \prod_{\ell = k+1}^{n_{j,i}} (\vartheta^{\ell}_{v_j, i}, \vartheta^{\ell}_{w_j, i}).$$

Note that the last index k in $\Upsilon(i, j, k)$ is the index of the variable $x^k_{i,j}$ which is at threshold. Therefore τ_x is a subset of the hyperplane $\{x^k_{j,i} = \vartheta^k_{j,i}\}$. In Figure 2, each extended domain κ has an x-face equal to $\kappa \times (0, \vartheta^1)$ or $\kappa \times (\vartheta^1, \infty)$.

We use the terms *left* and *right* x-faces and y-faces analogously to Definition 2.5, and (τ_x, κ), (τ_y, κ) will denote *x-walls* and *y-walls*, respectively. As before, we say that $\mathrm{sgn}(\tau, \kappa) = 1$ if τ is a left x- or y-face of κ and that $\mathrm{sgn}(\tau, \kappa) = -1$ if τ is a right x- or y-face of κ.

2.4 Generic parameters

The following corollary is an immediate consequence of the definition of domains and the piecewise constant form of $\tilde{\Lambda}_i$.

Corollary 2.9: *Fix a domain $\tilde{\kappa}$. Then the values $\tilde{\Lambda}_i(p)$, $i = 1, \ldots, N$ are constant over $p \in \tilde{\kappa}$.*

We denote these constants

$$\tilde{\Lambda}_i(\tilde{\kappa}) := \tilde{\Lambda}_i(p) \text{ for any } p \in \tilde{\kappa}.$$

Definition 2.10: The *parameter space* $\bar{Z}$ associated to **RN** is the set of all $(l, u, \theta, \gamma) \in \mathbb{R}^D$ such that $0 < \gamma_i, 0 < \theta_{j,i}$ and $0 < l_{j,i} \leq u_{j,i}$. A parameter $z = (l, u, \theta, \gamma) \in \bar{Z}$ is *regular* if, in addition,

1. all of these inequalities are strict,
2. for each $y_i \in V'$, the thresholds $\{\theta_{j,i}\}$ are distinct, and
3. for any $\tilde{\kappa} \in \mathcal{K}(z)$, $-\gamma_i \theta_{j,i} + \tilde{\Lambda}_i(\tilde{\kappa}) \neq 0$ for each hyperplane $\{y_i = \theta_{j,i}\}$ bordering $\tilde{\kappa}$.

We denote the set of regular parameters of **RN** by Z.

Remark 2.11: Note that at any parameter $z \in \bar{Z}$, the range of $\tilde{\Lambda}_i$ is a finite set for each $y_i \in V'$. As a consequence, the property that $-\gamma_i\theta_{j,i} + \tilde{\Lambda}_i(\tilde{\kappa}) \neq 0$ is generic.

The following fact will be important in proofs in future sections.

Corollary 2.12: *Let $z \in Z$. Let $\tilde{\kappa}, \tilde{\kappa}'$ be two domains separated by a face $\tilde{\tau}$ on which $y_i = \theta_{j,i}$ with $i \neq j$. Then $\tilde{\Lambda}_i(\tilde{\kappa}) = \tilde{\Lambda}_i(\tilde{\kappa}') = \tilde{\Lambda}_i(p)$ for any $p \in \tilde{\tau}$.*

Proof: Since the thresholds of y_i are distinct, the threshold $\theta_{j,i}$ is associated to the unique edge $(\beta_i, \beta_j) \in E$. Therefore, only the switching function associated to y_j may change across $\tilde{\tau}$: $\tilde{\Lambda}_j(\tilde{\kappa}) \neq \tilde{\Lambda}_j(\tilde{\kappa}')$. Since $i \neq j$, $\tilde{\Lambda}_i$ is constant across $\tilde{\tau}$.

Definition 2.13: We define the *parameter space* $\bar{H}$ associated to an extended system to be the set of all $h = (z, a, b) \in \mathbb{R}^{|D'|}$ such that $z \in \bar{Z}$ and $0 < a^k_{j,i}$ and $0 < b^k_{j,i}$ for all $x^k_{j,i} \in X$. A parameter $h = (z, a, b) \in \bar{H}$ is *regular* if $z \in \bar{Z}$ is regular. We denote the set of regular parameters by H.

Note as in Remark 2.11 for the switching system that H is generic in $\bar{H}$.

2.5 Wall-labeling function

We now explain how to assign a uniform direction of flow across each face in the phase space of a canonical extension at a regular parameter.

Lemma 2.14: *Let* **EX** *be a canonical extension of* **RN** *with extended system* (5) *and regular parameter $h = (z, a, b)$. Let $\kappa, \kappa' \in \mathscr{K}(h)$ be extended domains such that there exists $\tau_y \subset \{y_i = \theta_{j,i}\}$ with (τ_y, κ) a left y-face and (τ_y, κ') a right y-face. Then,*

$$\Lambda_i(\kappa) = \Lambda_i(p) = \Lambda_i(\kappa') \textit{ for all } p \in \tau_y. \quad (11)$$

Proof: First consider an edge $(\beta_i, \beta_j) \in E$ such that $X_{j,i} = \varnothing$. From Definition 2.8, we know that there exists a fixed $\widehat{\Xi} := \prod_{s=1}^{N} \Xi(y_s)$ such that

$$\kappa = \tilde{\kappa} \times \widehat{\Xi}, \quad \kappa' = \tilde{\kappa}' \times \widehat{\Xi}, \quad \tau_y = \tilde{\tau} \times \widehat{\Xi}$$

for some domains $\tilde{\kappa}, \tilde{\kappa}' \in \mathscr{K}(z)$ that share the face $\tilde{\tau}$. Since $i \neq j$ in a canonical extension when $X_{j,i} = \varnothing$ and since z is regular by Definition 2.13 then Corollary 2.12 tells us

$$\tilde{\Lambda}_i(\tilde{\kappa}) = \tilde{\Lambda}_i(p) = \tilde{\Lambda}_i(\tilde{\kappa}') \text{ for all } p \in \tilde{\tau}. \quad (12)$$

We note that $\tilde{\Lambda}_i(\tilde{\kappa})$ is not guaranteed to equal $\Lambda_i(\kappa)$. However, in $\hat{\Xi}$, there are no $x^k_{i,s}$ at threshold. In other words, y_i cannot be regulated by any y_ℓ or $x^k_{i,s}$ on $\kappa \cup \tau_y \cup \kappa'$. Therefore (12) implies (11).

Now we consider an edge $(\beta_i, \beta_j) \in E$ with $X_{j,i} \neq \varnothing$ with a corresponding chain $C_{j,i} = \{(\beta_i, \alpha^1), (\alpha^1, \alpha^2), \ldots, (\alpha^n, \beta_j)\}$. Then on τ_y where $y_i = \theta_{j,i}$, only the variable x^1 is regulated because h is regular; in particular we use condition (2) of Definition 2.10. Therefore $\Lambda_s(\kappa) = \Lambda_s(p) = \Lambda_s(\kappa')$ for all $p \in \tau_y$ and for all $s = 1, \ldots, N$, a stronger result than required.

Theorem 2.15: *Let* **EX** *be a canonical extension where the associated extended system has a regular parameter* $h = (z, a, b) \in H$. *Let* τ *be a face on which* $\mu \in V' \cup X'$ *is constant. Then* $\operatorname{sgn}(\dot{\mu})$ *is constant and nonzero on* τ.

Proof: Let $\mu \in V'$, so that $\mu = y_i$ and $\mu|_\tau = \theta_{j,i}$ for some i, j. Let κ be such that (τ, κ) is a wall. Combining Lemma 2.14 with (6) evaluated on τ, we have

$$\dot{\mu}\,|_\tau = -\gamma_i\theta_{j,i} + \Lambda_i(\kappa).$$

In other words, $\dot{\mu}|_\tau$ exists and is constant. Since $h \in H$, $\gamma_i\theta_{j,i} \neq \Lambda_i(\kappa)$ and so $\dot{\mu}$ is also nonzero on τ.

Now let $\mu \in X'$, so that $\mu = x^k_{j,i}$ and $\mu\,|_\tau = \vartheta^k_{j,i}$ for some i, j, k. If $k = 1$, then from (7) evaluated on τ,

$$\operatorname{sgn}(\dot{\mu})\,|_\tau = \operatorname{sgn}(-a^1_{j,i}\vartheta^1_{j,i} + b^1_{j,i}y_i) \Rightarrow \operatorname{sgn}(\dot{\mu})\,|_\tau = \operatorname{sgn}(y_i - \theta_{j,i}),$$

using Definition 2.6. We have $\tau \cap \{y_i = \theta_{j,i}\} = \varnothing$ by Definition 2.8, which states that exactly one variable is at threshold on each face. Thus y_i remains strictly on one side of $\theta_{j,i}$ on τ, so that $\operatorname{sgn}(\dot{\mu}) = \operatorname{sgn}(y_i - - \theta_{j,i})$ is constant and nonzero on τ.

If $k > 1$, then from (8) evaluated on τ,

$$\operatorname{sgn}(\dot{\mu})\,|_\tau = \operatorname{sgn}(-a^k_{j,i}\vartheta^k_{j,i} + b^k_{j,i}x^{k-1}) \Rightarrow \operatorname{sgn}(\dot{\mu})\,|_\tau = \operatorname{sgn}(x^{k-1}_{j,i} - \vartheta^{k-1}_{j,i}).$$

Similarly, $\tau \cap \{x^{k-1}_{j,i} = \vartheta^{k-1}_{j,i}\} = \varnothing$ implies that $\operatorname{sgn}(\dot{\mu})$ is constant and nonzero on τ.

Theorem 2.15 allows us to introduce a well-defined wall labeling scheme that indicates a direction of flow on either side of each face.

Definition 2.16: Let $z \in \bar{Z}$ and let $\mathscr{W}(z)$ be the set of walls in the phase space associated to z. Then the *wall-labeling function* $\tilde{\ell}: \mathscr{W}(z) \to \{-1, 0, 1\}$ is

$$\widetilde{\ell}(\widetilde{\tau},\widetilde{\kappa}) := \operatorname{sgn}(\widetilde{\tau},\widetilde{\kappa}) \cdot \operatorname{sgn}(-\gamma_i\theta_{j,i} + \widetilde{\Lambda}_i(\widetilde{\kappa})),$$

where $\widetilde{\tau} \subset \{y_i = \theta_{j,i}\}$.

Definition 2.17: Let $h \in H$ for the extended system of a canonical extension **EX** and let $\mathscr{W}(h)$ be the set of walls in phase space. We define the *wall-labeling function* $\ell: \mathscr{W}(h) \to \{-1, 0, 1\}$ by

$$\ell(\tau,\kappa) := \operatorname{sgn}(\tau,\kappa) \cdot \operatorname{sgn}(\dot{\mu})\,|_{\tau},$$

where μ is constant on τ and

$$\operatorname{sgn}(\dot{\mu})\,|_{\tau} = \begin{cases} \operatorname{sgn}(-\gamma_i\theta_{j,i} + \Lambda_i(\kappa)) & \text{if } \mu = y_i \text{ and } \tau \subset \{y_i = \theta_{j,i}\} \\ \operatorname{sgn}(y_i - \theta_{j,i})\,|_{\tau} & \text{if } \mu = x^1_{j,i} \text{ and } \tau \subset \{x^1_{j,i} = \vartheta^1_{j,i}\} \\ \operatorname{sgn}(x^{k-1}_{j,i} - \vartheta^{k-1}_{j,i})\,|_{\tau} & \text{if } \mu = x^k_{j,i} \text{ and } k > 1 \text{ and } \tau \subset \{x^k_{j,i} = \vartheta^k_{j,i}\} \end{cases}$$

A wall (τ, κ) is an *outgoing wall* if $\ell(\tau, \kappa) = -1$, an *incoming wall* if $\ell(\tau, \kappa) = 1$, and a *tangential* wall if $\ell(\tau, \kappa) = 0$. A domain κ is an *attracting* domain if every (τ, κ) is an incoming wall.

Observe that Definition 2.17 for an extended system reduces to Definition 2.16 for a switching system, when $X_{j,i} = \varnothing$ for all $(\beta_i, \beta_j) \in E$.

Remark 2.18: Note that Definitions 2.10 and 2.16 imply that there are no tangential walls for any regular parameter $z \in Z$ in the switching system. Theorem 2.15 says that $\operatorname{sgn}(\dot{\mu})/_{\tau} \neq 0$ for $h \in H$ a regular parameter and **EX** a canonical extension, so there are no tangential walls in the extended system either.

A canonical extension has an important additional property that every face is an outgoing wall on one side and an incoming wall on the other, as we now prove. Existence of white and black walls in the switching system shows that this result is not true for the switching system.

Corollary 2.19: *Let $h \in H$ be a regular parameter for the extended system of a canonical extension* **EX**. *Then for any x- or y-face τ between adjacent domains $\kappa, \kappa' \in \mathscr{K}(h)$, we have that*

$$\ell(\tau,\kappa) \cdot \ell(\tau,\kappa') = -1.$$

Proof: Let $\mu \in V \cup X$ be constant on τ. If (τ, κ) is a right face, it is necessary that (τ, κ') is a left face, so that $\mathrm{sgn}(\tau, \kappa) = -\mathrm{sgn}(\tau, \kappa')$. Since $\mathrm{sgn}(\dot{\mu})$ is constant and nonzero on τ by Theorem 2.15, we have that $\ell(\tau, \kappa) \cdot \ell(\tau, \kappa') = -1$.

The consequence of Corollary 2.19 is that flow is transverse across all faces in a canonical extension, as can be seen in phase space in Figures 2b and 2d.

2.6 Domain graph and Morse graph

We now construct graphs that represent the coarse dynamics of (3) and (5). We first build a state transition diagram, which we call a domain graph, and take the strongly connected components (recurrent components) as a Morse decomposition to build a Morse graph.

Definition 2.20: For each parameter $z \in Z$, we construct a *domain graph* $(\mathscr{V}, \mathscr{E}_Z)$ with vertex set $\mathscr{V}$ which is in one-to-one correspondence with the domains $\mathscr{K}(z)$ via a bijection $f_Z: \mathscr{K}(z) \to \mathscr{V}$. The edges $(u, v) \in \mathscr{E}_Z$ are defined via a multivalued map

$$\mathscr{F}_z : \mathscr{V} \rightrightarrows \mathscr{V}$$

where $v \in \mathscr{F}_Z(u)$ if and only if there is a directed edge $u \to v$ in the domain graph, $(u, v) \in \mathscr{E}_Z$. The map $\mathscr{F}_Z$ is defined as follows

1. if $u = v$, then $v \in \mathscr{F}_Z(u)$ if and only if $\tilde{\kappa} = f_z^{-1}(u)$ is attracting, or
2. if $u \neq v$, then $v \in \mathscr{F}_Z(u)$ if and only if there exists a face $\tilde{\tau}$ such that $(\tilde{\tau}, f_z^{-1}(u))$ is an outgoing wall and $(\tilde{\tau}, f_z^{-1}(v))$ is an incoming wall.

Using the wall-labeling function ℓ, we can now construct a domain graph associated to the extended system analogous to the domain graph $(\mathscr{V}, \mathscr{E}_Z)$ for the switching system of Definition 2.20.

Definition 2.21: Recall that there is a bijection $f_Z: \mathscr{K}(z) \to \mathscr{V}$ between the set of domains $\mathscr{K}(z)$ and the vertices $\mathscr{V}$ of the domain graph for the switching system. Each extended domain $\kappa \in \mathscr{K}(h)$ has the form

$$\kappa := \tilde{\kappa} \times \prod_{s=1}^{N} \Xi(y_s)$$

for some $\tilde{\kappa} \in \mathscr{K}(z)$, where $h = (z, a, b)$. We define a set of vertices for the domain graph of the extended system as

$$\mathfrak{X} := \mathscr{V} \times \{-1, 1\}^{|X|}.$$

Then there is bijection f_h: $\mathscr{K}(h) \to \mathfrak{X}$ defined by

$$f_h(\kappa) = (f_z(\tilde{\kappa}), D)$$

where $D \in \{-1, 1\}^{|X|}$ satisfies

$$D(i,j,k) = -1 \text{ if } x_{j,i}^{k} < \vartheta_{j,i}^{k} \text{ on } \kappa$$
$$D(i,j,k) = +1 \text{ if } x_{j,i}^{k} > \vartheta_{j,i}^{k} \text{ on } \kappa\,.$$

We define the *domain graph* $(\mathfrak{X}, \mathscr{E}_h)$ induced by the wall-labeling ℓ at the regular parameter $h \in H$ via a multivalued map

$$\mathscr{F}_h : \mathfrak{X} \rightrightarrows \mathfrak{X},$$

where an edge from $u \in \mathfrak{X}$ to $v \in \mathfrak{X}$ exists, $(u, v) \in \mathscr{E}_h$, if and only if $v \in \mathscr{F}_h(u)$. To define the multivalued map $\mathscr{F}_h$, we let $\kappa, \kappa' \in \mathscr{K}(h)$ and $(\xi, D) := f_h(\kappa)$, $(\xi', D') := f_h(\kappa')$ respectively. We say $(\xi, D) \in \mathscr{F}_h((\xi', D'))$ if and only if one of the following holds:

- $\kappa = \kappa'$ is an attracting extended domain, or
- $\kappa \neq \kappa'$ and there exists a y- or x-face τ such that $\ell(\tau, \kappa) = 1$ and $\ell(\tau, \kappa') = -1$.

Figure 3(a) and (c) show domain graphs for extended systems in Figure 2.

Definition 2.22: A *recurrent component* of a directed graph $\mathscr{G}$ is a maximal subgraph $\mathscr{C}$ of $\mathscr{G}$ such that for any vertices $u, v \in \mathscr{C}$, there exists a nonempty path in $\mathscr{C}$ from u to v.

We refer to a recurrent component of $(\mathscr{V}, \mathscr{E}_z)$ as a *Morse set* of $(\mathscr{V}, \mathscr{E}_z)$. Consider the collection of all Morse sets of $(\mathscr{V}, \mathscr{E}_z)$. Observe that there exists a partial order on $(\mathscr{V}, \mathscr{E}_z)$ arising from the reachability relation on $(\mathscr{V}, \mathscr{E}_z)$. We define an indexing set P for the elements of $(\mathscr{V}, \mathscr{E}_z)$, respecting this partial order. In particular, for any $p, q \in$ P, $q \leq p$ if and only if there exists a path in $(\mathscr{V}, \mathscr{E}_z)$ from an element of $\mathscr{C}_p$ to an element of $\mathscr{C}_q$. The collection of Morse sets of $(\mathscr{V}, \mathscr{E}_z)$ together with this partial order is called a *Morse decomposition* of $(\mathscr{V}, \mathscr{E}_z)$, which we denote by $(\mathrm{MD}((\mathscr{V}, \mathscr{E}_z)), \leq)$, or when the order is clear, by $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$.

The *Morse graph* of $(\mathscr{V}, \mathscr{E}_z)$, denoted $\mathrm{MG}((\mathscr{V}, \mathscr{E}_z))$, is the Hasse diagram of the poset $(\mathrm{MD}((\mathscr{V}, \mathscr{E}_z)), \leq)$. For this reason we also call the elements of $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ *Morse nodes* of $(\mathscr{V}, \mathscr{E}_z)$.

Using the map $\mathscr{F}_h$, we define the Morse decomposition and Morse graph of $(\mathfrak{X}, \mathscr{E}_h)$, $\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ and $\mathrm{MG}((\mathfrak{X}, \mathscr{E}_h))$ respectively, exactly as in Definition 2.22 by replacing $(\mathscr{V}, \mathscr{E}_z)$ by $(\mathfrak{X}, \mathscr{E}_h)$.

The Morse graph is a compact representation of the dynamics in phase space. The nodes of the graph represent the recurrent-like dynamics and the directed edges represent the gradient-like dynamics. Figures 3f and 3h show Morse graphs for the extended phase spaces in Figures 2b and 2d. The labels on these nodes are explained in Definition 4.1.

2.7 Combinatorial parameter graph

2.7.1 Switching system—For each parameter $z \in Z$, there is a Morse graph representing the coarse dynamics of the switching system (3). Furthermore, there is a finite partition of the parameter space $z \in Z$ into semi-algebraic sets such that within each component the Morse graph is constant [11]. This decomposition can be represented in the form of a *geometric parameter graph* and its combinatorial representation, a *combinatorial parameter graph*.

For clarity of presentation, we reproduce the construction of the latter from [11].

Definition 2.23: Define the *input combinations* of $\beta_j \in V$ to be the Cartesian product

$$\mathrm{ln}_j := \prod_{\beta_i \in S(\beta_j)} \{\mathrm{off}, \mathrm{on}\}.$$

Define the *valuation function* v_j: $\mathrm{In}_j \to \mathbb{R}^{|S(\beta_j)|}$ via

$$v_{j,i}(A) = \begin{cases} l_{j,i} & \text{whenever } A_i = \mathrm{off} \\ u_{j,i} & \text{whenever } A_i = \mathrm{on} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now define the *output combinations* of the node β_j to be the set

$$\mathrm{Out}_j := \{i \mid (\beta_j, \beta_i) \in E\}.$$

Definition 2.24: *A logic parameter* is a function

$$L: \coprod_{i=1}^{N} (\mathrm{In}_i \times \mathrm{Out}_i) \to \{-1, 1\}.$$

We denote the restriction of L onto $\mathrm{In}_i \times \mathrm{Out}_i$ as L_i. An *order parameter O* is a collection of total orderings O_i of Out_i for each $y_i \in V$. A *combinatorial parameter* is a pair $\phi = (L,O)$ where L is a logic parameter and O is an order parameter. We denote the collection of combinatorial parameters as Φ. The *combinatorial assignment function $\omega: Z \to \Phi$* is given by $\omega(z) := (L,O)$ where $O = O(z)$ is the total order of thresholds given by z and

$$L_i(A,B) = \operatorname{sgn}\left(M_i \circ v_i(A) - \gamma_i \theta_{B,i}\right) \text{ for all } 1 \le i \le N, \quad (13)$$

where M_i is the multilinear function from (3). For all $z \in Z$, we say that $\omega(z)$ is *the combinatorial parameter associated to the parameter z*. A combinatorial parameter $\phi \in \Phi$ is *realizable in Z* if there exists $z \in Z$ such that $\phi = \omega(z)$.

We impose an adjacency relation between combinatorial parameters to arrange them into a graph.

Definition 2.25: Let $\phi = (L,O)$ and $\phi' = (L',O')$ be two distinct combinatorial parameters in Φ. Denote the domain of the logic parameters by $\mathscr{D} := \coprod_{i=1}^{N} (\mathrm{In}_i \times \mathrm{Out}_i)$. Then ϕ and ϕ' are *adjacent* if and only if one of the following holds.

1. $O = O'$ and there exists $d \in \mathscr{D}$ such that L and L' are equal on all of $\mathscr{D}$ except at d.
2. $L = L'$, $O_i = O_i'$ for $i = 1, 2, \ldots, i^* - 1, i^* + 1, \ldots, N$ and the total orders O_{i^*} and O'_{i^*} differ only by a single swap of the ordering of consecutive thresholds.

The *combinatorial parameter graph* CPG_Z is the undirected graph on the realizable combinatorial parameters with an edge between two parameter nodes ϕ and ϕ' if and only if they are adjacent.

We now show that given a combinatorial parameter $\phi = (L,O)$, every z in its preimage $\omega^{-1}(\phi)$ is associated to the same domain graph and Morse graph. We start by defining a vector-valued *indicator* function $\tilde{\chi}_j: (0,\infty)^N \to \mathrm{In}_j$ such that the i^{th} component of $\tilde{\chi}_j$ is given by

$$\tilde{\chi}(y_j, y_i; \theta_{j,i}) = \begin{cases} \text{off} & \text{if } \beta_i \to \beta_j \text{ and } y_i < \theta_{j,i} \text{ or if } \beta_i \dashv \beta_j \text{ and } y_i > \theta_{j,i} \\ \text{on} & \text{if } \beta_i \to \beta_j \text{ and } y_i > \theta_{j,i} \text{ or if } \beta_i \dashv \beta_j \text{ and } y_i < \theta_{j,i} \\ \text{undefined} & \text{otherwise} \end{cases} \quad (14)$$

Note that with this definition, we have

$$\tilde{\sigma}_j = v_j \circ \tilde{\chi}_j.$$

So the valuation function v_j uses binary inputs provided by $\tilde{\chi}_j$ to define the values of $\tilde{\sigma}_j$. The binary inputs describe the relative position of y_i with respect to the thresholds $\theta_{j,i}$. Notice that $\tilde{\chi}_i$ is constant across a domain $\tilde{\kappa}$, so that $\tilde{\chi}_i(\tilde{\kappa})$ is well-defined, and we can associate to each $\tilde{\kappa}$ and i the value

$$L_i(\widetilde{\chi}_i(\widetilde{\kappa}), j) = \operatorname{sgn}(\widetilde{\Lambda}_i(\widetilde{\kappa}) - \gamma_i\theta_{j,i}) \quad (15)$$

Definition 2.26: For $z \in Z$ with $\omega(z) = \phi$ and $(\tilde{\tau}, \tilde{\kappa}) \in \mathscr{W}(z)$ with $\tilde{\tau} \subset \{y_i = \theta_{j,i}\}$, define the *combinatorial wall labeling* $\tilde{\mathscr{L}}: \mathscr{W}(z) \to \{-1, 0, 1\}$ for ϕ by

$$\widetilde{\mathscr{L}}(\widetilde{\tau}, \widetilde{\kappa}) = \operatorname{sgn}(\widetilde{\tau}, \widetilde{\kappa}) \cdot L_i(\widetilde{\chi}_i(\widetilde{\kappa}), j).$$

With this definition, the following is immediate:

Theorem 2.27: *Let $z \in Z$ with $\omega(z) = \phi$ and $(\tilde{\tau}, \tilde{\kappa}) \in \mathscr{W}(z)$ with $\tilde{\tau} \subset \{y_i = \theta_{j,i}\}$. Then $\tilde{\ell}(\tilde{\tau}, \tilde{\kappa}) = \tilde{\mathscr{L}}(\tilde{\tau}, \tilde{\kappa})$. That is, the wall-labeling functions for a parameter and its associated combinatorial parameter are identical over phase space.*

2.7.2 Extended system—We define a combinatorial parameter graph for **EX** in a way analogous to that of **RN**. The results of this section are a consequence of an observation that the parameters a and b in $h = (z, a, b)$ do not affect the combinatorial parameter associated to z. This is because every $x_{j,i}^k$ has only a single threshold $\vartheta_{j,i}^k$ and no associated values $l_{j,i}^k$ and $u_{j,i}^k$.

Note that In_i and v_i only depend on the number of sources of the node β_i, and Out_i only depends on the number of targets of β_i. Therefore In_i, Out_i and v_i are the same in the original and the extended system for every β_i. It follows that the set of combinatorial parameters Φ is the same in the original and the extended system.

Definition 2.28: We define the *combinatorial assignment function* $\psi: H \to \Phi$ for **EX** by

$$\psi := \omega \circ \pi \quad (16)$$

where $\pi: H \to Z$ is the projection $(z, a, b) \mapsto z$, and $\omega: Z \to \Phi$ is the combinatorial assignment function for the original system. Again we say a combinatorial parameter ϕ is *realizable in H* if there exists $h \in H$ such that $\phi = \psi(h)$.

The *combinatorial parameter graph* for **EX**, denoted CPG_H (as opposed to CPG_Z), is the undirected graph on the combinatorial parameters realizable in **EX** with an edge between two combinatorial parameters $\phi, \phi' \in \Phi$ if and only if ϕ and ϕ' are adjacent as in Definition 2.25.

Lemma 2.29: *A combinatorial parameter $\phi \in \Phi$ is realizable in Z if and only if it is realizable in H.*

Proof: Let $\phi \in \Phi$ be realizable in Z. Then there exists $z \in Z$ such that $\phi = \omega(z)$. Let $h := (z, 1, 1)$, by which we mean that we set $a^k_{j,i} = b^k_{j,i} = 1$ for all i, j, k where $X_{j,i} \neq \emptyset$. Clearly, $h \in H$ is regular. By (16),

$$\psi(h) = \omega(z) = \phi\,.$$

Therefore ϕ is realizable in H.

Let $\phi \in \Phi$ be realizable in H. Then there exists $h = (l, u, \theta, \gamma, a, b)$ such that $\phi = \psi(h)$. Let $z := (l, u, \theta, \gamma)$. By Definition 2.13, z is regular. By (16),

$$\omega(z) = \omega \circ \pi(h) = \psi(h) = \phi \in \Phi\,.$$

Therefore ϕ is realizable in Z.

Theorem 2.30: $\mathrm{CPG}_{\mathrm{Z}}$ *and* $\mathrm{CPG}_{\mathrm{H}}$ *are isomorphic.*

Proof: Recall that CPG_Z and CPG_H are undirected graphs on the same realizable subset of Φ by Lemma 2.29. The rules of adjacency are identical and depend only on the combinatorial parameter (see Definition 2.25).

We illustrate Theorem 2.30 on our running examples in Figure 1: the combinatorial parameter graphs are identical for the switching systems and their canonical extensions in Figure 1. This parameter graph is depicted in Figure 4. All three combinatorial parameters in Figure 4 are realizable. For example, if we take $\gamma_1 = 1$, $l_{1,1} = 2$ and $u_{1,1} = 4$, then the choices $\theta_{1,1} = 5$, $\theta_{1,1} = 3$, and $\theta_{1,1} = 1$ define regular realizations of the combinatorial parameters for vertices from left to right in Figure 4.

The phase spaces in Figure 2 and the domain and Morse graphs in Figure 3 correspond to the middle combinatorial parameter in Figure 4.

Remark 2.31: We emphasize that although the combinatorial parameter graphs are isomorphic, the associated Morse graphs of the switching and extended systems are not in general isomorphic. That is, given a parameter node in CPG_Z and the corresponding node in CPG_H under isomorphism, the coarse dynamics of the systems can differ at this parameter node. This means that choosing to use the extended system over the switching system may result in nontrivial changes in the long term dynamics. See for example the difference between Figures 3c, 3d and Figures 3g, 3h.

It remains to show that if $\psi(h) = \phi$, then we can recover the wall labeling associated to the parameter $h \in H$ using only $\phi \in \mathrm{CPG}_H$ in the same way that we recovered this information from $\phi \in \mathrm{CPG}_Z$ in Theorem 2.27. This implies that the combinatorial parameters obtained from ψ are sufficient to recover all information necessary to construct a domain graph for the extended system.

We begin by defining an indicator function for the extension, as in (14).

Definition 2.32: Let $C_{j,i} \subset E_X$ be a chain from β_i to β_j, and let $\chi_j = (\chi_{j,1}, \chi_{j,2}, \ldots, \chi_{j,N})$ be the *extended indicator function*, which we define here. If $X_{j,i} = \varnothing$, then

$$\chi_{j,i} := \widetilde{\chi}(y_j, y_i; \theta_{j,i}),$$

referring back to (14). If $X_{j,i}$ is non-empty, denote the last intermediary by x^n with artificial threshold ϑ^n. Then

$$\chi_{j,i} := \widetilde{\chi}(y_j, x^n; \vartheta^n).$$

If there is no edge $(\beta_i, \beta_j) \in E$, then $\chi_{j,i}$ is undefined.

As in (15), we note that χ_i is constant over any $\kappa \in \mathscr{K}(h)$, and

$$L_i(\chi_i(\kappa), j) = \operatorname{sgn}(\Lambda_i(\kappa) - \gamma_i \theta_{j,i}). \quad (17)$$

Using this fact, we define a new wall labeling function similar to that in Definition 2.26.

Definition 2.33: Let $(L, O) \in \Phi$ be realizable by $h \in H$ such that $O(h) = O$. Let $(\tau, \kappa) \in \mathscr{W}(h)$ be such that τ is a face of κ on which $\mu \in V' \cup X'$ is constant.

$$\mathscr{L}(\tau,\kappa) := \operatorname{sgn}(\tau,\kappa) \cdot \begin{cases} L_i(\chi_i(\kappa), j) & \text{if } \mu = y_i \text{ and } \tau \subset \{y_i = \theta_{j,i}\} \\ \operatorname{sgn}(y_i - \theta_{j,i})\,|_\tau & \text{if } \mu = x^1_{j,i} \text{ and } \tau \subset \{x^1_{j,i} = \vartheta^1_{j,i}\} \\ \operatorname{sgn}(x^{k-1}_{j,i} - \vartheta^{k-1}_{j,i})\,|_\tau & \text{if } \mu = x^k_{j,i} \text{ and } k > 1 \text{ and } \tau \subset \{x^k_{j,i} = \vartheta^k_{j,i}\} \end{cases}$$

Then by (17), the following is immediate.

Theorem 2.34: *Let $h \in H$ and let $\phi = \psi(h)$. Then if $(\tau, \kappa) \in \mathscr{W}(h)$, we have $\ell(\tau, \kappa) = \mathscr{L}(\tau, \kappa)$.*

2.8 Conclusion

Using the wall-labeling function ℓ, we constructed a domain graph $(\mathfrak{X}, \mathscr{E}_h)$ as in Definition 2.20, substituting a parameter $h \in H$, and extended walls and domains for switching walls and domains (see Definition 2.21). From this domain graph we can extract Morse graphs as in Definition 2.22. Each node in the isomorphic combinatorial parameter graphs is associated to a Morse graph. Therefore we can compare Morse graphs between isomorphic nodes in CPG_Z and CPG_H. In Section 3, we will explore the relationship between the Morse graphs of the original and extended system in detail.

3 Morse Graph Theorems

Throughout the section, we will fix a regular parameter $z \in Z$ for a switching system and $h = (z, a, b) \in H$ for the associated canonical extension. We will explore the relationship between the Morse decompositions of $(\mathscr{V}, \mathscr{E}_z)$ and $(\mathfrak{X}, \mathscr{E}_h)$. Our main result is the existence of an order preserving map of $(\mathrm{MD}((\mathscr{V}, \mathscr{E}_z)),\leq)$ into $(\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h)),\leq)$. In Section 4, we will provide several counterexamples to potential stronger relationships between the Morse decompositions.

Definition 3.1: For each node in the domain graph of the switching system, $\xi \in \mathscr{V}$, we define the *fiber over* ξ to be the set

$$\{(\xi, D) \mid D \in \{-1, 1\}^{|X|}\} \subset \mathfrak{X}.$$

For the two self-loops (Figures 1a and 1c) there are two domains, $(0, \theta_{1,1})$ and $(\theta_{1,1},\infty)$ in phase space, as can be seen in Figures 2a and 2c. Therefore two nodes, ξ_1 and ξ_2, in the respective domain graphs (Figures 3a and 3c. Hence in Figures 3e and 3g the fibers over ξ_1 and ξ_2 are $\{(\xi_1, 1), (\xi_1,-1)\}$ and $\{(\xi_2, 1), (\xi_2,-1)\}$, respectively.

Definition 3.2: For each $\xi \in \mathscr{V}$, we define the **-element* (ξ,D^*) of the fiber over ξ in the following way. Pick $D^* \in \{-1, 1\}^{|X|}$ such that for each $x^k_{j,i} \in X'$,

$$\begin{aligned}&\text{if } y_i < \theta_{j,i} \text{ on } f_z^{-1}(\xi), \text{ then } x^k_{j,i} < \vartheta^k_{j,i} \text{ on } f_h^{-1}((\xi, D^*))\\ &\text{if } y_i > \theta_{j,i} \text{ on } f_z^{-1}(\xi), \text{ then } x^k_{j,i} > \vartheta^k_{j,i} \text{ on } f_h^{-1}((\xi, D^*))\end{aligned}$$

Note that this is equivalent to the following by Definition 2.21.

$$\begin{aligned}&\text{if } y_i < \theta_{j,i} \text{ on } f_z^{-1}(\xi), \text{ then } D^*(i, j, k) = -1\\ &\text{if } y_i > \theta_{j,i} \text{ on } f_z^{-1}(\xi), \text{ then } D^*(i, j, k) = +1\end{aligned}$$

In our canonical examples in Figures 3e and 3g, the *-elements over ξ_1 and ξ_2 are respectively $(\xi_1,-1)$ and $(\xi_2, 1)$.

We now prove that the wall-labeling of $f_h^{-1}((\xi, D^*))$ is preserved from $f_z^{-1}(\xi)$ in the switching system.

Lemma 3.3: *Let* $\xi \in \mathscr{V}$ *with* $\tilde{\kappa} = f_z^{-1}(\xi)$ *and let* $\kappa = f_h^{-1}((\xi, D^*)) = \tilde{\kappa} \times \hat{\Xi}$ *for some* $\hat{\Xi}$. *Let* $\tau = \tilde{\tau} \times \hat{\Xi}$ *where* $\tilde{\tau}$ *is a face of* $\tilde{\kappa}$. *Then*

$$\tilde{\ell}(\tilde{\tau}, \tilde{\kappa}) = \ell(\tau, \kappa).$$

Proof: By Definitions 2.5 and 2.8, there exist $y_i \in V'$ and $y_j \in T(y_i)$ such that $y_i = \theta_{j,i}$ on $\tilde{\tau}$ and τ. By Definitions 2.16 and 2.17, it suffices to show that sgn($\tilde{\tau}$, $\tilde{\kappa}$) = sgn(τ, κ) and $\tilde{\Lambda}_i(\tilde{\kappa}) = \Lambda_i(\kappa)$.

It is clear from Definition 2.8 that $\tilde{\tau}$ is a left face of $\tilde{\kappa}$ if and only if τ is a left y-face of κ, and $\tilde{\tau}$ is a right face of $\tilde{\kappa}$ if and only if τ is a right y-face of κ. Thus sgn($\tilde{\tau}$, $\tilde{\kappa}$) = sgn(τ, κ).

For each $y_s \in S(y_i)$, if $X_{i,s} = \emptyset$, recall that $\tilde{\Lambda}_i(\tilde{\kappa}) = \Lambda_i(\kappa)$. If $X_{i,s}$ is nonempty denote the final intermediary in the chain $C_{i,s}$ by x^{n_s} and the associated threshold by ϑ^{n^s}. By Definition 3.2, if $y_s < \theta_{i,s}$ on $\tilde{\kappa}$ then $x^{n_s} < \vartheta^{n_s}$ on κ and if $y_s > \theta_{i,s}$ on $\tilde{\kappa}$ then $x^{n_s} > \vartheta^{n_s}$ on κ. Recalling the definitions of $\tilde{\Lambda}_i$ and Λ_i, it is now clear that $\tilde{\Lambda}_i(\tilde{\kappa}) = \Lambda_i(\kappa)$.

Although in the above proof we only use the fact that the terminal signs match (sgn($y_s - \theta_{i,s}$) = sgn($x^{n_s} - \vartheta^{n_s}$)), the proof of Lemma 3.6 requires that all signs match in the *-element.

Lemma 3.3 implies that the labelings of the set of y-walls of $\tilde{\kappa} \times \hat{\Xi} = f_h^{-1}((\xi, D^*))$ are fully determined by the labelings of the set of walls of $\tilde{\kappa} = f_z^{-1}(\xi)$. Lemma 3.4 relies on similar reasoning.

Recall the multi-valued map $\mathcal{F}_h$: $\mathfrak{X} \rightrightarrows \mathfrak{X}$ where the edge $u \in \mathfrak{X}$ to $v \in \mathfrak{X}$ exists if and only if $v \in \mathcal{F}_h(u)$. In what follows, we will use the notation $(u \to v) \in \mathcal{E}_h$ when we want to emphasize the graph theoretic point of view of the dynamics.

Lemma 3.4: *If $(\xi \to \xi') \in \mathcal{E}_z$ and (ξ, D^*) is the *-element of the fiber over ξ, then $\mathcal{E}_h$ contains the edge*

$$(\xi, D^*) \to (\xi', D^*).$$

Proof: Let $\tilde{\kappa} := f_z^{-1}(\xi)$ and $\tilde{\kappa}' := f_z^{-1}(\xi')$. Because $(\xi \to \xi') \in \mathcal{E}_z$, $\tilde{\kappa}$ and $\tilde{\kappa}'$ share a face $\tilde{\tau}$ such that $\tilde{\ell}(\tilde{\tau}, \tilde{\kappa}) = -1$. If $\kappa := f_h^{-1}((\xi, D^*))$ and $\kappa' := f_h^{-1}((\xi', D^*))$, then κ and κ' share a y-face $\tau = \tilde{\tau} \times \hat{\Xi}$, and by Lemma 3.3, $\ell(\tau, \kappa) = -1$. Now, by Corollary 2.19, $\ell(\tau, \kappa') = 1$, so the result follows.

The assumption of Lemma 3.4 is not satisfied in our ongoing switching system examples from Figures 1a and 1c, since the single face $\{\theta_{1,1}\}$ in the phase space does not admit a transverse flow, as can be seen by the inward and outward pointing arrows in the phase lines in Figures 2a and 2c. This results in a domain graph with no edges, as can be seen in Figure 3c.

Having established properties of the y-walls of the *-element, we now address the x-walls.

Definition 3.5: We say $(\xi, D) \in \mathfrak{X}$ is *attracting in the fiber* if every x-wall (τ_x, κ) is an incoming wall for $\kappa := f_h^{-1}((\xi, D))$.

Lemma 3.6: *The *-element of the fiber is attracting in the fiber.*

Proof: It is sufficient to show the following: let $\xi \in \mathscr{V}$ and $D \in \{-1, 1\}^{|X|}$ be such that $\kappa' := f_h^{-1}((\xi, D))$ shares an x-face τ_x with $\kappa := f_h^{-1}((\xi, D^*))$. Then $(\xi, D^*) \in \mathscr{F}_h((\xi, D))$. Our hypotheses imply there exist i, j, k such that $x^k \in X'_{j,i}$ with associated threshold ϑ^k satisfying either $x^k < \vartheta^k$ on κ and $x^k > \vartheta^k$ on κ' or vice versa. Assume the first case for now. Then τ_x is a right face of κ and a left face of κ', so $\mathrm{sgn}(\tau_x, \kappa) = -1$ and $\mathrm{sgn}(\tau_x, \kappa') = 1$.

Now, by Definition 3.2, since $x^k < \vartheta^k$ on κ, we also have $y_i < \theta_{j,i}$ on κ. If $k > 1$, then $x^{k-1} < \vartheta^{k-1}$ on κ as well. Observe that because κ and κ' share an x-face, all variables in $(V' \cup X') \setminus \{x^k\}$ are fixed between the same respective pair of consecutive thresholds on κ' as on κ. Thus on both κ and κ', $\mathrm{sgn}(y_i - \theta_{j,i}) = -1$ and if x^{k-1} exists, then $\mathrm{sgn}(x^{k-1} - \vartheta^{k-1}) = -1$. By Definition 2.17, this means $\ell(\tau_x, \kappa) = 1$ and $\ell(\tau_x, \kappa') = -1$. The result now follows from Definition 2.21.

If instead $x^k > \vartheta^k$ on κ and $x^k < \vartheta^k$ on κ', the argument is analogous.

Recall that in Figures 3e and 3g the *-elements of the domain graphs are given by $(\xi_1, -1)$ and $(\xi_2, 1)$, respectively. As predicted by Lemma 3.6, we see the edges $(\xi_1, 1) \rightarrow (\xi_1, -1)$ and $(\xi_2, -1) \rightarrow (\xi_2, 1)$, showing attractiveness in the fiber.

We will now state another property of the *-element in the fiber over a particular $\xi \in \mathscr{V}$, namely that this element is reachable from every other element of the same fiber via a path contained in that fiber. The proof of Lemma 3.7 is given in Appendix B.

Lemma 3.7: *Let $\xi \in \mathscr{V}$. For each element (ξ, D) in the fiber over ξ, there exists a path from (ξ, D) to (ξ, D^*) such that for each (ξ', D') in the path, we have $\xi = \xi'$.*

Lemma 3.7 investigates how the domains in the fiber over a single domain ξ are connected. Clearly, in addition to connections within the fiber, there are additional connections between these and domains in neighboring fibers that differ in a y-coordinate.

Lemmas 3.3, 3.4, 3.6, and 3.7 establish key properties of the *-element of each fiber. It will become important that the *-element is the unique element of the fiber with the latter two properties (see Theorem 3.12 below). The next two results show that this is indeed the case.

Corollary 3.8: *The *-element is the unique element in the fiber satisfying Lemma 3.7*

Proof: By Lemma 3.6, the *-element is attracting in the fiber. Therefore there is no path restricted to the fiber from (ξ, D^*) to any other element (ξ, D).

Corollary 3.9: *The *-element is the unique element in the fiber that is attracting in the fiber.*

Proof: Because every element $(\xi, D) \neq (\xi, D^*)$ has a path to (ξ, D^*) by Lemma 3.7, each such (ξ, D) must have an outgoing x-wall. Hence it is not attracting in the fiber.

Definition 3.10: A recurrent component $\mathscr{C}$ in $(\mathrm{MD}(G), \leq)$, where G is either $(\mathscr{V}, \mathscr{E}_z)$ or $(\mathfrak{X}, \mathscr{E}_h)$, is called a *fixed point* if it contains a single node ξ. A recurrent component is an *attractor* if there is no $\mathscr{C}' \in \mathrm{MD}(G)$ such that $\mathscr{C}'$ is reachable from $\mathscr{C}$.

Lemma 3.11: *Every fixed point in* $(\mathrm{MD}(G), \leq)$, *where G is either* $(\mathscr{V}, \mathscr{E}_z)$ *or* $(\mathfrak{X}, \mathscr{E}_h)$, *is an attractor.*

Proof: This is immediate from Definitions 2.20 and 2.21, because $\{u\}$ a recurrent component in $\mathrm{MD}(G)$ is equivalent to $u \in \mathcal{F}(u)$, where $\mathcal{F}$ is either $\mathcal{F}_z$ or $\mathcal{F}_h$. This in turn is equivalent to $f^{-1}(u)$ attracting, where f is either f_z or f_h.

Theorem 3.12: *There exists a fixed point* $\mathscr{C} = \{\xi\} \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ *if and only if* $\hat{\mathscr{C}} = \{(\xi, D^*)\} \in \mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ *is a fixed point. If there exists a fixed point* $\mathscr{C} = \{(\xi, D)\}$ *in any fiber, it is the unique fixed point in that fiber.*

Proof: Let ξ be the sole element of $\mathscr{C} \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ and let $\tilde{\kappa} := f_z^{-1}(\xi)$ and $\kappa := f_h^{-1}(\xi, D^*)$. Then $\xi \in \mathcal{F}_z(\xi)$, and therefore $\tilde{\kappa}$ is an attracting domain. Therefore, for any face $\tilde{\tau}$ of $\tilde{\kappa}$, $\tilde{\ell}(\tilde{\tau}, \tilde{\kappa}) = 1$. Then by Lemma 3.3, for each y-face τ of κ, $\ell(\tau, \kappa) = 1$. Furthermore, Lemma 3.6 implies that for each x-face τ_x of κ, $\ell(\tau_x, \kappa) = 1$. Thus κ is an attracting extended domain, so $(\xi, D^*) \in \mathcal{F}_h((\xi, D^*))$.

Now conversely let $\hat{\mathscr{C}} \in \mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ have a single element (ξ, D). By Lemma 3.9, the *-element is the only element in the fiber that is attracting in the fiber. Therefore since $\hat{\mathscr{C}} \in \mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ has a single element (ξ, D), we have $D = D^*$, giving uniqueness. Let $\kappa := f_h^{-1}(\xi, D^*)$ and $\tilde{\kappa} = f_z^{-1}(\xi)$. We have that κ is an attracting extended domain by Definition 2.21 and so $\ell(\tau, \kappa) = 1$ for every face τ of κ. In particular, this holds for each y-face of κ. Then by Lemma 3.3, for each face $\tilde{\tau}$ of $\tilde{\kappa}$, $\tilde{\ell}(\tilde{\tau}, \tilde{\kappa}) = 1$, so $\tilde{\kappa}$ is an attracting domain. Thus $\xi \in \mathcal{F}_z(\xi)$, so the result follows.

Definition 3.13: Consider domain graphs $(\mathscr{V}, \mathscr{E}_z)$ and $(\mathfrak{X}, \mathscr{E}_h)$. Recall that $\mathfrak{X} = \mathscr{V} \times \{-1, 1\}^{|X|}$. Let

$$\Pi' : \mathfrak{X} \to \mathscr{V}$$

be a projection on the first component. Let $\mathscr{T}$ be collection of all directed edges on vertex set $\mathscr{V}$. Π' induces a projection between edges

$$\Pi'' : \mathscr{E}_h \to \mathscr{T}$$

via

$$\Pi''((\xi_1, D_1), (\xi_2, D_2)) := \begin{cases} (\Pi'(\xi_1, D_1), \Pi'(\xi_2, D_2)) & \text{if } \xi_1 \neq \xi_2 \text{ or } (\xi_1, D_1) = (\xi_2, D_2) \\ \varnothing & \text{otherwise} \end{cases}$$

Let $\mathscr{S}(G)$ denote the collection of subgraphs of G. Then Π' and Π'' induce a projection

$$\Pi: \mathscr{S}((\mathfrak{X}, \mathscr{E}_h)) \to \mathscr{S}((\mathscr{V}, \mathscr{T}))$$

via

$$\Pi((X, E)) = (\Pi'(X), \Pi''(E))$$

for $G = (X,E)$ a subgraph of $(\mathfrak{X}, \mathscr{E}_h)$.

Consider $\mathscr{C} \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$, a recurrent component of $(\mathscr{V}, \mathscr{E}_z)$. By Lemma 3.4, for each edge $\xi \to \xi'$ in $\mathscr{C}$, $(\mathfrak{X}, \mathscr{E}_h)$ contains the edge $(\xi, D^*) \to (\xi', D^*)$. Furthermore, by Lemma 3.7, there exists a path from (ξ', D^*) to (ξ', D'^*) in $(\mathfrak{X}, \mathscr{E}_h)$, contained in the fiber over ξ'. Denote the set of such paths from (ξ', D^*) to (ξ', D'^*) by $\mathscr{Q}(\xi, \xi')$, where an element of $\mathscr{Q}(\xi, \xi')$ is given by:

$$(\xi', D^*) = (\xi', D_1) \to (\xi', D_2) \to \cdots \to (\xi', D_m) = (\xi', D'^*)$$

Definition 3.14: If $\mathscr{C} = (\mathscr{V}_{\mathscr{C}}, \mathscr{E}_{\mathscr{C}}) \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ is a recurrent component of $(\mathscr{V}, \mathscr{E}_z)$, then the **-subgraph over* $\mathscr{C}$ is the subgraph $\mathscr{C}^* := (\mathfrak{X}^*_{\mathscr{C}}, \mathscr{E}^*_{\mathscr{C}})$ of $(\mathfrak{X}, \mathscr{E}_h)$ defined by

$$\mathscr{E}^*_{\mathscr{C}} := \bigcup_{(\xi, \xi') \in \mathscr{E}_{\mathscr{C}}} \{((\xi, D^*), (\xi', D^*))\} \cup \mathscr{Q}(\xi, \xi')$$

$$\mathfrak{X}^*_{\mathscr{C}} := \{(\xi, D) \mid ((\xi, D), (\xi', D')) \in \mathscr{E}^*_{\mathscr{C}} \text{ or } ((\xi', D'), (\xi, D)) \in \mathscr{E}^*_{\mathscr{C}} \text{ for some } \xi' \text{ and } D'\}.$$

We now prove our first main result. We will show that for the *-subgraphs $\mathscr{C}^* \subset (\mathfrak{X}, \mathscr{E}_h)$, the projection $\Pi(\mathscr{C}^*)$ is contained in $\mathscr{S}((\mathscr{V}, \mathscr{E}_z))$ and not just $\mathscr{S}((\mathscr{V}, \mathscr{T}))$. That is, the projection of each edge in $\mathscr{C}^*$ is an edge in the set $\mathscr{E}_z$. Moreover, $\mathscr{C}^*$ is a strongly connected subgraph that *projects onto a Morse set* of the switching domain graph.

Theorem 3.15: *Let* $\mathscr{C} = (\mathscr{V}_{\mathscr{C}}, \mathscr{E}_{\mathscr{C}}) \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$. *Then the *-subgraph* $\mathscr{C}^* = (\mathfrak{X}^*_{\mathscr{C}}, \mathscr{E}^*_{\mathscr{C}})$ *over* $\mathscr{C}$ *is a strongly connected subgraph of* $(\mathfrak{X}, \mathscr{E}_h)$ *such that*

$$\Pi(\mathscr{C}^*) = \mathscr{C}.$$

Proof: By Definitions 3.13 and 3.14

$$\Pi''(\mathscr{E}^*_{\mathscr{C}}) = \bigcup_{\mathscr{E}_{\mathscr{C}}} \{(\xi, \xi')\}$$

$$\Pi'(\mathfrak{X}^*_{\mathscr{C}}) = \bigcup_{\mathscr{V}_{\mathscr{C}}} \{\xi\}$$

so that $\mathscr{C}^*$ projects onto $\mathscr{C}$, $\Pi(\mathscr{C}^*) = \mathscr{C}$.

It remains to be shown that $\mathscr{C}^*$ is strongly connected. Let $u, v \in \mathscr{C}^*$. Because $\mathscr{C}$ is a recurrent component, there exists a path in $\mathscr{C}$ from $\Pi'(u)$ to $\Pi'(v)$. Let $\xi_1 := \Pi'(u)$ and $\xi_m := \Pi'(v)$ and denote this path by $\xi_1 \to \xi_2 \to \cdots \to \xi_m$. Because this path is contained in $\mathscr{C}$, $\mathscr{C}^*$ contains the edge $(\xi_p, D^*_p) \to (\xi_{p+1}, D^*_p)$ for $p = 1, 2, \ldots, m - 1$ by Lemma 3.4 and Definition 3.14. By Lemma 3.7, there exists a path in the fiber over ξ_{p+1} (and therefore in $\mathscr{C}^*$) from (ξ_{p+1}, D^*_p) to (ξ_{p+1}, D^*_{p+1}). By our assumptions, u is in the fiber over ξ_1, so $\mathscr{C}^*$ also contains a path from u to (ξ_1, D^*_1), and likewise there exists a path from (ξ_m, D^*_{m-1}) to v. Concatenating these paths together, we see that $\mathscr{C}^*$ contains a path

$$u \to \cdots \to (\xi_1, D^*_1) \to (\xi_2, D^*_1) \to \cdots \to (\xi_2, D^*_2) \to \cdots \to (\xi_m, D^*_{m-1}) \to \cdots \to v$$

Remark 3.16: Note that Theorem 3.15 does *not* imply the stronger result that if there exists a recurrent component $\mathscr{C} \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$, then there exists a *recurrent component* $\hat{\mathscr{C}} \in \mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ that projects onto $\mathscr{C}$. The subgraph $\mathscr{C}^*$ in Theorem 3.15 may not be a *maximal* strongly connected component, as Definition 2.22 of recurrent component requires. We present in Example 4.4 a case where disjoint components $\mathscr{C}_1, \mathscr{C}_2 \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ have *-subgraphs $\mathscr{C}^*_1, \mathscr{C}^*_2$ that are subgraphs of a single strongly connected component.

<u>Theorem 3.17:</u> *Let $\mathscr{C}_p, \mathscr{C}_q \in \mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ be two recurrent components of $(\mathscr{V}, \mathscr{E}_z)$ and let $\mathscr{C}^*_p$ and $\mathscr{C}^*_q$ be the *-subgraphs over $\mathscr{C}_p$ and $\mathscr{C}_q$, respectively. If there exists a path in $(\mathscr{V}, \mathscr{E}_z)$ from an element of $\mathscr{C}_p$ to an element of $\mathscr{C}_q$, then there exists a path in $(\mathfrak{X}, \mathscr{E}_h)$ from an element of $\mathscr{C}^*_p$ to an element of $\mathscr{C}^*_q$.*

Proof: Let $\xi_1 \in \mathscr{C}_p$ and $\xi_m \in \mathscr{C}_q$ and assume that $(\mathscr{V}, \mathscr{E}_z)$ contains a path $\xi_1 \to \xi_2 \to \cdots \to \xi_m$. By Definition 3.14, $\mathscr{C}^*_p$ and $\mathscr{C}^*_q$ contain (ξ_1, D^*_1) and (ξ_m, D^*_m) as vertices, respectively. By Lemmas 3.4 and 3.7, $(\mathfrak{X}, \mathscr{E}_h)$ contains a path

$$(\xi_1, D^*_1) \to (\xi_2, D^*_1) \to \cdots \to (\xi_2, D^*_2) \to \cdots \to (\xi_m, D^*_{m-1}) \to \cdots \to (\xi_m, D^*_m)$$

using a similar concatenation technique to that used in the proof of Theorem 3.15.

Definition 3.18: We define the *component map* $\varphi : \mathrm{MD}((\mathcal{V}, \mathcal{E}_z)) \to \mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$ such that for each $\mathcal{C} \in \mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$, $\varphi(\mathcal{C})$ is the Morse node of $(\mathfrak{X}, \mathcal{E}_h)$ containing the *-subgraph $\mathcal{C}^*$ over $\mathcal{C}$.

Theorem 3.19: $\varphi : \mathrm{MD}((\mathcal{V}, \mathcal{E}_z)) \to \mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$ *is an order-preserving map.*

Proof: First we show that φ is well-defined. Let $\mathcal{C} \in \mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$. Then $\mathcal{C}^*$ is a strongly connected component of $(\mathfrak{X}, \mathcal{E}_h)$. Therefore, $\mathcal{C}^*$ is contained in a unique recurrent component of $(\mathfrak{X}, \mathcal{E}_h)$.

Now let $\mathcal{C}_p, \mathcal{C}_q \in (\mathrm{MD}((\mathcal{V}, \mathcal{E}_z)),\leq)$ be such that $q \leq p$. Then there exist elements $\xi \in \mathcal{C}_p$ and $\xi' \in \mathcal{C}_q$ and a path in $(\mathcal{V}, \mathcal{E}_z)$ from ξ to ξ' by Definition 2.22. If $\mathcal{C}_p^*$ and $\mathcal{C}_q^*$ are the *-subgraphs over these two components, then by Theorem 3.17, there exists a path from an element of $\mathcal{C}_p^*$ to an element of $\mathcal{C}_q^*$. Let $\hat{\mathcal{C}}_p$ and $\hat{\mathcal{C}}_q$ be the Morse nodes of $(\mathfrak{X}, \mathcal{E}_h)$ containing these *-subgraphs. Then this same path is from an element of $\hat{\mathcal{C}}_p$ to one of $\hat{\mathcal{C}}_q$. Thus $q \leq p$ in $(\mathrm{MD}((\mathfrak{X}, \mathcal{E}_h)),\leq)$.

Theorem 3.19 shows the existence of an order-preserving map $\varphi : \mathrm{MD}((\mathcal{V}, \mathcal{E}_z)) \to \mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$. A natural question is whether there is a stronger relationship between $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$ and $\mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$. We show in the following section that φ is not in general either injective or surjective. In particular, we give an example where $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$ consists of a single element and $\mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$ contains more than one element; and we give another example where two elements of $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$ map to a single element of $\mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$.

4 Examples

In this section we provide three examples that demonstrate that, in general, a few natural candidates for a stronger Theorem 3.19 are not valid.

1. In general, there does not exist a surjection, order preserving or otherwise, from $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$ onto $\mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$. As a consequence $\mathrm{MG}((\mathcal{V}, \mathcal{E}_z))$ and $\mathrm{MG}((\mathfrak{X}, \mathcal{E}_h))$ are not isomorphic. We provide two examples. In the first example, there are two Morse sets $\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$, but only one Morse set $\mathcal{C} \in \mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$. Clearly, since the cardinality of $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$ is less than the cardinality of $\mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$, there can be no surjection from the first to the second. However, we also show that $\Pi(\mathcal{C}_1) = \mathcal{C}$ and $\Pi(\mathcal{C}_2) \in \mathcal{S}(\mathcal{C}, \mathcal{T} \mid \mathcal{E}_z))$. In other words, the vertices of $\mathcal{C}_2$ project onto vertices of $\mathcal{C}$, but none of the edges in $\Pi(\mathcal{C}_2)$ appear in the domain graph $(\mathcal{V}, \mathcal{E}_z)$.

   In the second example, we likewise show the lack of a surjection in which there is a Morse set $\mathcal{C}_1 \in \mathrm{MD}((\mathfrak{X}, \mathcal{E}_h))$ that does not project onto any Morse set of $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$. In this case, there are vertices in $\mathcal{C}_1$ that do not project onto vertices of any Morse set in $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$

2. The map φ is not, in general, injective. In the third example, we show that two Morse sets in $\mathrm{MD}((\mathcal{V}, \mathcal{E}_z))$ have *-subgraphs that embed into the same recurrent component of $(\mathfrak{X}, \mathcal{E}_h)$.

We will make the following annotations to the Morse sets when displaying the Morse graphs for the parameterized ODE systems of a regulatory network **RN** and its canonical extension **EX**.

Definition 4.1: Let $\mathscr{C}$ be a recurrent component of either $(\mathscr{V}, \mathscr{E}_z)$ or $(\mathfrak{X}, \mathscr{E}_h)$. Recall from Definition 3.10 that $\mathscr{C}$ is a fixed point if and only if $\mathscr{C}$ contains a single node. In this case we annotate $\mathscr{C}$ by FP. Otherwise we say $\mathscr{C}$ is a *cycle* and annotate it by C.

Example 4.2: The first example will show that in general there is no surjection from $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ onto $\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ and therefore no isomorphism between $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ and $\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$.

Let **RN** = (V, E) where $V = \{\beta_1, \beta_2\}$ and $E = \{\beta_1 \dashv \beta_2, \beta_2 \to \beta_1\}$, as shown in Figure 5a. The associated switching system is on the left below (on the right is the extended system associated to a canonical extension defined below).

$$
\begin{aligned}
\dot{y}_1 &= -\gamma_1 y_1 + \tilde{\Lambda}_1(y) & \dot{y}_1 &= -\gamma_1 y_1 + \Lambda_1(y, x) \\
& & \dot{x}_{2,1} &= -a_{2,1} x_{2,1} + b_{2,1} y_1 \\
\dot{y}_2 &= -\gamma_2 y_2 + \tilde{\Lambda}_2(y) & \dot{y}_2 &= -\gamma_2 y_2 + \Lambda_2(y, x) \\
& & \dot{x}_{1,2} &= -a_{1,2} x_{1,2} + b_{1,2} y_2
\end{aligned}
$$

Note that there exist four domains in the phase space of the switching system, as seen in Figure 5b.

We must make a choice of combinatorial parameter in order to construct the domain graph $(\mathscr{V}, \mathscr{E}_z)$. By Defintion 2.24, a combinatorial parameter choice in this example is equivalent to a choice of order between each $\gamma_1\theta_{2,1}$ and the values of $\tilde{\Lambda}_1$, a choice of order between $\gamma_2\theta_{1,2}$ and the values of $\tilde{\Lambda}_2$, and a choice of order on the set of thresholds associated to each $y_i \in V$. We will pick the combinatorial parameter (L, O) such that $l_{1,2} < \gamma_1\theta_{2,1} < u_{1,2}$ and $l_{2,1} < \gamma_2\theta_{1,2} < u_{2,1}$. This results in the domain graph $(\mathscr{V}, \mathscr{E}_z)$ in Figure 5c. Clearly the entire domain graph comprises a recurrent component. The result is the Morse graph $\mathrm{MG}((\mathscr{V}, \mathscr{E}_z))$ with a single component C, as shown in Figure 5d.

Let **EX** = $(V \cup X, E_X)$ where $X = \{\alpha_{2,1}, \alpha_{1,2}\}$ and E_X is the union of two chains $C_{2,1} := \{\beta_1 \to \alpha_{2,1}, \alpha_{2,1} \dashv \beta_2\}$ and $C_{1,2} := \{\{\beta_2 \to \alpha_{1,2}, \alpha_{1,2} \to \beta_1\}$ as shown in Figure 6a. The canonical extension of the regulatory network, maintaining the combinatorial parameter for **RN** above, produces the domain graph seen in Figure 6b. It is relatively easy to see that the subgraphs induced by the two vertex sets

$$
\begin{aligned}
\mathfrak{X}(\mathrm{C}_0) &:= \{(\xi_1, -1, -1), (\xi_3, -1, -1), (\xi_3, -1, 1), (\xi_4, -1, 1), (\xi_4, 1, 1), (\xi_2, 1, 1), (\xi_2, 1, -1), (\xi_1, 1, -1)\} \\
\mathfrak{X}(\mathrm{C}_1) &:= \{(\xi_1, 1, 1), (\xi_1, -1, 1), (\xi_2, -1, 1), (\xi_2, -1, -1), (\xi_4, -1, -1), (\xi_4, 1, -1), (\xi_3, 1, -1), (\xi_3, 1, 1)\}
\end{aligned}
$$

are recurrent components, C_0 and C_1 respectively, and that these are the only recurrent components of the system. Moreover, there is a path from C_1 to C_0 in the domain graph, leading to the Morse graph shown in Figure 6c. We leave the reader to check that C_0 is the *-subgraph over the single recurrent component C of $(\mathscr{V}, \mathscr{E}_z)$. Since C_1 cycles in the opposite direction through the same projected nodes, $\Pi'(\mathfrak{X}(C_0)) = \Pi'(\mathfrak{X}(C_1))$, we have that $C_1 \in \mathscr{S}((\mathscr{V}, \mathscr{E}_z)^T)$, where $(\mathscr{V}, \mathscr{E}_z)^T$ is the transpose graph of $(\mathscr{V}, \mathscr{E}_z)$ where all directed edges are reversed. Thus $\Pi(C_1)$ shares no edges with $(\mathscr{V}, \mathscr{E}_z)$.

Note that the dynamics observed in this example are consistent with results about cyclic feedback systems [34, 22, 19]. **RN** is a negative cyclic feedback system with two variables and **EX** is a negative cyclic feedback system with four variables.

Example 4.3: Note that by Example 4.2, not every element of $MD((\mathfrak{X}, \mathscr{E}_h))$ projects onto an element of $MD((\mathscr{V}, \mathscr{E}_z))$ via our projection map Π, as the edges in the non-attracting cycle in $MD((\mathfrak{X}, \mathscr{E}_h))$ do not project onto the edges of the unique Morse node of $MD((\mathscr{V}, \mathscr{E}_z))$. However, the vertex set of the former does project onto that of the latter. The following example shows that there may be Morse sets of $MD((\mathfrak{X}, \mathscr{E}_h))$ that do not even share nodes with a Morse set of $MD((\mathscr{V}, \mathscr{E}_z))$.

Let **RN** = (V, E) where $V = \{\beta_1, \beta_2\}$ and $E = \{\beta_1 \rightarrow \beta_2, \beta_2 \rightarrow \beta_1\}$ as shown in Figure 7a. Since the only difference between this example and Example 4.2 is the sign of the feedback from β_1 to β_2, the domain structure of the phase space for **RN** is the same between these two examples. The direction of arrows in the domain graphs will be different between these examples. We pick the same combinatorial parameter (L, O) as in Example 4.2: $l_{1,2} < \gamma_1\theta_{2,1} < u_{1,2}$ and $l_{2,1} < \gamma_2\theta_{1,2} < u_{2,1}$. The results are the domain graph and Morse graph in Figures 7b and 7c.

Let **EX** = $(V \cup X, E_X)$ where $X = \{\alpha_{2,1}, \alpha_{1,2}\}$ and E_X is the union of two chains $C_{2,1} := \{\beta_1 \rightarrow \alpha_{2,1}, \alpha_{2,1} \rightarrow \beta_2\}$ and $C_{1,2} := \{\beta_2 \rightarrow \alpha_{1,2}, \alpha_{1,2} \rightarrow \beta_1\}$, as shown in Figure 8a. Again, we maintain the combinatorial parameter for **RN** from above. The canonical extension gives rise to the domain graph in Figure 8b.

Note that as predicted by Theorem 3.12, there exist exactly two one-element recurrent components, namely the subgraphs induced by $\{(\xi_1, -1, -1)\}$ and $\{(\xi_4, 1, 1)\}$. However, there exists an additional recurrent component: the subgraph induced by the set

$$\{(\xi_1, -1, 1), (\xi_1, 1, -1), (\xi_2, -1, -1), (\xi_2, -1, 1), (\xi_2, 1, -1), (\xi_2, 1, 1), (\xi_3, -1, -1), (\xi_3, -1, 1), (\xi_3, 1, -1), (\xi_3, 1, 1), (\xi_4, -1, 1), (\xi_4, 1, -1)\}.$$

Note there also exist paths, for instance, from $(\xi_1, -1, 1)$ to $(\xi_1, -1, -1)$ and from $(\xi_4, 1, -1)$ to $(\xi_4, 1, 1)$. The result is the Morse graph in Figure 8c. Note also that even the vertex set of the cycle does not project onto that of any recurrent component of $(\mathscr{V}, \mathscr{E}_z)$.

As in Example 4.2, the dynamics observed in this example are consistent with results about cyclic feedback systems [34, 22, 19]. **RN** is a positive cyclic feedback system with two variables and **EX** is a positive cyclic feedback system with four variables.

Example 4.4: In this final example, we show that the map φ is not, in general, injective. Recall that φ maps a recurrent component $\mathscr{C}$ of $(\mathscr{V}, \mathscr{E}_Z)$ to the recurrent component of $(\mathfrak{X}, \mathscr{E}_h)$ containing the *-subgraph $\mathscr{C}^*$ over $\mathscr{C}$. In this example the *-subgraphs over the two distinct Morse nodes of $(\mathscr{V}, \mathscr{E}_Z)$ are contained in the same recurrent component of $(\mathfrak{X}, \mathscr{E}_h)$.

Let **RN** = (V, E) where $V = \{\beta_1, \beta_2, \beta_3\}$ and

$$E = \{\beta_1 \to \beta_1, \beta_1 \to \beta_2, \beta_1 \to \beta_3, \beta_2 \to \beta_1, \beta_3 \dashv \beta_1\}$$

For any arbitrary regular parameter $z \in Z$, the switching system associated to **RN** is as follows

$$\begin{aligned} \dot{y}_1 &= -\gamma_1 y_1 + \tilde{\Lambda}_1(y) \\ \dot{y}_2 &= -\gamma_2 y_2 + \tilde{\Lambda}_2(y) \\ \dot{y}_3 &= -\gamma_1 y_3 + \tilde{\Lambda}_3(y) \end{aligned}$$

The phase space of the switching system is partitioned into sixteen domains, as shown in Figure 9b. For clarity, we divide the domains into two “layers.” On the left are the domains on which $y_3 < \theta_{1,3}$, while on the right, $y_3 > \theta_{1,3}$.

We will assume that the logic function M_1 associated to y_1 is additive; that is, $\tilde{\Lambda}_1 = M_1 \circ \tilde{\sigma}_1 = \tilde{\sigma}_{1,1} + \tilde{\sigma}_{1,2} + \tilde{\sigma}_{1,3}$. Furthermore, we will pick the combinatorial parameter that stipulates the following.

$$\begin{array}{lll} \theta_{2,1} < \theta_{1,1} < \theta_{3,1} & & \\ l_{1,1} + l_{1,2} + l_{1,3} & < \gamma_1\theta_{2,1} & \gamma_1\theta_{2,1} < l_{1,1} + l_{1,2} + u_{1,3} < \gamma_1\theta_{1,1} \\ u_{1,1} + l_{1,2} + l_{1,3} & < \gamma_1\theta_{2,1} & \gamma_1\theta_{1,1} < l_{1,1} + u_{1,2} + u_{1,3} < \gamma_1\theta_{3,1} \\ l_{1,1} + u_{1,2} + l_{1,3} & < \gamma_1\theta_{2,1} & \gamma_1\theta_{3,1} < u_{1,1} + l_{1,2} + u_{1,3} \\ u_{1,1} + u_{1,2} + l_{1,3} & < \gamma_1\theta_{2,1} & \gamma_1\theta_{3,1} < u_{1,1} + u_{1,2} + u_{1,3} \\ l_{2,1} < \gamma_2\theta_{1,2} < u_{2,1} & & \\ l_{3,1} < \gamma_3\theta_{1,3} < u_{3,1} & & \end{array}$$

This parameter choice gives rise to the domain graph in Figure 9c.

It is relatively easy to find that the sets $\{\xi_1, \xi_2, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{16}\}$ and $\{\xi_3, \xi_4, \xi_{11}, \xi_{12}\}$ both induce subgraphs that are recurrent components of $(\mathscr{V}, \mathscr{E}_Z)$, C_0 and C_1 respectively. Furthermore, there exists a path from ξ_3 to ξ_7, resulting in the Morse graph in Figure 9d.

Now let **EX** = ($V \cup X$, E_X) where $X = \{\alpha_{1,1}, \alpha_{2,1}, \alpha_{3,1}, \alpha_{1,2}, \alpha_{1,3}\}$ and E_X is the union of the five chains shown in Figure 9e. The extended system for $h \in H$ is

$$\begin{aligned}
\dot{y}_1 &= -\gamma_1 y_1 + \Lambda_1(x)\\
\dot{x}_{2,1} &= -a_{2,1}x_{2,1} + b_{2,1}y_1\\
\dot{x}_{3,1} &= -a_{3,1}x_{3,1} + b_{3,1}y_1\\
\dot{y}_2 &= -\gamma_2 y_2 + \Lambda_2(x)\\
\dot{x}_{1,2} &= -a_{1,2}x_{1,2} + b_{1,2}y_2\\
\dot{y}_3 &= -\gamma_3 y_3 + \Lambda_3(x)\\
\dot{x}_{1,3} &= -a_{1,3}x_{1,3} + b_{1,3}y_3
\end{aligned}$$

Consider now the *-subgraphs C_1^* and C_0^* in ($\mathfrak{X}$, $\mathscr{E}_h$) over the two Morse nodes C_1 and C_0 in MD(($\mathscr{V}$, $\mathscr{E}_Z$)), respectively, while maintaining our combinatorial parameter and assuming that all elements of the parameter sets a and b are strictly positive. Because both C_1^* and C_0^* project onto a recurrent component of ($\mathscr{V}$, $\mathscr{E}_Z$), each is a strongly connected graph. However, as we now show, each fails to be a *maximal* strongly connected component of ($\mathfrak{X}$, $\mathscr{E}_h$). We first note that Theorem 3.17 implies that there exists a path in ($\mathfrak{X}$, $\mathscr{E}_h$) from C_1^* to C_0^*. As we will show now, there is also a path from C_0^* to C_1^* which shows that both of them are parts of a single strongly connected component of ($\mathfrak{X}$, $\mathscr{E}_h$).

First note that (ξ_5, D_5^*) is a node in C_0^* and that (ξ_3, D_3^*) is a node in C_1^*. We will show that there is a path

$$(\xi_5, D_5^*) \to (\xi_1, D_5^*) \to (\xi_2, D_5^*) \to (\xi_3, D_5^*) \to \cdots \to (\xi_3, D_3^*)$$

in ($\mathfrak{X}$, $\mathscr{E}_h$). The first edge follows from Lemma 3.4 and the final path from (ξ_3, D_5^*) to (ξ_3, D_3^*) follows from Lemma 3.7. We have no result in general for the second and third edges, but we justify these edges as follows.

Let $\kappa := f_h^{-1}((\xi_1, D_5^*))$, $\kappa' := f_h^{-1}((\xi_2, D_5^*))$, and notice that these two domains are adjacent in phase space separated by a y-face τ where $y_1 = \theta_{2,1}$ (see Figure 9b). Also from Figure 9b, we see that $y_1 < \theta_{2,1}$ on κ and $y_1 > \theta_{2,1}$ on κ', so

$$\operatorname{sgn}(\tau, \kappa) = -1 \text{ and } \operatorname{sgn}(\tau, \kappa') = 1$$

by the definition of left and right faces. Moreover, from the definition of the *-element D_5^* and the location of $\tilde{\kappa}_5$ we see that on κ and κ', $x_{1,1} < \vartheta_{1,1}$, $x_{1,2} > \vartheta_{1,2}$, and $x_{1,3} < \vartheta_{1,3}$. Then $\Lambda_1(\kappa) = \Lambda_1(\kappa') = l_{1,1} + u_{1,2} + u_{1,3} > \gamma_1 \theta_{2,1}$. Hence

$$\operatorname{sgn}(-\gamma_1\theta_{2,1} + \Lambda_1(\kappa)) = \operatorname{sgn}(-\gamma_1\theta_{2,1} + \Lambda_1(\kappa')) = 1,$$

and so

$$\ell(\tau, \kappa) = -1 \text{ and } \ell(\tau, \kappa') = 1,$$

This satisfies the condition for an edge from $f_h(\kappa)$ to $f_h(\kappa')$ in $(\mathfrak{X}, \mathscr{E}_h)$.

We similarly show the existence of the edge $(\xi_2, D_5^*) \to (\xi_3, D_5^*)$ by first letting $\kappa' = f_h^{-1}((\xi_2, D_5^*))$ as before, and $\kappa'' := f_h^{-1}((\xi_3, D_5^*))$. These two domains are adjacent in phase space separated by a face τ' where $y_1 = \theta_{1,1}$, and

$$\operatorname{sgn}(\tau', \kappa') = -1 \text{ and } \operatorname{sgn}(\tau', \kappa'') = 1.$$

Because no intermediaries cross their associated thresholds at τ', we have

$$\Lambda_1(\kappa') = \Lambda_1(\kappa'') = l_{1,1} + u_{1,2} + u_{1,3} > \gamma_1\theta_{1,1},$$

implying

$$\ell(\tau', \kappa') = -1 \text{ and } \ell(\tau', \kappa'') = 1$$

which gives the edge $f_h(\kappa')$ to $f_h(\kappa'')$ in $(\mathfrak{X}, \mathscr{E}_h)$, and completes the path.

Let C^* be the recurrent component of $(\mathfrak{X}, \mathscr{E}_h)$ which contains $C_0^* \cup C_1^*$. Then

$$\varphi(C_0) = \varphi(C_1) = C^*,$$

so that φ is not injective.

5 Discussion

We have explored the dynamical behavior of the extension of a nonlinear switching system into a mixed nonlinear-linear model that allows differential modeling of protein and mRNA in gene regulatory networks. Mathematically, the switching system permits a partition of phase space into distinct domains with the property that neighboring domains have a well defined direction of flow between them. This gives rise to a state transition graph, which is a discrete representation of the flow. The strongly connected components of the state transition diagram are the nodes of a graph called a Morse graph with edges determined by reachability conditions in the state transition diagram. This is a coarse representation of the long-term dynamics of the system, in which nodes represent recurrent dynamics and edges represent gradient-like dynamics. The lower sets of the Morse graph correspond to attracting regions in phase space. Furthermore, the coarse description of the dynamics leads to an explicit decomposition of the parameter space into regions that admit the same state transition graph, and hence the same Morse graph. This decomposition is characterized by a parameter graph, whose nodes represent the parameter regions and whose edges represent co-dimension 1 boundaries between the regions [11].

We show that the extended system also permits the construction of a similar partition, state transition diagram, Morse graph, and parameter graph. A surprising result is that the parameter graph of a switching system and any of its extensions is the same. This result shows that even though there are a different number of parameters in each system, the partition of the parameter space into regions with the same essential dynamics is the same. We then compare the Morse graphs of the switching system and any canonical extension for the parameters that correspond to the same parameter node in the common parameter graph. We show that there is an order-preserving map between the Morse graph of the switching system and its extension, but that stronger relationships do not hold. In particular, there are neither injections nor surjections in general from the switching Morse graph to the extended Morse graph. We conclude that there are nontrivial changes in the long-term dynamics between systems, and we give several examples demonstrating this fact.

There are two main barriers to proving a stronger result than Theorem 3.19. The more technical reason is the fact that while the direction of the edge $(\xi_1, D_1^*) \to (\xi_2, D_1^*)$ in $(\mathfrak{X}, \mathscr{E}_h)$ is the same as the direction of the edge $\xi_1 \to \xi_2$ in $(\mathscr{V}, \mathscr{E}_z)$, we do not have control over edges between (ξ_1, D) and (ξ_2, D) in $(\mathfrak{X}, \mathscr{E}_h)$ for $D \neq D_1^*$. Since in principle these edges can go in the opposite direction $(\xi_2, D) \to (\xi_1, D)$, there are limits on the relationship between $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ and $\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$.

Another more intriguing reason may be the fact that the Morse decomposition $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ is incomplete in the sense that a perturbation of a switching system to a smooth approximation can increase the number of Morse sets. While we are not considering such perturbations in this paper, we will illustrate this idea on Example 4.3, where the Morse decomposition $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ consists of two attracting fixed points A_1, A_2. It is easy to see by a topological argument that any small perturbation to a smooth system will contain not only these attracting fixed points, but also contain a saddle point whose stable manifold separates the basins of attraction of the two attracting fixed points. If we now consider the extension $(\mathfrak{X}, \mathscr{E}_h)$ of the system $(\mathscr{V}, \mathscr{E}_z)$, its Morse decomposition $\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$ contains two attracting fixed points (as required by Theorem 3.12) that project to A_1 and A_2 respectively. However, it also contains a saddle-like set C that separates their basins of attraction and which has no corresponding Morse set in $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$. This suggests an intriguing conjecture that there is a stronger relationship between an appropriately defined "completion" of the Morse decomposition $\mathrm{MD}((\mathscr{V}, \mathscr{E}_z))$ and the Morse decomposition $\mathrm{MD}((\mathfrak{X}, \mathscr{E}_h))$. Such a completion would include Morse sets of all sufficiently near smooth perturbations of the switching system associated to **RN**. A closely related conjecture is that the solutions of a Filippov extension of a switching system [7], viewed as a multivalued flow, admit a Morse decomposition with a stronger relationship to MD(($\mathcal{X}$, $\mathcal{E}_h$)) than that described in Theorem 3.19 between MD(($\mathcal{V}$, $\mathcal{E}_z$)) of the switching system and MD(($\mathcal{X}$, $\mathcal{E}_h$)).

Our results indicate that explicitly modeling the translation process from mRNA to protein may fundamentally change the prediction of the model, and that the choice of model between switching system and extension is a nontrivial decision. Therefore experimentalists can directly compare the outcome of a switching model versus an extension and decide on the best model given observed data.

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References

1. Acary V, De Jong H, Brogliato B. Numerical simulation of piecewise-linear models of gene regulatory networks using complementarity systems. Physica D: Nonlinear Phenomena. 2014; 269:103–119.
2. Albert R, Collins JJ, Glass L. Introduction to Focus Issue: Quantitative approaches to genetic networks. Chaos. Jun.2013 23(2):025001. [PubMed: 23822498]
3. Albert R, Othmer HG. The topology and signature of the regulatory interactions predict the expression pattern of the segment polarity genes in Drosophila melanogaster. Journal of Theoretical Biology. 2003; 223(1):1–18. [PubMed: 12782112]
4. Arai Z, Kalies W, Kokubu H, Mischaikow K, Oka H, Pilarczyk P. A database schema for the analysis of global dynamics of multiparameter systems. SIAM Journal on Applied Dynamical Systems. 2009; 8(3):757–789.
5. Belta C, Habets LCGJM. Controlling a class of nonlinear systems on rectangles. IEEE Transactions on Automatic Control. 2006; 51(11):1749–1759.
6. Berenguier D, Chaouiya C, Monteiro PT, Naldi A, Remy E, Thieffry D, Tichit L. Dynamical modeling and analysis of large cellular regulatory networks. Chaos. Jun.2013 23(2):025114. [PubMed: 23822512]
7. Casey R, De Jong H, Gouzé JL. Piecewise-linear models of genetic regulatory networks: equilibria and their stability. Journal of mathematical biology. 2006; 52(1):27–56. [PubMed: 16195929]
8. Chaves M, Albert R. Studying the effect of cell division on expression patterns of the segment polarity genes. J Royal Society Interface. 2008; 5(S1):S71–S84.
9. Chaves M, Albert R, Sontag ED. Robustness and fragility of boolean models for genetic regulatory networks. J Theoretical Biology. 2005; 235(3):431–449.
10. Conley, C. Isolated Invariant Sets and the Morse Index. American Mathematical Society; 1978. Regional conference series in mathematics. R.I
11. Cummins B, Gedeon T, Harker S, Mischaikow K, Mok K. Combinatorial representation of parameter space for switching networks. SIAM Journal on Applied Dynamical Systems. 2016; 15(4):2176–2212.
12. de Jong H. Modeling and simulation of genetic regulatory systems: a literature review. Journal of Computational Biology. Jan; 2002 9(1):67–103. [PubMed: 11911796]
13. Edwards R. Analysis of continuous-time switching networks. Physica D: Nonlinear Phenomena. 2000; 146(March):165–199.

14. Edwards R, Farcot E, Foxall E. Explicit construction of chaotic attractors in Glass networks. Chaos, Solitons & Fractals. May; 2012 45(5):666–680.
15. Edwards R, Ironi L. Periodic solutions of gene networks with steep sigmoidal regulatory functions. Physica D: Nonlinear Phenomena. Jul.2014 282:1–15.
16. Edwards R, Machina A, McGregor G, van den Driessche P. A modelling frame-work for gene regulatory networks involving transcription and translation. Bull Math Biol. 2015; 77:953–983. [PubMed: 25758753]
17. Fan G, Cummins B, Gedeon T. Convergence properties of post-translationally modified protein-protein switching networks with fast decay rates. Bull Math Biol. Online first.
18. Farcot E. Geometric properties of a class of piecewise affine biological network models. Journal of Mathematical Biology. 2006; 52(3):373–418. [PubMed: 16382313]
19. Gedeon T. Cyclic feedback systems. Memoirs of AMS. 1998; 134(637)
20. Gedeon T, Cummins G, Heys J. Effect of model selection on prediction of periodic behavior in gene regulatory networks. Bull Math Biology. 2012; 74(8)
21. Gedeon T, Harker S, Kokubu H, Mischaikow K, Oka H. Global Dynamics for Steep Sigmoidal Nonlinearities in Two Dimensions. Physica D. Jan 15.2017 339:18–38. [PubMed: 28392611]
22. Gedeon T, Mischaikow K. Structure of the global attractor of cyclic feedback systems. J Dynam Diff Eq. 1995; 7:141–190.
23. Glass L, Kauffman SA. Co-operative components, spatial localization and oscillatory cellular dynamics. Journal of Theoretical Biology. Feb; 1972 34(2):219–37. [PubMed: 5015702]
24. Glass L, Kauffman SA. The logical analysis of continuous, non-linear biochemical control networks. Journal of Theoretical Biology. Apr; 1973 39(1):103–29. [PubMed: 4741704]
25. Glass L, Pasternack JS. Stable oscillations in mathematical models of biological control systems. Journal of Mathematical Biology. 1978; 6:207–223.
26. Habets LCGJM, van Schuppen JH. Control of piecewise-linear hybrid systems on simplices and rectangles. Lecture notes in computer science. 2001; 2034:261–274.
27. Habets, LCGJM., van Schuppen, JH. Control Conference (ECC), 2001 European. IEEE; 2001. A controllability result for piecewise-linear hybrid systems; p. 3870-3873.
28. Habets LCGJM, van Schuppen JH. A control problem for affine dynamical systems on a full-dimensional polytope. Automatica. 2004; 40(1):21–35.
29. Harker, S., Goullet, A., Cummins, B. DSGRN: Dynamic Signatures for Gene Regulatory Networks. 2012–2015. http://chomp.rutgers.edu/Projects/DSGRN/DB
30. Hudson D, Edwards R. Dynamics of transcription-translation networks. Physica D. 2016; 331:102–113.
31. Ironi L, Panzeri L, Plahte E, Simoncini V. Dynamics of actively regulated gene networks. Physica D: Nonlinear Phenomena. 2011; 240(8):779–794.
32. Kreeger PK, Lauffenburger DA. Cancer systems biology: a network modeling perspective. CARCINOGENESIS. Jan; 2010 31(1):2–8. [PubMed: 19861649]
33. Levine M, Davidson EH. Gene regulatory networks for development. PNAS. Apr 5; 2005 102(14): 4936–4942. [PubMed: 15788537]
34. Mallet-Paret J, Smith HL. The Poincare-Bendixson theorem for monotone cyclic feedback systems. Journal of Dynamics and Differential Equations. 1990; 2(4):367–421.
35. Tournier L, Chaves M. Uncovering operational interactions in genetic networks using asynchronous boolean dynamics. J Theor Biol. 2009; 260(2):196–209. [PubMed: 19524598]
36. Veflingstad SR, Plahte E. Analysis of gene regulatory network models with graded and binary transcriptional responses. Biosystems. 2007; 90(2):323–339. [PubMed: 17118528]

Appendix A Nullclines

In this appendix we provide some geometric motivation for our choice of thresholds in Definition 2.6 for the intermediaries $x_{j,i}^{k}$. This motivation revolves around the nullclines of the extended system (5).

Definition A.1

Let **EX** be a canonical extension of a regulatory network **RN**, and let h be a regular parameter for **EX**. By Definition 2.4 there exists an extended system associated to **EX** at h. Let $y_i \in V'$. We define the y_i-*nullcline* under h to be the set

$$\mathscr{Y}_i(h) := \{p \in (0,\infty)^{|V \cup X|} \mid \dot{y}_i|_p = 0\}$$

Let $\beta_j \in T(\beta_i)$ and let $X_{j,i} = \{\alpha^1, \alpha^2, \ldots, \alpha^n\}$ with associated variables $\{x^k\}$. For $k = 1, 2, \ldots, n$, the x^k-*nullcline* under h is the set

$$\mathscr{X}^k(h) := \{p \in (0,\infty)^{|V \cup X|} \mid \dot{x}^k|_p = 0\}$$

Remark A.2

Observe that by Definition 2.4, for each $y_i \in V'$ the y_i-nullcline under h can be characterized as

$$\mathscr{Y}_i(h) = \left\{p \in (0,\infty)^{|V \cup X|} \middle| y_i|_p = \frac{\Lambda_i(p)}{\gamma_i}\right\}.$$

Also observe that by Definition 2.6, the x^k-nullcline under h can be characterized in a similar way for $k = 1, 2, \ldots, n$:

$$\begin{aligned} \mathscr{X}^1(h) &= \left\{p \in (0,\infty)^{|V \cup X|} \middle| x^1|_p = \frac{b^1}{a_1} y_i|_p\right\} \\ \mathscr{X}^k(h) &= \left\{p \in (0,\infty)^{|V \cup X|} \middle| x^k|_p = \frac{b^k}{a^k} x^{k-1}|_p\right\}, \quad k = 2, 3, \ldots, n. \end{aligned}$$

From Remark A.2 it can be seen that the choice of thresholds $\vartheta^1, \ldots, \vartheta^n$ for the intermediaries $x^1, \ldots, x^n$, respectively, is due directly to the locations of these nullclines in phase space. In particular, we choose ϑ^1 such that the intersection of the two hyperplanes $\{y_i = \theta_{j,i}\}$ and $\{x^1 = \vartheta^1\}$ is contained in the x_1-nullcline. Similarly, for $k = 2, \ldots, n$, the intersection of the hyperplanes $\{x^{k-1} = \vartheta^{k-1}\}$ and $\{x^k = \vartheta^k\}$ is contained in the x^k-nullcline. We give a key consequence of these observations in Lemma A.3. This leads to an alternative derivation of Theorem 2.15 and consequently Corollary 2.19, which we provide here.

Lemma A.3

Fix a regular parameter $h \in H$ for **EX** *and let $y_i \in V'$ and $y_j \in T(y_i)$. Consider the chain $C_{j,i} = \{(\beta_i, \alpha^1), (\alpha^1, \alpha^2), \ldots, (\alpha^n, \beta_j)\}$. Let τ_y be a y-face on which $y_i = \theta_{j,i}$. Let τ_x be an x-face on which $x^k = \vartheta^k$. Then*

$$\tau_y \cap \mathscr{Y}_i(h) = \tau_x \cap \mathscr{X}^k(h) = \varnothing$$

Proof

By Lemma 2.14, we note that for each $p \in \tau_y$,

$$\Lambda_i(\kappa) = \Lambda_i(\kappa') = \Lambda_i(p)$$

Because h is a regular parameter,

$$\theta_{j,i} \neq \frac{\Lambda_i(\kappa)}{\gamma_i}$$

and therefore at each $p \in \tau_y$,

$$y_i|_p \neq \frac{\Lambda_i(p)}{\gamma_i}$$

so we conclude that $p \notin \mathscr{Y}_i(h)$ for any $p \in \tau_y$.

Now let $p \in \tau_x$. If $k = 1$, then by Definitions 2.6 and 2.8 $x^1 = \vartheta^1 = \frac{b^1}{a^1}\theta_{j,i}$ at p. Because $p \in \tau_x$, $y_i \in (\theta_{v_j,i}, \theta_{w_j,i})$ at p, for some pair of consecutive thresholds $\theta_{v_j,i}$ and $\theta_{w_j,i}$, and hence $y_i|_p \neq \theta_{j,i}$. Therefore $x^1 \neq \frac{b^1}{a^1}y_i$. We conclude $p \notin \mathscr{X}^1(h)$ for any $p \in \tau_x$.

In a very similar fashion, we observe that if $p \in \tau_x$ and $k > 1$, then at p, $x^k = \frac{b^k}{a^k}\vartheta^{k-1}$ and $x^{k-1} \neq \vartheta^{k-1}$. This implies $x^k \neq \frac{b^k}{a^k}x^{k-1}$ at any $p \in \tau_x$, and so $p \notin \mathscr{X}^k(h)$.

Using Lemma A.3, we can now prove Theorem 2.15 geometrically, with the theorem restated here for reference.

Theorem

Let **EX** *be a canonical extension where the associated extended system has a regular parameter* $h = (z, a, b) \in H$. *Let* τ *be a face on which* $\mu \in V' \cup X'$ *is constant. Then* $\operatorname{sgn}(\dot{\mu})$ *is constant and nonzero on* τ.

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Proof

The first part of the proof of Lemma A.3 establishes that $\dot{y}_i$ is constant and nonzero on any face where y_i is constant, implying that $\mathrm{sgn}(\dot{y}_i)$ is constant and nonzero as well.

Consider now $x^k = \vartheta^k$ on a wall τ_x with nullcline $\mathscr{X}^k(h)$. By Lemma A.3, $\tau_x \cap \mathscr{X}^k(h) = \varnothing$. Therefore $\dot{x}^k$ and therefore $\mathrm{sgn}(\dot{x}^k)$ are nonzero everywhere on τ^x. Suppose for a contradiction that $\mathrm{sgn}(\dot{x}^k)$ is nonconstant. By the continuity of $\dot{x}^k$ and the Intermediate Value Theorem, there then exists a point $r \in \tau_x$ such that $\dot{x}^k/_r = 0$, a contradiction.

Appendix B Proof of Lemma 3.7

For reference, we repeat the lemma here:

Lemma

Let $\xi \in \mathscr{V}$. For each element (ξ, D) in the fiber over ξ, there exists a path from (ξ, D) to (ξ, D^*) such that for each (ξ', D') in the path, we have $\xi = \xi'$.

The proof of this lemma is equivalent to a proof of correctness of Algorithm 1:

Algorithm 1

Path-Finding Problem

procedure PathFinder((ξ, D_1))
  $\mathscr{P} \leftarrow$ list((ξ, D_1))
  for $y_i \in V'$ **do**
    for $y_j \in \mathcal{T}(y_i)$ **do**
      $\kappa \leftarrow f_h^{-1}(\mathscr{P}(\text{last}))$
      $\mathscr{T} \leftarrow \{\tau \,/\, \tau$ is an x-face of $\kappa\}$
      $\mathscr{T}' \leftarrow$ OppositeSigns($\mathscr{T}, \kappa, y_i, y_j$)
      for $\tau \in \mathscr{T}'$ **do**
        $\kappa' \leftarrow$ Neighbor(τ, κ)
        Append $f_h(\kappa')$ to $\mathscr{P}$
      end for
    end for
  end for
  return $\mathscr{P}$
end procedure

procedure OppositeSigns($\mathscr{T}, \kappa, y_i, y_j$)
  $\mathscr{T}' \leftarrow$ list()
  for $\tau \in \mathscr{T}$ **do**
    Identify k such that $x^k_{j,i} = \vartheta^k_{j,i}$ on τ
    $S \leftarrow \mathrm{sgn}\,(y_i - \theta_{j,i})\mathrm{sgn}(x^k_{j,i} - \vartheta^k_{j,i})\big|_\kappa$
    if S is -1 **then**
      Append τ to $\mathscr{T}'$
    end if
  end for
  $\mathscr{T}' \leftarrow \mathscr{T}'$ sorted by increasing k
  return $\mathscr{T}'$
end procedure

procedure Neighbor(τ, κ)
  where (τ, κ') is a wall **and** $\kappa' \neq \kappa$
    return κ'
end procedure

Lemma B.1

Given $\xi \in \mathscr{V}$ and any (ξ, D_1) in the fiber over ξ, Algorithm 1 *constructs a path*

$$(\xi, D_1) \to (\xi, D_2) \to \cdots \to (\xi, D^*) .$$

The procedure PathFinder takes an initial starting point (ξ, D_1) and builds an ordered list of nodes. The key to the proof of correctness of Algorithm 1 is to show first that this list is a path in the domain graph restricted to the fiber over ξ, and secondly to show that the terminal element of this list is (ξ, D^*). We note that the algorithm must terminate with a list of some nonzero length, since it iterates over finite sets and consists at least of the initial node (ξ, D_1).

The procedure PathFinder in Algorithm 1 builds a list $\mathcal{P}$ starting with (ξ,D_1) by iterating over every variable y_i and all of the targets $T(y_i)$ of y_i in **RN** (that is, the iteration occurs over all of the y_j such that (β_i, β_j) ∈ E). For each y_i, y_j pair, we take the last node (ξ_m, D_m) in $\mathcal{P}$ and find the corresponding domain κ_m. Then we collect all of the x-faces of κ_m, each of which is a subset of a hyperplane $\{ x^k_{j,i} = \vartheta^k_{j,i} \}$ for some k. Notice that there will be exactly $|X|$ x-faces of κ_m for every m, because $x^k_{j,i}$ has exactly one threshold and therefore the hyperplane $\{ x^k_{j,i} = \vartheta^k_{j,i} \}$ borders every $\kappa \in \mathscr{K}(h)$.

Subsequently, the procedure OppositeSigns finds the subset of the x-faces where $\operatorname{sgn}(x^k_{j,i} - \vartheta^k_{j,i})$ disagrees with sgn($y_i - \theta_{j,i}$) on κ_m. This is returned as a list sorted in order of increasing k. For each x-face τ_k with a disagreeing sign on κ_m, the procedure Neighbor finds the unique neighboring domain κ_{m+1} that shares face τ_k with κ_m. The node (ξ_{m+1},D_{m+1}) = f_h(κ_{m+1}) then appended to the output list of PathFinder. We prove that every node in the output list of PathFinder is necessarily in the fiber over ξ, that the list is a path in the domain graph, and that the list terminates at (ξ, D^*). We now proceed to the proof of Lemma B.1.

Proof

Consider the output list $\mathcal{P}$ = PathFinder((ξ, D_1)). We need to show that

1. every element (ξ_m, D_m) ∈ $\mathcal{P}$ is well-defined;
2. every element (ξ_m, D_m) ∈ $\mathcal{P}$ satisfies $\xi_m = \xi$;
3. every consecutive pair of elements (ξ_m, D_m), (ξ_{m+1}, D_{m+1}) ∈ $\mathcal{P}$ satisfies

$$((\xi_m, D_m), (\xi_{m+1}, D_{m+1})) \in \mathscr{E}_h;$$

4. $\mathcal{P}$ is finite and terminates in (ξ, D^*).

Statement 1

Let (ξ_m, D_m) ∈ $\mathcal{P}$. We will show via induction that if $D_m \neq D^*$, then the subsequent element (ξ_{m+1}, D_{m+1}) exists. Base case: (ξ, D_1) ∈ $\mathcal{P}$, and by definition, $\kappa := f_h^{-1}((\xi, D_1))$ exists. Since

X is nonempty, κ has a collection of x-faces $\mathcal{T}$. Since $D_1 \neq D^*$, there exists at least one triple i, j, k such that τ_x is an x-face of κ where $x^k_{j,i} = \vartheta^k_{j,i}$ and

$$\operatorname{sgn}(y_i - \theta_{j,i})\operatorname{sgn}(x^k_{j,i} - \vartheta^k_{j,i})\,\big|_\kappa = -1, \quad (18)$$

by the definition of the *-element. This is equivalent to $\tau_x \in$ OppositeSigns($\mathcal{T}$, κ, y_i, y_j). By the definition of faces, there exists exactly one κ' that shares the face τ_x with κ; that is, κ' = Neighbor(τ_x, κ) is uniquely defined. Thus $f_h(\kappa')$ is uniquely defined and becomes $(\xi_2, D_2) \in \mathcal{P}$. If we now assume $(\xi_m, D_m) \in \mathcal{P}$ is well-defined with $D_m \neq D^*$, then the induction follows an identical argument.

Statement 2

Let $(\xi_m, D_m) = f_h(\kappa_m)$ and $(\xi_{m+1}, D_{m+1}) = f_h(\kappa_{m+1})$ be consecutive elements in $\mathcal{P}$. Then κ_{m+1} = Neighbor(τ_x, κ_m) is a neighbor of κ across an x-face. In other words, if

$$\kappa_m = \tilde{\kappa} \times \prod_{s=1,\, s \neq k}^{N} \Xi(s) \times \Xi(k)$$

then

$$\kappa_{m+1} = \tilde{\kappa} \times \prod_{s=1,\, s \neq k}^{N} \Xi(s) \times \Xi'(k)$$

where $\Xi(k) \neq \Xi'(k)$, but all other terms are identical. Then,

$$f_h(\kappa_m) = (f_z(\tilde{\kappa}), D_m) = (\xi_m, D_m); \quad f_h(\kappa_{m+1}) = (f_z(\tilde{\kappa}), D_{m+1}) = (\xi_{m+1}, D_{m+1})$$

so that $\xi_m = \xi_{m+1}$, which must be ξ by induction.

Statement 3

Let $(\xi, D_m) = f_h(\kappa_m)$ and $(\xi, D_{m+1}) = f_h(\kappa_{m+1})$ be consecutive elements in $\mathcal{P}$ such that κ_m and κ_{m+1} are separated by a face τ_k where $x^k_{j,i} = \vartheta^k_{j,i}$. Recalling the definitions of left and right faces, we note that

$$\operatorname{sgn}(\tau_k, \kappa_m)\operatorname{sgn}(\tau_k, \kappa_{m+1}) = -1.$$

Then we recall the wall labeling function

$$\ell(\tau_k, \kappa_m) = \operatorname{sgn}(\tau_\kappa, \kappa_m)\operatorname{sgn}(\mu - \theta)\,|_{\tau_k}$$
$$\ell(\tau_k, \kappa_{m+1}) = -\operatorname{sgn}(\tau_k, \kappa_m)\operatorname{sgn}(\mu - \theta)\,|_{\tau_k},$$

where

$$\mu - \theta = \begin{cases} y_i - \theta_{j,i} & \text{if } k = 1 \\ x_{j,i}^{k-1} - \vartheta_{j,i}^{k-1} & \text{if } k > 1 \end{cases}$$

So $\ell(\tau_x, \kappa_m)\ell(\tau_x, \kappa_{m+1}) = -1$ and there is an edge going from one domain to the other. We still require the direction of the arrow.

By necessity, $\tau_k \in$ OppositeSigns($\mathscr{T}$, κ_m, y_i, y_j), which implies that

$$\operatorname{sgn}(y_i - \theta_{j,i})\operatorname{sgn}(x_{j,i}^k - \vartheta_{j,i}^k)\Big|_{\kappa_m} = -1$$
$$\operatorname{sgn}(y_i - \theta_{j,i})\operatorname{sgn}(x_{j,i}^k - \vartheta_{j,i}^k)\Big|_{\kappa_{m+1}} = 1.$$

These expressions summarize the fact that if $\tau_k \in$ OppositeSigns($\mathscr{T}$, κ_m, i, j), then by construction $\tau_k \notin$ OppositeSigns($\mathscr{T}$, κ_{m+1}, y_i, y_j). Since we cross an x-wall, it must be true that sgn($y_i - \theta_{j,i}$) is constant on $\kappa_m \cup \kappa_{m+1}$, and so the change in sign occurs in $\operatorname{sgn}(x_{j,i}^k - \vartheta_{j,i}^k)$ across the two domains.

First suppose $y_i > \theta_{j,i}$; then $x_{j,i}^k < \vartheta_{j,i}^k$ on κ_m, so that τ_k is a right face of κ_m. By definition of a right face,

$$\operatorname{sgn}(\tau_k, \kappa_m) = -1.$$

If $k = 1$, then

$$\operatorname{sgn}(\mu - \theta)\,|_{\tau_k} = \operatorname{sgn}(y_i - \theta_{j,i})\Big|_{\kappa_m \cup \kappa_{m+1}} = 1$$

and

$$\ell(\tau_k, \kappa_m) = -1; \quad \ell(\tau_k, \kappa_{m+1}) = 1$$

as desired for an edge $(\xi, D_m) \to (\xi, D_{m+1})$ in the domain graph.

On the other hand, if $k > 1$, then we know that

$$\operatorname{sgn}(y_i - \theta_{j,i})\operatorname{sgn}(x_{j,i}^{k-1} - \vartheta_{j,i}^{k-1})\Big|_{\kappa_m} = 1,$$

either because $\tau_{k-1} \notin$ OppositeSigns($\mathscr{T}$, κ_m, y_i, y_j), or because the procedure OppositeSigns returns a sorted list in increasing k. In other words, if $\tau_{k-1} \in$ OppositeSigns($\mathscr{T}$, κ_m, y_i, y_j), then it is resolved before τ_k. Thus, for $y_i > \theta_{j,i}$, τ_{k-1} is a left face of κ_m. Then $x_{j,i}^{k-1} > \vartheta_{j,i}^{k-1}$ on $\kappa_m \cup \kappa_{m+1}$. Using that sgn($\tau_k$, κ_m) = −1, we have as before

$$\ell(\tau_k, \kappa_m) = -1; \quad \ell(\tau_k, \kappa_{m+1}) = 1.$$

The argument for $y_i < \theta_{j,i}$ is similar and returns the same result. So we have shown that each consecutive pair of elements has an edge between them, and thus the list $\mathscr{P}$ is a path in the domain graph.

Statement 4

$\mathscr{P}$ is necessarily finite because it is the result of iterations over finite sets. After the last element (ξ,D_n) is added, it is true that OppositeSigns($\mathscr{T}$, κ_n, y_i, y_j) is empty for all i, j. If not, then the algorithm has not yet terminated, since if $\tau_k \in$ OppositeSigns($\mathscr{T}$, κ_n, y_i, y_j), then by construction there exists κ_{n+1} where $\tau_k \notin$ OppositeSigns($\mathscr{T}$, κ_{n+1}, y_i, y_j). So by Definition 3.2, $D_n = D^*$.

Highlights

- We study switching systems (SS), where nonlinearities are piece-wise constant
- We characterize global structure of dynamics in parameter space by parameter graphs
- Extensions of SS replace some of direct interactions by chains of linear equations
- We show that parameter graphs of SS and their extensions are identical
- We discuss relationship between Morse graphs of SS and their extensions.

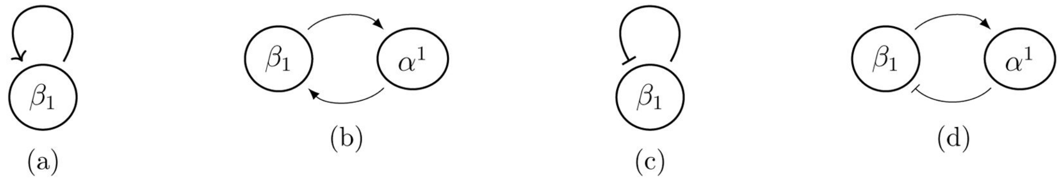


Figure 1.
(a) **RN** for positive self-loop, (b) Canonical extended network **EX** for (a), (c) **RN** for negative self-loop and (d) Canonical extended network **EX** for (c).

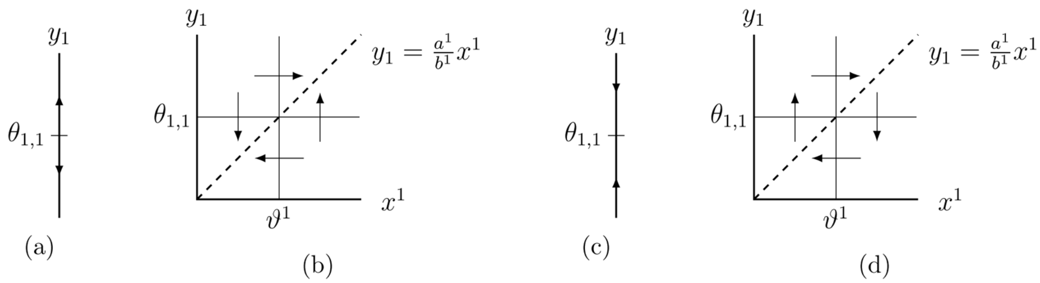


Figure 2.

Phase space for (a) the positive self-loop **RN** in Figure 1a; (b) the positive feedback extension **EX** in Figure 1b; (c) the negative self loop **RN** in Figure 1c; and (d) the negative feedback extension **EX** in Figure 1d, when for all systems parameters satisfy $l_{1,1} < \gamma_1\theta_{1,1} < u_{1,1}$. The arrows represent direction of transversal flow on the interior of each face. In the extended systems (b) and (d), the vertical arrows are determined by the switching function $\Lambda_1^{\pm}$, and the horizontal arrows are a consequence of the fact that $y_1 = \frac{a^1}{b^1}x^1$ is a nullcline of the second, linear equation.

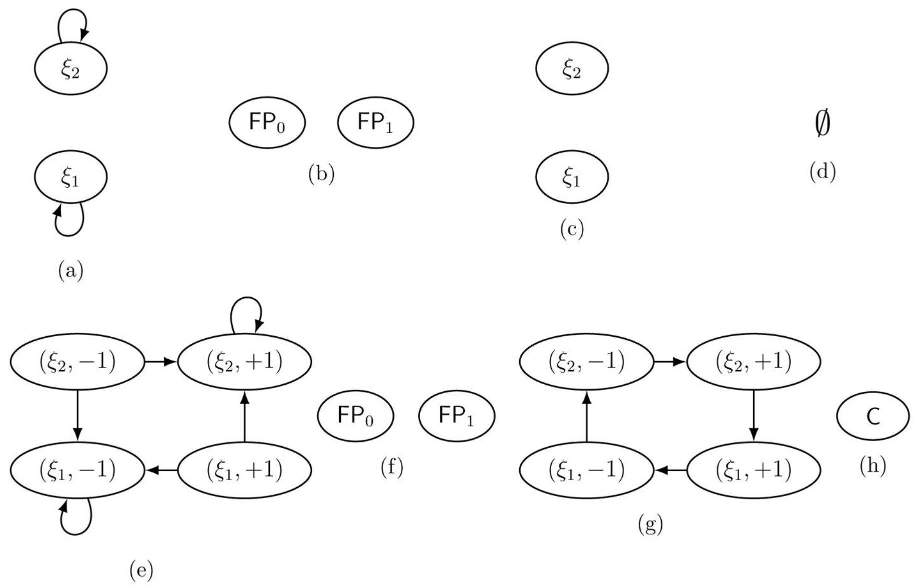


Figure 3.

Domain graphs and Morse graphs for the phase spaces in Figure 2. (a), (b) Domain and Morse graphs for the switching phase space in Figure 2a for the positive self-loop in Figure 1a. (c), (d) Domain and (empty) Morse graphs for the switching phase space in Figure 2c for the negative self-loop in Figure 1c. (e), (f) Domain and Morse graphs for the extended phase space in Figure 2b for the positive feedback in Figure 1b. (g), (h) Domain and Morse graphs for the extended phase space in Figure 2d for the positive feedback in Figure 1d. In all cases, ξ_1 is the domain graph node corresponding to the switching domain where $y_1 < \theta_{1,1}$, and ξ_2 corresponds to $y_1 > \theta_{1,1}$.

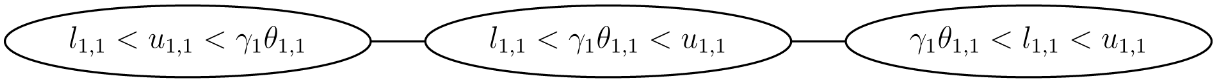


Figure 4.
Combinatorial parameter graph for all networks in Figure 1.

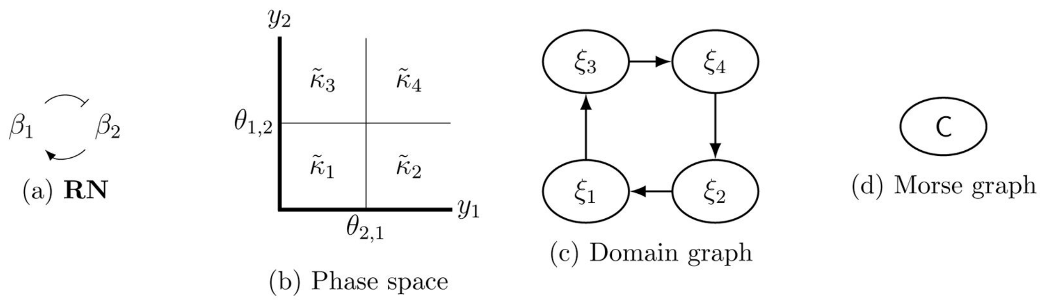


Figure 5.

Example 4.2: (a) **RN**, (b) phase space of the switching system, (c) domain graph of the switching system with the parameter in the text, and (d) Morse graph of the domain graph.

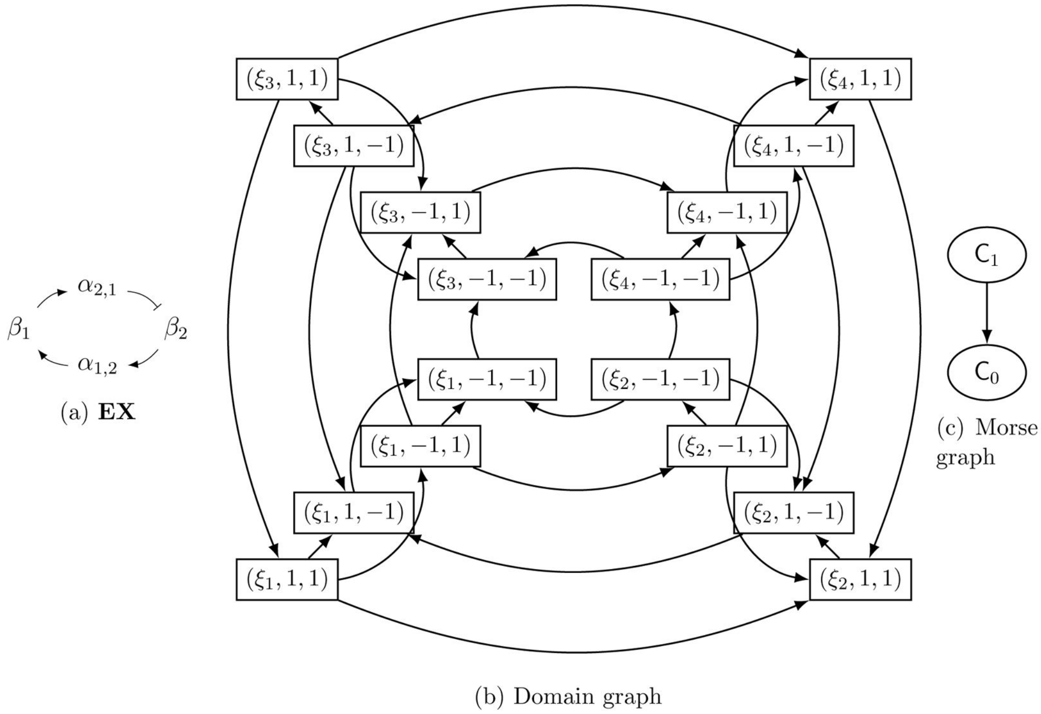


Figure 6.

Example 4.2: (a) **EX**, (b) domain graph of the extended system with the parameter in the text, and (c) Morse graph of the domain graph.

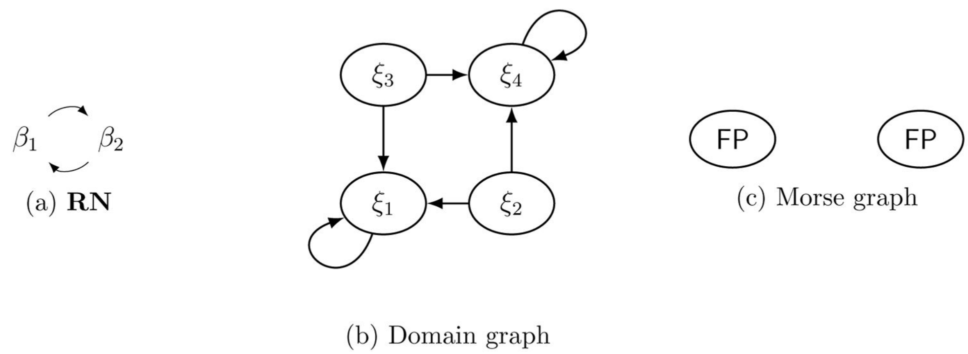
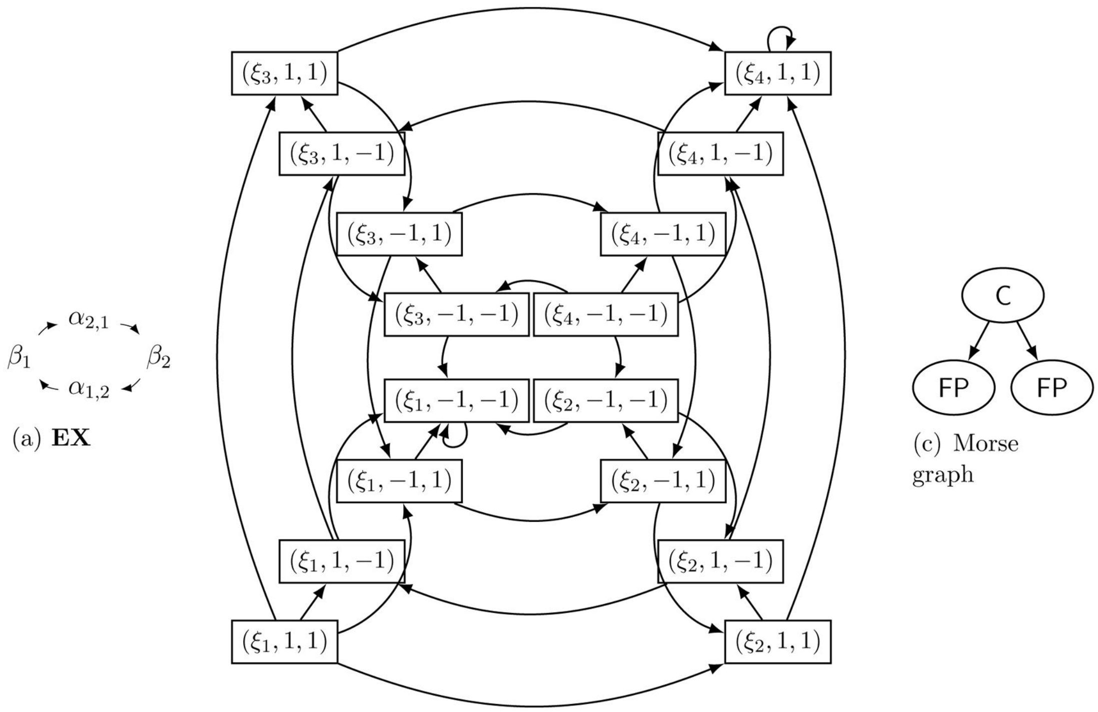


Figure 7.

RN and the domain and Morse graphs of the switching system for Example 4.3. The phase space of the associated switching system is the same as in Example 4.2 (see Figure 5b).



(a) **EX**

(b) Domain graph

(c) Morse graph

Figure 8.

Example 4.3: (a) **EX**, (b) domain graph of the extended system with the parameter in the text, and (c) Morse graph of the domain graph.

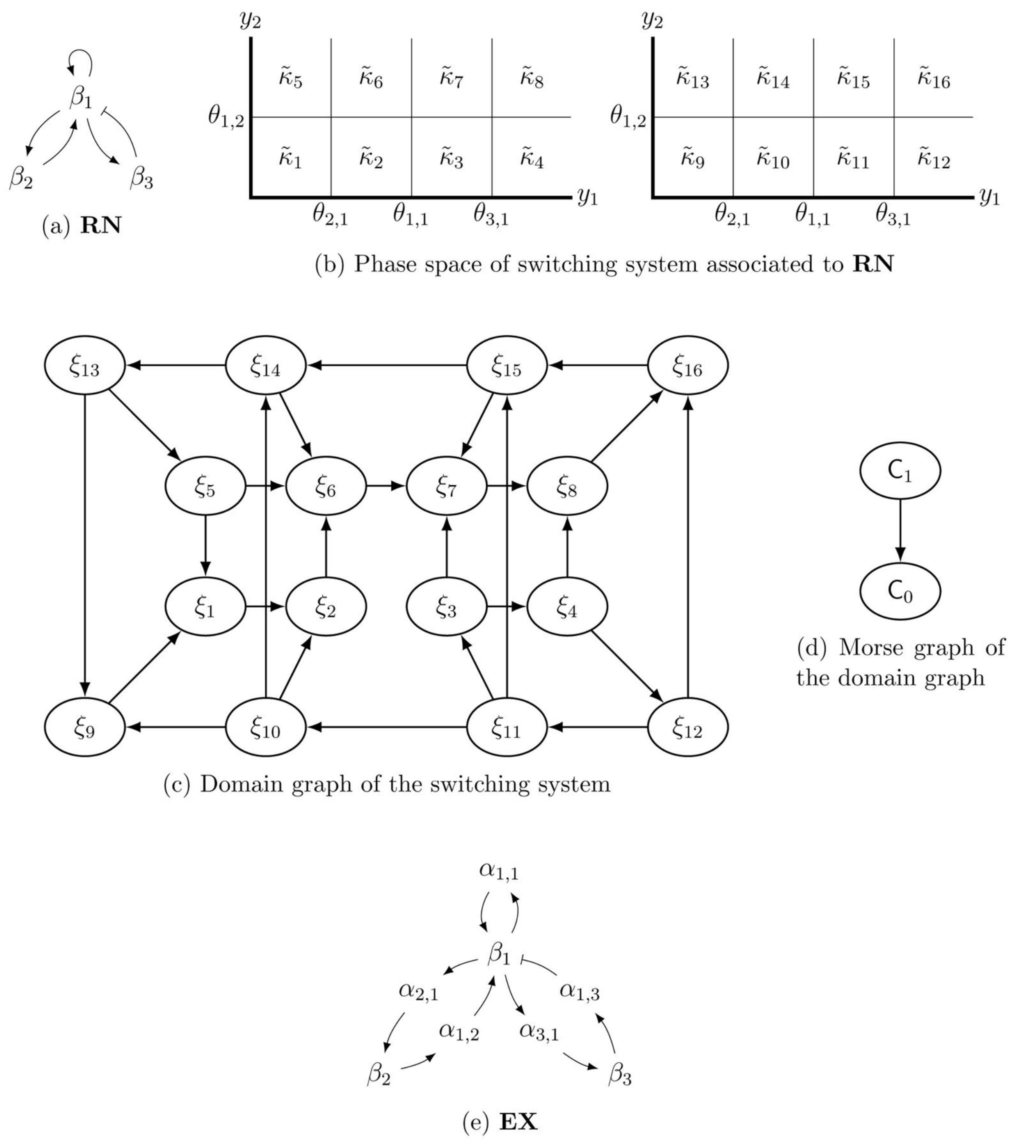


Figure 9.

RN, the phase space of the associated switching system, the domain and Morse graphs for the parameterized switching system, and **EX** in Example 4.4.