Self-similar intermediate asymptotics for nonlinear degenerate parabolic free-boundary problems that occur in image processing

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In the boundary layers around the edges of images, basic nonlinear parabolic equations for image intensity used in image processing assume a special degenerate asymptotic form. An asymptotic self-similar solution to this degenerate equation is obtained in an explicit form. The solution reveals a substantially nonlinear effect—the formation of sharp steps at the edges of the images, leading to edge enhancement. Positions of the steps and the time shift parameter cannot be determined by direct construction of a self-similar solution; they depend on the initial condition of the pre-self-similar solution. The free-boundary problem is formulated describing the image intensity evolution in the boundary layer.

nonlinear partial differential equations | self-similar solutions

Modern computer vision studies are based on a differential-geometric approach having roots in ideas presented in the inaugural lecture of B. Riemann at the Philosophical Faculty of Göttingen University in 1854. This lecture was earmarked in fact for a single listener, K. F. Gauss, who selected the topic from three that were proposed by Riemann and whose ideas concerning geometric theory of surfaces received in this lecture far-reaching development.

In an illuminating essay by B. Kagan (1), a review of the development of Riemann's geometric ideas in an active period up to the mid-thirties is presented most comprehensively together with a detailed bibliography. I want to mention here an instructive moment. Riemann's lecture was published by a German mathematician, R. Dedekind, long after Riemann's death. The title of the lecture was "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" (on the hypotheses which lie at the foundation of the geometry). Soon after publication of Riemann's lecture, there appeared a paper by H. von Helmholtz having a title practically coinciding with the title of Riemann's lecture except for a single word: "Ueber die Tatsachen welche der Geometrie zu Grunde liegen" (on the facts which lie at the foundation of the geometry). Helmholtz claimed in this paper that he came to the ideas presented in Riemann's lecture independently, and what is most interesting now, by a completely different motivation, trying to construct a physiological model of vision (Helmholtz's basic profession was physiology and medicine). It is instructive to see how these ideas are resurrected in computer vision science!

Rather early it was recognized in computer vision studies [see especially the paper by Perona and Malik (2)] that the technique of image processing leads to solving nonlinear parabolic partial differential equations. What is important (it was emphasized in ref. 2), that a properly selected nonlinearity, i.e., the image intensity flux, can lead to an enhancement of image edges even if the flux is as usually directed opposite to the image intensity gradient. A different approach to the edge enhancement problem was proposed by Alvarez *et al.* (3). They selected the image flux direction orthogonal to the image intensity gradient. The basic partial differential equation for the image intensity obtained in ref. 3 is also a nonlinear parabolic one, but it does not belong to the class outlined in ref. 2.

In the present note the appearance of the edge enhancement in the technique proposed by Malladi and Sethian and their colleagues is investigated. In refs. 4 and 5, these authors arrived at the following equations for image intensity ϕ by using the differential-geometric approach and various assumptions concerning the image intensity flux:

$$\partial_t\phi = \frac{(1+(\partial_y\phi)^2)\partial^2_{xx}\phi - 2\partial_x\phi\partial_y\phi\partial^2_{xy}\phi + (1+(\partial_x\phi)^2)\partial^2_{yy}\phi}{1+(\partial_x\phi)^2+(\partial_y\phi)^2} \quad [1]$$

[mean curvature flow (4)], and

$$\partial_t\phi = \frac{(1+(\partial_y\phi)^2)\partial^2_{xx}\phi - 2\partial_x\phi\partial_y\phi\partial^2_{xy}\phi + (1+(\partial_x\phi)^2)\partial^2_{yy}\phi}{[1+(\partial_x\phi)^2+(\partial_y\phi)^2]^2} \quad [2]$$

[Beltrami flow (5)]. Here, x and y are the Cartesian coordinates in the image plane, t is time. Thus, according to refs. 4 and 5, image processing is reduced to the solution of the chosen equation under an initial condition $\phi(x, y, t_0) = \phi_0(x, y)$ corresponding to a grey level of the image being processed. I note that later the equation (**2**) was also published by Yezzi (6), who used a different model for the image processing.

As a result of a certain degeneracy of the asymptotic forms of Eqs. **1** and **2**, it is appropriate to consider a more general class of equations

$$\partial_t\phi = \gamma\frac{(\beta^2+(\partial_y\phi)^2)\partial^2_{xx}\phi - 2\partial_x\phi\partial_y\phi\partial^2_{xy}\phi + (\beta^2+(\partial_x\phi)^2)\partial^2_{yy}\phi}{[\beta^2+(\partial_x\phi)^2+(\partial_y\phi)^2]^{1+\alpha}}, \quad [3]$$

where $\alpha \geq 0$ and β, γ are positive constants. Both Eqs. **1** and **2** belong to this class.

Boundary Layer Effect in Image Processing and the Asymptotic Form of the Basic Equation

An analysis of images presented in refs. 4 and 5 showed that near the edges of the images always exists a boundary layer (see Fig. 1), where the normal component of the image intensity gradient is large. We use the local Cartesian coordinates in the boundary layer: x, along the normal to its midline, and y, along the midline. It can be assumed that $(\partial_x\phi)^2 \sim 1/h^2$ in the boundary layer is, generally speaking, much larger than β^2. It can be assumed also that in the boundary layer, $(\partial_y\phi)^2 \ll 1/H^2$ is much less than β^2. Therefore, I can neglect $(\partial_y\phi)^2$ in comparison with β^2. Eq. **3** in the boundary layer is reduced to the one-dimensional form

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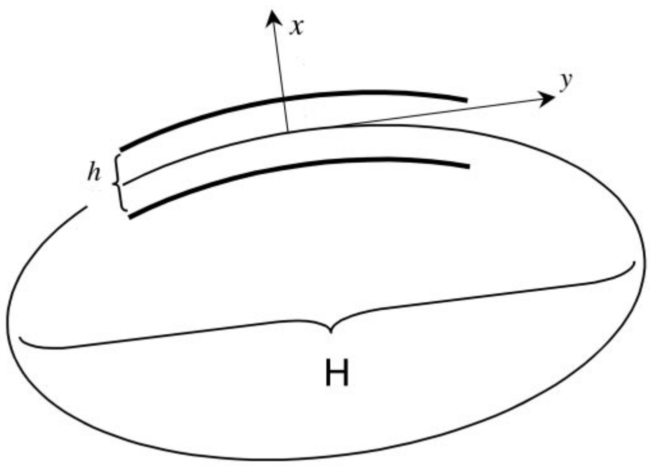

Fig 1. The boundary layer at the image edge.

$$\partial_t \phi = \kappa \frac{\partial^2_{xx}\phi}{[\beta^2 + (\partial_x \phi)^2]^{1+\alpha}}. \quad [4]$$

Here, $\kappa = \gamma\beta^2$. Eq. **4** belongs to a general class of equations considered in the article by Bertsch and Dal Passo (7). If $(\partial_x\phi)^2$ is much larger than β^2, I can neglect β^2 in the denominator of Eq. **4**, and an asymptotic form of Eq. **3** is obtained

$$\partial_t \phi = \kappa \frac{\partial^2_{xx}\phi}{[(\partial_x \phi)^2]^{1+\alpha}}, \quad [5]$$

governing the evolution of the image intensity in the boundary layer.

I note a certain connection between Eq. **5** and the Bertsch equation (see ref. 8)

$$\partial_t \psi = \psi \partial^2_{xx}\psi - (c-1)(\partial_x\psi)^2$$

(c is a constant). Indeed, assuming $\phi = \psi^{(1+2\alpha)/2(1+\alpha)}$, I reduce Eq. **5** to a similar form

$$\partial_t \psi = \kappa \frac{[\psi\partial^2_{xx}\psi - [1/2(1+\alpha)](\partial_x\psi)^2]}{(1+2\alpha/2(1+\alpha))^{2(1+\alpha)}(\partial_x\psi)^{2(1+\alpha)}}.$$

This form is also more convenient for numerical computations.

Intermediate-Asymptotic Solution

For a useful comparison, I present at first briefly a derivation based on the dimensional analysis of the classic intermediate-asymptotic solution to the linear equation $\partial_t\phi = \kappa\partial^2_{xx}\phi$ (formally corresponding to Eq. **5** for $\alpha = -1$) for a "smoothed step" initial-boundary value problem

$$\phi(x, -t_0) = \begin{cases} \phi_1, & -\infty < x \le -a \\ \phi_0(x), & -a \le x \le b \\ \phi_2, & b \le x < \infty \end{cases} \quad [6]$$

$$\phi(-\infty, t) = \phi_1,\ \phi(\infty, t) = \phi_2.$$

Here, $\phi_1 > \phi_2 \ge 0$ and $a, b > 0$ are constant parameters of the problem, and the function $\phi_0(x)$ is assumed to be smooth at $-a \le x \le b$, so that $\phi_0(-a) = \phi_1$, $\phi_0(b) = \phi_2$. Also, it is assumed that $\phi_0'(-a)$, $\phi_0'(b)$ are ≤ 0. Without loss of generality, ϕ_2 can be assumed to be equal to zero.

A priori an intermediate-asymptotic solution to the problem Eq. **6** can depend only on the quantities κ, $t + t_0$, ϕ_1, and $x - x_0$. The constant x_0 which enters due to the invariance of the equation to shift $x' = x + \text{const}$; it remains, however, undetermined in a direct construction of the self-similar intermediate asymptotics. The dimensions of the involved quantities are $[\phi] = [\phi_1] = \Phi$, $[x - x_0] = L$, $[t + t_0] = T$, and for the linear case under consideration, $[\kappa] = L^2T^{-1}$. (Maxwell's notation is used for the dimension of z.) Here, Φ is the independent dimension of ϕ, L and T are dimensions of length and time. Dimensional analysis shows that the intermediate-asymptotic solution can be represented in the form $\phi = \phi_1 f(\xi)$, where, for the linear case under consideration, a dimensionless independent variable is inversely proportional to $\sqrt{t + t_0}$: $\xi = (x - x_0)/\sqrt{\kappa(t + t_0)}$. Substituting $\phi = \phi_1 f(\xi)$ to the linear equation (Eq. **5** for $\alpha = -1$), I obtain a linear ordinary differential equation for the function f. Easy integration under boundary conditions $f(-\infty) = 1$, $f(\infty) = 0$ allows one to obtain the function f in an explicit form, and the intermediate-asymptotic solution appears in the classic form

$$\phi = \phi_1 \frac{1}{\sqrt{\pi}} \int_{(x-x_0)/2\sqrt{\kappa(t+t_0)}}^{\infty} e^{-z^2} dz. \quad [7]$$

Solution **7** demonstrates that for a linear case ($\alpha = -1$), the smoothed stepwise initial distribution extends with time; its properly defined width increases with time proportionally to $\sqrt{t + t_0}$, and the maximum of the derivative modulus $|\partial_x\phi|$ decreases with time as $1/(t + t_0)^{1/2}$.

I repeat now the above argument for the case of nonlinear asymptotic Eq. **5** corresponding to $\alpha = 1$ [the Beltrami flow, Malladi and colleagues (5)]. The essential difference is that in this case, the dimension of coefficient κ is different:

$$\kappa = \Phi^4 L^{-2} T^{-1}. \quad [8]$$

This difference leads to a dramatic change in the solution. As before, the solution is represented in the form $\phi = \phi_1 f(\xi)$; however in this case ($\alpha = 1$),

$$\xi = (x - x_0)\sqrt{\kappa(t + t_0)}/\phi_1^2, \quad [9]$$

so that the dimensionless argument of the function f is directly proportional, not inversely proportional, to $\sqrt{t+t_0}$. Eq. **5** assumes for the case $\alpha = 1$ the form

$$\partial_t\phi = \kappa\partial^2_{xx}\phi/(\partial_x\phi)^4. \quad [10]$$

Substituting to Eq. **10** $\phi = \phi_1 f(\xi)$, we obtain for $f(\xi)$ the ordinary differential equation:

$$\tfrac{1}{2}\xi\frac{df}{d\xi} = \frac{d^2f}{d\xi^2}\left(\frac{df}{d\xi}\right)^{-4}, \quad [11]$$

where ξ is determined by Eq. **9**. Easy integration gives

$$\frac{df}{d\xi} = -\frac{1}{\xi_f^{1/2}[1 - (\xi^2/\xi_f^2)]^{1/4}}. \quad [12]$$

Here, ξ_f is an integration constant. Further integration and the boundary conditions $f(-\xi_f) = 1$, $f(\xi_f) = 0$ give

$$f = 1 - \xi_f^{1/2}\int_{-1}^{\xi/\xi_f} \frac{d\zeta}{(1 - \zeta^2)^{1/4}},\ -\xi_f \le \xi \le \xi_f. \quad [13]$$

The integration constant ξ_f is obtained from the condition $f(\xi_f) = 0$, so that

APPLIED MATHEMATICS

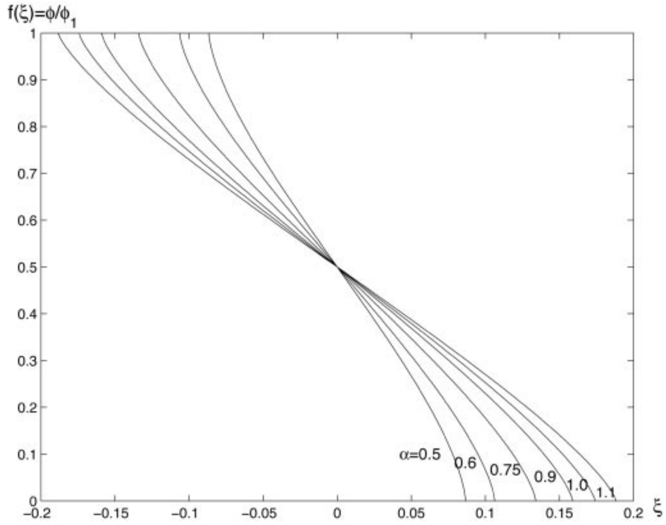

Fig 2. The self-similar solutions for different values of the parameter α.

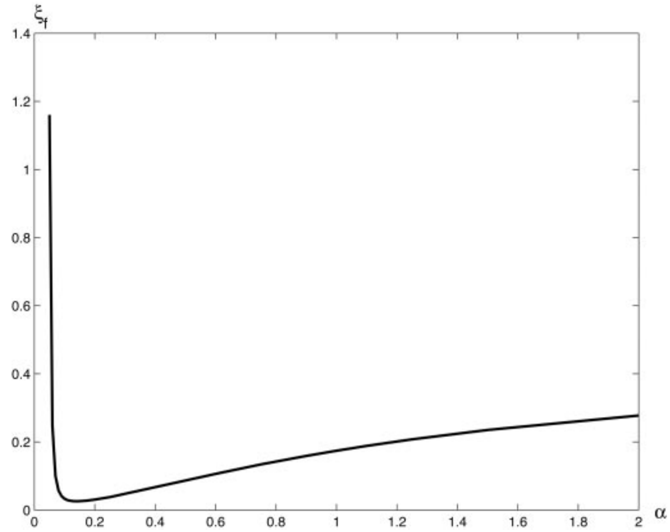

Fig 3. The dependence of the dimensionless width of the transition region ξ_f on α.

$$\xi_f = \frac{1}{\left[2\int_0^1 \frac{d\zeta}{(1-\zeta^2)^{1/4}}\right]^2} \cong 0.174. \qquad \textbf{[14]}$$

Thus, the intermediate-asymptotic solution assumes the form (see Fig. 2):

$$\phi = \phi_1\left(1 - \xi_f^{1/2}\int_{-1}^{[(x-x_0)\sqrt{\kappa(t+t_0)}/\xi_f\phi_1^2]} \frac{d\zeta}{(1-\zeta^2)^{1/4}}\right) \qquad \textbf{[15]}$$

for $x_f^- = x_0 - \xi_f(\phi_1^2)/(\sqrt{\kappa(t+t_0)}) \le x \le x_f^+ = x_0 + \xi_f(\phi_1^2/\sqrt{\kappa(t+t_0)})$. It is seen that, contrary to the linear case presented above, this solution is a local solution. At free boundaries $x = x_f^-$ and $x = x_f^+$, the image intensity is continuous but the derivative $\partial_x\phi$ suffers an infinite jump. The condition $\partial_x\phi = -\infty$ can be interpreted as the zero flux condition.

Relation **15** reveals important asymptotic properties of the image evolution in the boundary layer at the image edge. First of all, the width of the transition region $x_f^+ - x_f^-$ equal to $2\xi_f\phi_1^2[\kappa(t+t_0)]^{-1/2}$ decreases with time; the step forms from an initially noisy image (Relation **6**) and the edge enhancement takes place. Furthermore, the value of $\phi(x_0, t)$ remains constant and equal to $\phi_1/2$. (I emphasize that position x_0 cannot be obtained in the presented construction and requires a matching with pre-self-similar solution, e.g., by a numerical computation.) Finally, the value of $|\partial_x\phi|$ at $x = x_0$ equal to $\sqrt{\kappa(t+t_0)}/\phi_1\,\xi_f^{1/2}$, which is the minimum of $|\partial_x\phi|$, is growing with time, therefore the validity of the asymptotic Eq. **5** improves with time.

Solution **15** suggests the following free-boundary problem for determination of the image intensity evolution in the boundary layer. At the initial moment, the points $x_f^-(t_0) = -a$ and $x_f^+(t_0) = b$—the boundaries of the uncertainty belt—are prescribed, so that $\phi \equiv \phi_1$ for $x \le x_f^-(t_0)$ and $\phi \equiv 0$ for $x \ge x_f^+(t_0)$ (Relation **6**). At $t > t_0$, the image intensity $\phi(x, t)$ and the free boundaries $x_f^-(t), x_f^+(t)$ should be determined so that Eq. **4**, initial condition **6**, and the conditions at free boundaries

$$\phi \equiv \phi_1,\ \partial_x\phi = -\infty \text{ at } x = x_f^-(t);\ \phi \equiv 0,\ \partial_x\phi = -\infty \text{ at } x = x_f^+(t) \qquad \textbf{[16]}$$

should be satisfied. The condition $\partial_x\phi = -\infty$ at $x = x_f^-(t)$ and $x = x_f^+(t)$ can be interpreted as zero flux condition. This one-dimensional free-boundary problem can be implemented to two-dimensional problems.

Intermediate-Asymptotic Solution for Arbitrary $\alpha > 0$

I return to the general Eqs. **4** and **5**. In this case, $[\kappa] = \Phi^{2(1+\alpha)}L^{-2\alpha}T^{-1}$, so that

$$\phi = \phi_1 f(\xi),\ \xi = (x - x_0)(\kappa(t+t_0))^{1/2\alpha}/\phi_1^{(1+\alpha)/\alpha}, \qquad \textbf{[17]}$$

and the equation for the function $f(\xi)$ takes the form

$$\frac{1}{2\alpha}\,\xi\,\frac{df}{d\xi} = \frac{d^2f}{d\xi^2}\left(\frac{df}{d\xi}\right)^{-2(1+\alpha)}. \qquad \textbf{[18]}$$

Integrating and using the boundary condition $f(-\xi_f) = 1$, I obtain

$$f(\xi) = 1 - \left(\frac{2\alpha}{1+\alpha}\right)^{1/2(1+\alpha)} \xi_f^{\alpha/1+\alpha}\int_{-1}^{\xi/\xi_f} \frac{d\zeta}{(1-\zeta^2)^{1/2(1+\alpha)}} \qquad \textbf{[19]}$$

for $-\xi_f \le \xi \le \xi_f$. By using the boundary condition $f(\xi_f) = 0$, the relation for ξ_f can be obtained:

$$\xi_f = \left[2\left(\frac{2\alpha}{1+\alpha}\right)^{1/2(1+\alpha)} \int_0^1 \frac{d\zeta}{(1-\zeta^2)^{1/2(1+\alpha)}}\right]^{-(1+\alpha)/\alpha}. \qquad \textbf{[20]}$$

The function $\xi_f(\alpha)$ is nonmonotonic (see Fig. 3); $\xi_f(0)$ is equal to infinity. At first, $\xi_f(\alpha)$ decreases with growing α, reaches a minimum, and then starts to grow.

The intermediate-asymptotic solution takes for arbitrary positive α the form $\phi \equiv \phi_1$, for

$$x < x_f^- = x_0 - \xi_f\phi_1^{(1+\alpha)/\alpha}(\kappa(t+t_0))^{-1/2\alpha};$$

$$\phi = \phi_1\left[1 - \left(\frac{2\alpha}{1+\alpha}\right)^{1/2(1+\alpha)} \xi_f^{\alpha/1+\alpha}\int_{-1}^{[(x-x_0)(\kappa(t+t_0))^{1/2\alpha}]/[\phi_1^{1+\alpha/\alpha}\xi_f]} \frac{d\zeta}{(1-\zeta^2)^{1/2(1+\alpha)}}\right], \qquad \textbf{[21]}$$

for $x_f^- < x < x_f^+$, $x_f^+ = x_0 + \xi_f \phi_1^{1+\alpha/\alpha} (\kappa(t + t_0))^{-1/2\alpha}$; and $\phi \equiv 0$, for $x > x_f^+$. For the width of the transition region, the relation **21** suggests the expression

$$x_f^+ - x_f^- = 2\xi_f \frac{\phi_1^{1+\alpha/\alpha}}{(\kappa(t + t_0))^{1/2\alpha}}. \quad [22]$$

So, qualitatively the situation for any $\alpha > 0$ is the same as in the case of the Beltrami flow $\alpha = 1$: the edge enhancement will take place if any equation of this class will be used.

A. E. Chertock performed a series of numerical computations of the solutions to the suggested free-boundary problem for the Eq. **4**. The function $\phi_0(x)$ in some runs was nonmonotonic. Parameter α assumed the values $\alpha = 1$ and other values including small positive ones. Computations demonstrated that the self-similar solution **21** was an intermediate asymptotics of the solutions computed numerically (A. Chertock, unpublished data). On Fig. 4, the evolution of the image intensity distribution in time is presented for a nonmonotonic initial condition in the case $\alpha = 1$ (Beltrami flow).

The case of the mean curvature flow (Eq. **1**) corresponding to $\alpha = 0$ requires additional analysis.

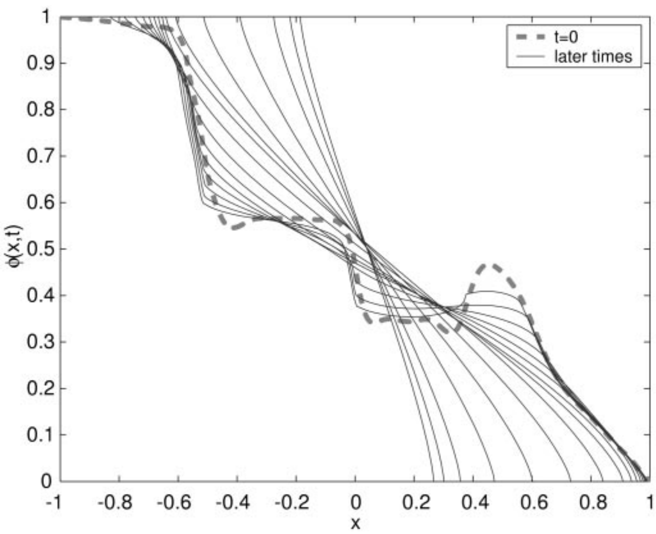

Fig 4. The evolution of the image intensity distribution for $\alpha = 1$ (Beltrami flow).

Conclusion

A free-boundary problem is formulated for the image intensity evolution in the boundary layer around the edge of the image. Analysis of intermediate-asymptotic solutions for the image evolution in the boundary layer of an image demonstrated that the edge enhancement takes place for the class of flows under consideration. The rate of enhancement depends on the parameter, i.e., on the hypotheses concerning the image intensity flow.

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