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Quantum walks on regular uniform hypergraphs

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Quantum walks on graphs have shown prioritized benefits and applications in wide areas. In some scenarios, however, it may be more natural and accurate to mandate high-order relationships for hypergraphs, due to the density of information stored inherently. Therefore, we can explore the potential of quantum walks on hypergraphs. In this paper, by presenting the one-to-one correspondence between regular uniform hypergraphs and bipartite graphs, we construct a model for quantum walks on bipartite graphs of regular uniform hypergraphs with Szegedy's quantum walks, which gives rise to a quadratic speed-up. Furthermore, we deliver spectral properties of the transition matrix, given that the cardinalities of the two disjoint sets are different in the bipartite graph. Our model provides the foundation for building quantum algorithms on the strength of quantum walks on hypergraphs, such as quantum walks search, quantized Google's PageRank, and quantum machine learning.

As a quantum-mechanical analogs of classical random walks, quantum walks have become increasingly popular in recent years, and have played a fundamental and important role in quantum computing. Owing to quantum superpositions and interference effects, quantum walks have been effectively used to simulate quantum phenomena¹, and realize universal quantum computation^{2,3}, as well as develop extensively quantum algorithms⁴. A wide variety of discrete quantum walk models have been successively proposed. The first quantization model of a classical random walk, which is the coined discrete-time model and which is performed on a line, was proposed by Aharonov⁵ in the early 1990s. Aharonov later studied its generalization for regular graphs in ref.⁶. Szegedy⁷ proposed a quantum walks model that quantizes the random walks, and its evolution operator is driven by two reflection operators on a bipartite graph. Moreover, in discrete models, the most-studied topology on which quantum walks are performed and their properties studied are a restricted family of graphs, including line^{8,9}, cycle^{10,11}, hypercube^{12,13}, and general graphs^{14–17}. Indeed, most of the existing quantum walk algorithms are superior to their classical counterparts at executing certain computational tasks, e.g., element distinctness^{18,19}, triangle finding^{20,21}, verifying matrix products²², searching for a marked element^{16,23}, quantized Google's PageRank²⁴ and graph isomorphism^{25,26}. In addition to above scientific documents, there are many good introductions and reviews written by Kempe²⁷, Kendon^{28,29}, Venegas-Andraca³⁰, and Wang³¹ relevant to deepening into the physical, mathematical and algorithmic properties of quantum walks.

In mass scenarios, a graph-based representation is incomplete, since graph edges can only represent pairwise relations between nodes. However, hypergraphs are a natural extension of graphs that allow modeling of higher-order relations in data. Because the mode of representation is even nearer to the human visual grouping system, hypergraphs are more available and effective than graphs for solving many problems in several applications. Owing to Zhou's random walks on hypergraphs for studying spectral clustering and semi-supervised ranking³², hypergraphs have made recent headlines in computer vision^{33,34}, information retrieval^{35–37}, and categorical data clustering³⁸. Many interesting and promising findings were covered in random walks on hypergraphs, and quantum walks provide a method to explore all possible paths in a parallel way, due to constructive quantum interference along the paths. Therefore, paying attention to quantum walks on hypergraphs is a natural choice.

Inspired by these latter developments, we focus on discrete-time quantum walks on regular uniform hypergraphs. In ref.³⁹, Konno defined a two-partition quantum walk and the quantum walk on hypergraph, also he showed that the quantum walk on hypergraph is also a two partition quantum walk. In this paper, by analyzing the mathematical formalism of hypergraphs and three existing discrete-time quantum walks⁴⁰ (coined quantum walks, Szegedy's quantum walks, and staggered quantum walks), we find that discrete-time quantum walks on regular uniform hypergraphs can be transformed into Szegedy's quantum walks on bipartite graphs that are used

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to model the original hypergraphs. Furthermore, the mapping is one to one. That is, we can study Szegedy’s quantum walks on bipartite graphs instead of the corresponding quantum walks on regular uniform hypergraphs. In ref.⁷ Szegedy proved that his schema brings about a quadratic speed-up. Hence, we construct a model for quantum walks on bipartite graphs of regular uniform hypergraphs with Szegedy’s quantum walks. In the model, the evolution operator of an extended Szegedy’s walks depends directly on the transition probability matrix of the Markov chain associated with the hypergraphs.

In more detail, we first introduce the classical random walks on hypergraphs, in order to get the vertex-edge transition matrix and the edge-vertex transition matrix. We then define a bipartite graph that is used to model the original hypergraph. Lastly, we construct quantum operators on the bipartite graph using extended Szegedy’s quantum walks, which is the quantum analogue of a classical Markov chain. In this work, we deal with the case that the cardinalities of the two disjoint sets can be different from each other in the bipartite graph. In addition, we deliver a slightly different version of the spectral properties of the transition matrix, which is the essence of the quantum walks. As a result, our work generalizes quantum walks on regular uniform hypergraphs by extending the classical Markov chain, due to Szegedy’s quantum walks.

The paper is organized as follows. There three subsection in Sec. Results. In Sec. Results: *Random walks on hypergraphs*, we briefly introduce random walks on hypergraphs needed to present the quantum version of it. In Sec. Results: *Quantum walks on hypergraphs*, we construct a method for quantizing Markov chain to create discrete-time quantum walks on regular uniform hypergraphs. In Sec. Results: *Spectral analysis of quantum walks on hypergraphs*, we analyze the eigen-decomposition of the operator. Sec. Discussion is devoted to conclusions.

Results

Random walks on hypergraphs. Let $HG = (V, E)$ denote a hypergraph, where V is the vertex set of the hypergraph and $E \subset 2^V \setminus \{\emptyset\}$ is the set of hyperedges. Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$, where $n = |V|$ is used to denote the number of vertices in the hypergraph and $m = |E|$ the number of hyperedges. Given a hypergraph, define its incidence matrix $H \in R^{n \times m}$ as follows:

$$h(i, j) = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{if } v_i \notin e_j \end{cases} \tag{1}$$

Then, the vertex and hyperedge degrees are defined as follows:

$$d(v) = \sum_{e \in E} h(v, e) = |E(v)|, \tag{2}$$

$$E(v) = \{e \in E: v \in e\}, \tag{3}$$

$$\delta(e) = \sum_{v \in V} h(v, e) = |e|, \tag{4}$$

where $E(v)$ is the set of hyperedges incident to v . A hypergraph is d -regular if all its vertices have the same degree. Also, a hypergraph is k -uniform if all its hyperedges have the same cardinality. In this paper, we will restrict our reach to quantum walks on d -regular and k -uniform hypergraphs from now on, denoting them as $HG_{k,d}$.

A random walk on a hypergraph $HG = (V, E)$ is a Markov chain on the state space V with its transition matrix P . The particle can move from vertex v_i to vertex v_j if there is a hyperedge containing both vertices. According to ref.³², a random walk on a hypergraph is seen as a two-step process. First, the particle chooses a hyperedge e incident with the current vertex v . Then, the particle picks a destination vertex u within the chosen hyperedge satisfying the following: $v, u \in e$. Therefore, the probability of moving from vertex v_i to v_j is:

$$P_{ij} = P(v_i, v_j) = \sum_{k=1}^m \frac{h_{ik}h_{jk}}{d(v_i)\delta(e_k)} = \frac{1}{d(v_i)} \sum_{k=1}^m \frac{h_{ik}h_{jk}}{\delta(e_k)}, \tag{5}$$

or, more accurately, the equation can be written as

$$P = \sum_{\substack{e \in E, \\ \{v, u\} \subseteq e}} \frac{1}{d(v)\delta(e)} = \frac{1}{d(v)} \sum_{\substack{e \in E, \\ \{v, u\} \subseteq e}} \frac{1}{\delta(e)}. \tag{6}$$

Let P_{VE} denote the vertex-edge transition matrix

$$P_{VE} = D_v^{-1}H \tag{7}$$

and P_{EV} the edge-vertex transition matrix

$$P_{EV} = D_e^{-1}H^T \tag{8}$$

with transition probability

$$\sum_{e \in E} P_{ve} = 1, \forall v \in V, \tag{9}$$

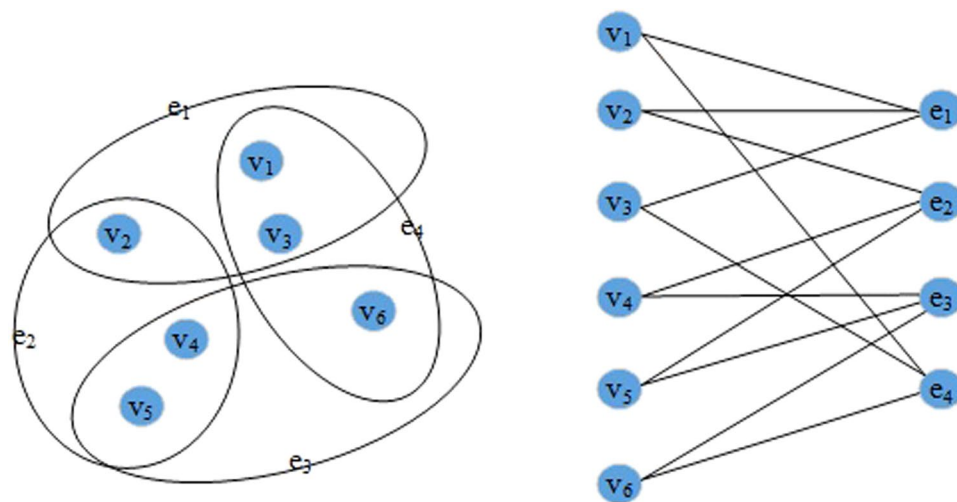


Figure 1. Example of a hypergraph with six vertices and four hyperedges, and its associated bipartite graph.

$$\sum_{v \in V} p_{ev} = 1, \forall e \in E, \tag{10}$$

where D_v and D_e are the diagonal matrices of the degrees of the vertices and edges, respectively. Naturally, we can indicate P in matrix form, as

$$P = D_v^{-1} H D_e^{-1} H^T = P_{VE} P_{EV}, \tag{11}$$

Quantum walks on hypergraphs. In this section, we design quantum walks on regular uniform hypergraphs by means of Szegedy’s quantum walks. We first convert the hypergraph into its associated bipartite graph, which can be used to model the hypergraph. We then define quantum operators on the bipartite graph using Szegedy’s quantum walks, which are a quantization of random walks.

Deriving Bipartite graphs from hypergraphs. Proposition 1 Any regular uniform hypergraph $HG_{k,d}$ can be represented usefully by a bipartite graph $BG_{n,m}$: the vertices V and the edges E of the hypergraph are the partitions of BG , and (v_i, e_j) are connected with an edge if and only if vertex v_i is contained in edge e_j in HG .

Proof. Hypergraph can be described by binary edge-node incidence matrix H with elements (1). To the incidence matrix of a regular uniform hypergraph $HG_{k,d}$ corresponds a bipartite incidence graph $B(H) = G(V \cup E, E_B)$ which is defined as follows. The vertices V and the edges E of the hypergraph are the partitions of BG , and $(v_i, e_j) \in E_B$ iff $h_{ij} = 1$. It is evident that B is bipartite, and the biadjacency matrix describing $B(H)$ is the following $(n+m) \times (n+m)$ matrix:

$$A_B = \begin{pmatrix} 0 & H \\ H^T & 0 \end{pmatrix}. \tag{12}$$

Under this correspondence, the biadjacency matrices of bipartite graphs are exactly the incidence matrices of the corresponding hypergraphs.

A similar reinterpretation of adjacency matrices may be used to show a one-to-one correspondence between regular uniform hypergraphs and bipartite graphs. That is, discrete-time quantum walks on regular uniform hypergraphs can be transformed into quantum walks on bipartite graphs that are used to model the original hypergraphs. The transformation process is outlined in detail below. If there is a hyperedge e_k containing both vertices v_i and v_j in the original hypergraph $HG = (V, E)$, convert it into two edges (v_i, e_k) and (e_k, v_j) in the bipartite graph.

As a concrete example, we consider a 3 – uniform and 2 – regular hypergraph with the vertices set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the set of hyperedges $E = \{e_1, e_2, e_3, e_4\}$. Then, a bipartite graph $BG_{6,4}$ with partite sets $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{e_1, e_2, e_3, e_4\}$ can represent the hypergraph $HG_{3,2}$, which is depicted in Fig. 1.

Theorem 1 Let $HG = (V, E)$ be a hypergraph, and we have

$$\sum_{v \in V} d(v) = \sum_{e \in E} d(e). \tag{13}$$

Proof. Let $B(H) = G(V \cup E, E_B)$ be the incidence graph of $HG = (V, E)$. We sum the degrees in the part E and in the part V in $B(H)$. Since the sums of the degrees in these two parts are equal, we obtain the result.

In particular, if the hypergraph is d – regular and k – uniform, we obtain $nd = mk$.

Szegedy quantum walks on the bipartite graphs. Since we have transformed the hypergraph $HG_{k,d}$ into its bipartite graph $BG_{n,m}$, we now describe Szegedy quantum walks that take place on the obtained bipartite graph $BG_{n,m}$ by extending the class of possible Markov chains. The quantum walks on the hypergraph HG start by considering an associated Hilbert space that is a linear subspace of the vector $H^{n^2m} = H_v^n \otimes H_e^m \otimes H_v^n$, where $n = |V|$, $m = |E|$. The computational basis of H^{n^2m} is $\{|v_i, e, v_j\rangle : e \in E, v_i, v_j \in V, v_i, v_j \in e\}$. In addition, quantum walks on the bipartite graph $BG_{n,m}$ with biadjacent matrix (12) have an associated Hilbert space $H_A = H_v^n \otimes H_e^m$ and $H_B = H_e^m \otimes H_v^n$.

To identify quantum analogues of Markov chains - that is, the classical random walks with probability matrices (7) and (8) with entries of (9) and (10) - we define the vertex-edge transition operators: $A: H^n \rightarrow H^{nd}$ and edge-vertex transition operators: $B: H^m \rightarrow H^{mk}$ as follows:

$$A = \sum_{v \in V} |\alpha_v\rangle \langle v|, \tag{14}$$

$$B = \sum_{e \in E} |\beta_e\rangle \langle e|, \tag{15}$$

where

$$|\alpha_v\rangle = |v\rangle \otimes \left(\sum_{e \in E} \sqrt{P_{ve}} |e\rangle \right), \tag{16}$$

$$|\beta_e\rangle = \left(\sum_{v \in V} \sqrt{P_{ev}} |v\rangle \right) \otimes |e\rangle. \tag{17}$$

The transition operators are defined on the Hilbert space H_A and H_B separately, where the computational basis of H^{nd} is $\{|v, e\rangle : v \in V, e \in E\}$ and the computational basis of H^{mk} is $\{|e, v\rangle : e \in E, v \in V\}$. The states $|\alpha_v\rangle$ and $|\beta_e\rangle$ as superpositions that start from vertex v to hyperedge e and from hyperedge e to vertex v , respectively. Obviously, the dimensions of A and B are $nd \times n$ and $mk \times m$, respectively. Note that $nd = mk$ from theorem 1. Using (16) and (17) along with (9) and (10), we obtain the following properties:

$$\langle \alpha_v | \alpha_{v'} \rangle = \delta_{vv'}, \tag{18}$$

$$\langle \beta_e | \beta_{e'} \rangle = \delta_{ee'}, \tag{19}$$

as well as

$$A^T A = I_n, \tag{20}$$

$$B^T B = I_m. \tag{21}$$

One can easily verify that $|\alpha_v\rangle$ and $|\beta_e\rangle$ are unit vectors due to the stochasticity of P_{vE} and P_{EV} . Distinctly, these equations imply that the action of A preserves the norm of the vectors. The same is true regarding B .

We now immediately define the projectors Π_A and Π_B as follows:

$$\Pi_A = AA^T = \sum_{v \in V} |\alpha_v\rangle \langle \alpha_v|, \tag{22}$$

$$\Pi_B = BB^T = \sum_{e \in E} |\beta_e\rangle \langle \beta_e|. \tag{23}$$

Using Eqs (22) and (23), it is easy to see that Π_A projects onto subspace H_A spanned by $\{|\alpha_v\rangle : v \in V\}$, and Π_B projects onto subspace H_B spanned by $\{|\beta_e\rangle : e \in E\}$. After obtaining the projectors, we can define the associated reflection operators, which are

$$R_A = 2\Pi_A - I_{nd}, \tag{24}$$

$$R_B = 2\Pi_B - I_{mk}, \tag{25}$$

where $I_{nd} = I_{mk}$ is the identity operator.

Note that the reflection operators R_A and R_B are unitary, we can get the following Proposition.

Proposition $2R_A$ is a reflection operator about the space H_A . Similarly, $2R_B$ is a reflection operator about the space H_B . *Proof.* We prove that R_A is a reflection operator about the space H_A . The other claim is analogous.

$$\begin{aligned}
 R_A|\Psi\rangle &= (2\Pi_A - I)|\Psi\rangle \\
 &= \left(2\sum_{v\in V} |\alpha_v\rangle\langle\alpha_v| - I\right)|\Psi\rangle \\
 &= \begin{cases} |\Psi\rangle & \text{if } \Psi \in H_A \\ -|\Psi\rangle & \text{if } \Psi \in H_A^\perp \end{cases}
 \end{aligned}
 \tag{26}$$

Definition 1 (Quantization of M.C. P)

$$W = R_B R_A \tag{27}$$

A single step of the quantum walks is given by the unitary evolution operator W based on the transition matrix P . In the bipartite graph, an application of W corresponds to two quantum steps of the walk from v to e and from e to v . At time t , the whole operator of the quantum walks is W^t .

Spectral analysis of quantum walks on hypergraphs. In many classical algorithms, the eigen-spectrum of the transition matrix P plays a critical role in the analysis of Markov chains. In a similar way, we now proceed to study the quantitative spectrum of the quantum walks unitary operator W .

Szegedy proved a spectral theorem for quantum walks, $W = ref_2ref_1$, in ref.⁷. In this section, we deliver a slightly different version in that the cardinality of set X may be different from the cardinality of set Y in the bipartite graph. In order to analyze the spectrum, we need to study the spectral properties of an $n \times m$ matrix D , which indeed establishes a relation between the classical Markov chains and the quantum walks. This matrix is defined as follows:

Definition 2 (Discriminant Matrix) The discriminant matrix for W is

$$D_{nm} = A^T B. \tag{28}$$

It also follows from the definition that $D = \sqrt{P_{VE} \circ P_{EV}}$ with entries

$$D_{ve} = \sqrt{p_{ve} p_{ev}}, \forall v \in V, \forall e \in E. \tag{29}$$

Suppose that the discriminant matrix D has the singular value decomposition $D = U\Sigma V^T = \sum_i \sigma_i \mu_i \nu_i^T$. The left singular vectors $|\mu_k\rangle$ satisfy

$$D|\nu_k\rangle = \sigma_k|\mu_k\rangle \tag{30}$$

and the right singular vectors

$$\langle\mu_k|D = \langle\nu_k|\sigma_k \tag{31}$$

with σ_k the singular value.

Theorem 2 For any σ_k the singular value of D , $0 \leq \sigma_k \leq 1$.

Proof. First, let $Dk = \sigma_k k$. Then we obtain

$$\begin{aligned}
 |\sigma_k|^2 \|k\|^2 &= \|Dk\|^2 \\
 &= \langle Dk, Dk \rangle \\
 &= \langle A^T Bk, A^T Bk \rangle \\
 &= \langle Bk, AA^T Bk \rangle \\
 &\leq \langle Bk, Bk \rangle \\
 &= \langle k, B^T Bk \rangle \\
 &= \langle k, k \rangle = \|k\|^2.
 \end{aligned}
 \tag{32}$$

Thus, $|\sigma_k| \leq 1$. Since $\langle k, D^T Dk \rangle \geq 0$ for all k , we have $0 \leq \sigma_k$. Therefore, $0 \leq \sigma_k \leq 1$.

Observing theorem 2, we can write the singular value σ_k as $\cos \theta_k$, where θ_k is the principal angle between subspace H_A and H_B . In the earlier years, Björck and Golub⁴¹ deduced the relationship between the singular value decomposition and the principal angle θ_k between subspace H_A and H_B . That is, $\cos(\theta_k) = \sigma_k$.

In the remainder of this section, we will explore the eigen-decomposition of the operator W , which can be calculated from the singular value decomposition of D .

First, we turn to the dimensionality of the spaces. Suppose that $n \geq m$. We learned earlier that nd is the dimension of edge Hilbert space about the bipartite graph $BG_{n,m}$, and the discriminant matrix D has m singular values, only some of which are non-zero. Space H_A spanned by $\{|\alpha_v\rangle; v \in V\}$, and space H_B spanned by $\{|\beta_e\rangle; e \in E\}$, are $n - dimension$ and $m - dimension$ subspaces of H^{nd} , respectively. Let H_{AB} be the space spanned by $\{|\alpha_v\rangle; v \in V\}$

and $\{|\beta_e\rangle: e \in E\}$, and the dimension of H_{AB} is $rank(H_{AB})$. Therefore, the dimension of space $(H_A \cup H_B)^\perp$ is given by $nd - rank(H_{AB})$, there are $2m$ eigenvectors in the space $H_A \cap H_B$, and $rank(H_{AB}) - 2m$ remaining vectors in the space $H_A \cup H_B - H_A \cap H_B$.

Using A to left-multiply (30) and B to left-multiply (31), We have

$$AD|\nu_k\rangle = (AA^T)B|\nu_k\rangle = \sigma_k A|\mu_k\rangle, \tag{33}$$

$$BD^T|\mu_k\rangle = (BB^T)A|\mu_k\rangle = \sigma_k B|\nu_k\rangle. \tag{34}$$

As we mentioned before, the action of A and B preserve the norm of the vectors, and $|\nu_k\rangle$ and $|\mu_k\rangle$ are unit vectors, so $A|\mu_k\rangle$ and $B|\nu_k\rangle$ also are unit vectors.

Proposition 3 On space $(H_A \cup H_B)^\perp$, W acts as the identity.

Proof. Suppose a vector $v \in (H_A \cup H_B)^\perp$. Then v lies in the space orthogonal to both H_A and H_B . Since R_i reflects a vector orthogonal to H_i , then v is reflected by both R_A and R_B . Then applying a walk operator W does not change v . Hence $W|v\rangle = |v\rangle$, and the subspace spanned by such v is an invariant subspace.

Observing Proposition 3, we only need to analyze the behavior of W in $H_A \cup H_B$.

Theorem 3 The subspace $span\{A|\mu_k\rangle, B|\nu_k\rangle\}$ is invariant under the action W .

Proof. Suppose we have $|v\rangle \in H_A^n$ and $|w\rangle \in H_B^m$, then we want to analyze the action of R_B on $A|v\rangle$ and the action of R_A on $B|w\rangle$. (Since $A|v\rangle \in H_A$, the action of R_A on $A|v\rangle$ is identity). Since we have

$$R_B A|v\rangle = 2BB^T A|v\rangle - A|v\rangle \tag{35}$$

where the first term of the last line lies in H_B and the second term in H_A , and similarly

$$R_A B|w\rangle = 2AA^T B|w\rangle - B|w\rangle \tag{36}$$

Hence, we only need to consider the action of W on $A|\mu_k\rangle$ and $B|\nu_k\rangle$ and that $A|\mu_k\rangle$ is invariant by R_A and $B|\nu_k\rangle$ is invariant by R_B .

$$\begin{aligned} WA|\mu_k\rangle &= R_B R_A A|\mu_k\rangle \\ &= R_B A|\mu_k\rangle \\ &= 2\sigma_k B|\nu_k\rangle - A|\mu_k\rangle \\ &= (\cos(\theta_k)B|\nu_k\rangle) - (A|\mu_k\rangle - \cos(\theta_k)B|\nu_k\rangle) \end{aligned} \tag{37}$$

The first term in the last line is the component of $A|\mu_k\rangle$ that is along $B|\nu_k\rangle$ and the second term is the component of $A|\mu_k\rangle$ that is orthogonal to $B|\nu_k\rangle$. So, on $A|\mu_k\rangle$, R_A is a reflection about $A|\mu_k\rangle$ (identity in this case), and R_B is a reflection about $B|\nu_k\rangle$ (because of the orthogonal component) in the subspace. Similarly, on $B|\nu_k\rangle$, R_A is a reflection about $A|\mu_k\rangle$ (because of the orthogonal component), and R_B is a reflection about $B|\nu_k\rangle$ (identity) in the subspace.

Next we shall create eigenvectors for W with the corresponding eigenvalues. Suppose that

$$W|k\rangle = \lambda_k |k\rangle \tag{38}$$

and

$$|k\rangle = aA|\mu_k\rangle + bB|\nu_k\rangle. \tag{39}$$

Simply plugging (39) into formulas (38), we obtain the following equation:

$$W|k\rangle = \lambda_k aA|\mu_k\rangle + \lambda_k bB|\nu_k\rangle. \tag{40}$$

Then, left-multiplying (39) by W , we have

$$\begin{aligned} W|k\rangle &= W(aA|\mu_k\rangle + bB|\nu_k\rangle) \\ &= -(a + 2b\sigma_k)A|\mu_k\rangle + [2a\sigma_k^2 + b(4\sigma_k - 1)]B|\nu_k\rangle. \end{aligned} \tag{41}$$

Comparing formulas (40) and (41), we can obtain the following equations:

$$\lambda_k a = -(a + 2b\sigma_k), \tag{42}$$

Eigenvector of W	Eigenvalue of W	Singular values of D
$\frac{A \mu_k\rangle - e^{\pm i\theta_k}B \nu_k\rangle}{\sqrt{2}\sin\theta_k}$	$\lambda_k = e^{\pm 2i\theta_k}$	$\sigma_k = \cos\theta_k$
$A \mu_k\rangle$	1	1
$noexpr.$	-1	0

Table 1. Eigenvalues of W obtained from the singular values of D , and angles θ_k obtained from the formula $\sigma_k = \cos\theta_k$, where $k = 1, 2, \dots, m$.

$$\lambda_k b = 2a\sigma_k^2 + b(4\sigma_k - 1). \quad (43)$$

Concerning unit vectors $A|\mu_k\rangle$ and $B|\nu_k\rangle$, we consider two cases with respect to non-collinearity and collinearity, as follows.

Case 1. First, we consider that $A|\mu_k\rangle$ and $B|\nu_k\rangle$ are linearly independent.

Using $\sigma_k = \cos\theta_k$, we obtain

$$\lambda_k = e^{\pm 2i\theta_k} \quad (44)$$

through a series of algebraic operations. Furthermore, we have the corresponding eigenvectors

$$|k\rangle = \frac{A|\mu_k\rangle - e^{\pm i\theta_k}B|\nu_k\rangle}{\sqrt{2}\sin\theta_k}. \quad (45)$$

Case 2. Then, we consider that $A|\mu_k\rangle$ and $B|\nu_k\rangle$ are collinear. However, since $A|\mu_k\rangle$ is invariant under the action of Π_A , $B|\nu_k\rangle$ also is; and vice versa, since $B|\nu_k\rangle$ is invariant under Π_B , and $A|\mu_k\rangle$ also is. Therefore, $A|\mu_k\rangle$ and $B|\nu_k\rangle$ are invariant under the action of W , and $A|\mu_k\rangle$ are eigenvectors of W with eigenvalue 1.

As a consequence, we obtain the following theorem:

Theorem 4 Let W be the unitary evolution operator on $BG_{n,m}$. Then W has eigenvalues 1 and -1 in the one-dimensional subspaces, and eigenvalues $\lambda_k = e^{\pm 2i\theta_k}$ ($0 < \theta_k < \frac{\pi}{2}$) where ($k = 1, 2, \dots, m$) and the eigenvectors $|k\rangle = \frac{A|\mu_k\rangle - e^{\pm i\theta_k}B|\nu_k\rangle}{\sqrt{2}\sin\theta_k}$ in the two-dimensional subspaces Table 1.

Discussion

Quantum walks are one of the elementary techniques of developing quantum algorithms. The development of successful quantum walks on graphs-based algorithms have boosted such areas as element distinctness, searching for a marked element, and graph isomorphism. In addition, the utility of walking on hypergraphs has been probed deeply in several contexts, including natural language parsing, social networks database design, or image segmentation, and so on. Therefore, we put our attention on quantum walks on hypergraphs considering its promising power of inherent parallel computation.

In this paper, we developed a new schema for discrete-time quantum walks on regular uniform hypergraphs using extended Szegedy's walks that naturally quantize classical random walks and yield quadratic speed-up compared to the hitting time of classical random walks. We found the one-to-one correspondence between regular uniform hypergraphs and bipartite graphs. Through the correspondence, we convert the regular uniform hypergraph into its associated bipartite graph on which extended Szegedy's walks take place. In addition, we dealt with the case that the cardinality of the two disjoint sets may be different from each other in the bipartite graphs. Furthermore, we delivered spectral properties of the transition matrix, which is the essence of quantum walks, and which has prepared for followup studies.

Our work presents a model for quantum walks on regular uniform hypergraphs, and the model opens the door to quantum walks on hypergraphs. We hope our model can inspire more fruitful results in quantum walks on hypergraphs. Our model provides the foundation for building up quantum algorithms on the strength of quantum walks on hypergraphs. Moreover, the algorithms of quantum walks on hypergraphs will be useful in quantum computation such as quantum walks search, quantized Google's PageRank, and quantum machine learning, based on hypergraphs.

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Author Contributions

Ying Liu initiated the idea and designed the model; Ying Liu, Jiabin Yuan, Bojia Duan and Dan Li provided the theoretical proposal and analysis. All authors reviewed the manuscript.

Additional Information

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