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An H(div)-conforming Finite Element Method for Biot's Consolidation Model

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Abstract

In this paper, we develop an H(div)-conforming finite element method for Biot's consolidation model in poroelasticity. In our method, the flow variables are discretized by an H(div)-conforming mixed finite elements. For relaxing the H^1-conformity of the displacement, we approximate the displacement by using an H(div)-conforming finite element method, in which the tangential components are discretized in the interior penalty discontinuous Galerkin framework. For both the semi-discrete and the fully discrete schemes, we prove the existence and uniqueness theorems of the approximate solutions and derive the optimal convergence rate for each variable.

Keywords

poroelasticity; mixed finite element; H(div)-conforming; discontinuous Galerkin method

1. Introduction

Poroelasticity [3] is of increasing interest because of its vital importance in various science and engineering applications. For example, the mathematical models for carbon sequestration in environment engineering, seismic wave propagation in earthquake prediction, surface subsidence, evolution of fractured reservoirs during gas production, and biomechanical descriptions of tissues and bones are all poroelastic models. These models describe the interactions between a fluid flow and a deformable elastic porous medium which is saturated in the fluid. In this work, we are interested in Biot's consolidation model. In the model, the motion of fluid in the porous medium is described by Darcy's law, whereas the deformation of the porous medium is governed by linear elasticity.

Because of the complex nature of Biot's model and the domain is usually irregular, it is not easy to obtain an analytical solution of this model. Thus, many researchers turn their

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attention to computational simulations. However, noting that both fluid dynamics and elasticity are involved in Biot's model, it is very important to design effective numerical methods that can mimic the physical laws involved in. Unfortunately, all numerical difficulties that existing in linear elasticity and fluid mechanics will also arise in numerical approximations of Biot's model. For the linear elasticity part, continuous Galerkin approximation of the displacement may cause locking or nonphysical pressure oscillations [5,27,30]. For eliminating the locking phenomenon in solving linear elasticity model, one can apply mixed Finite element method [23, 37] or employ discontinuous Galerkin (DG) method [30], or use the nonconforming finite element [36], or use weak Galerkin methods [9,17,32]. In numerical methods for incompressible fluid flow models, the standard Stokes elements (such as Taylor-Hood element and Mini element) have the shortcoming in that they do not satisfy the divergence constraint strongly or globally and therefore are not mass conservative [11,12,18].

In this work, we follow the strategy developed in [11,12,33] and adopt an H(div)-conforming finite element for the displacement. The purpose is to relax the H^1-conformity of displacement. The advantages of adopting such a discretization are two-fold: On one hand, the normal components of displacement across elements are continuous and therefore are locally conservative; On the other hand, the tangential components are discretized through an interior penalty discontinuous Galerkin method. As it is discontinuous Galerkin approximation, such a discretization enables us to overcome the locking phenomenon and the pressure oscillation [18,31,38]. We comment here that applying H(div)-conforming finite elements in a DG framework was initially proposed in [11] (see also [12,33]) for solving Stokes equations in fluid mechanics. Later, this method is extended to solve the Darcy-Stokes interface problems [10,19], Brinkman problem [20] and magnetic induction model [8]. In Biot's model, for the fluid part, we note that the governing equation is Darcy's law. If the mixed form of Darcy's law is used, it is natural to apply an H(div)-conforming finite element discretization to approximate the flow variables pressure because such a discretization can guarantee the mass conservation. In this work, we adopt Brezzi-Douglas-Marini (BDM_k) space for both the flow variables and the displacement. Moreover, we present a unified treatment of both flow variables and the displacement in our Finite Element method. This work can be regarded as a further development of H(div)-conforming finite element methods for solving Biot's problem. By using the framework presented in [28,29,36,38], we give a detailed analysis of our method. In particular, for both the semi-discrete and the fully discrete schemes for Biot's model, we prove the existence and uniqueness theorems of the approximate solutions and derive the optimal convergence rate for each variable.

The rest of this paper is organized as follows. In Section 2, we describe Biot's consolidation model, the functional spaces and the corresponding weak formulation. A spatial semi-discrete scheme based on H(div)-conforming elements is proposed in Section 3. The existence and uniqueness theorems for the semi-discrete numerical scheme are proved. Moreover, we derive the a priori error estimates of the solution of the semi-discrete scheme. In Section 4, a fully discrete numerical scheme based on the backward Euler time discretization is presented and analyzed. Conclusions are drawn in Section 5.

2. Biot's consolidation model and its weak formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with a Lipschitz boundary $\partial\Omega$. We consider the following Biot's consolidation model in Ω over a time interval $(0, T]$:

$$\frac{\partial}{\partial t}\left(c_0 p + \alpha \nabla \cdot \mathbf{u}\right) + \nabla \cdot \mathbf{q} = \psi, \quad \text{in } \Omega \times (0, T], \tag{2.1a}$$

$$\mathbf{q} = -\mathbf{K}\nabla p, \quad \text{in } \Omega \times (0, T], \tag{2.1b}$$

$$-\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}, \quad \text{in } \Omega \times (0, T]. \tag{2.1c}$$

Here, $\mathbf{u}(x,t)$ is the displacement of the solid phase, $p(x,t)$ is the fluid pressure, and $\mathbf{q}(x,t)$ is the Darcy volumetric fluid flux,

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}(\epsilon(\mathbf{u}))\mathbf{I} + 2\mu\epsilon(\mathbf{u}) - \alpha p \mathbf{I}, \text{ with } \epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right). \tag{2.2}$$

In the above expressions, $\boldsymbol{\sigma}(x,t)$ is the total stress tensor with λ and μ being the Lamé constants, $c_0 \geq 0$ is the storage coefficient, and α is the Biot.-Willis constant [3], ψ is a source term, $\mathbf{f}$ is the external force, $\mathbf{K}(x)$ is a symmetric and uniformly positive definite tensor satisfying

$$k_{\min}\xi^T\xi \leq \xi^T\mathbf{K}(x)\xi \leq k_{\max}\xi^T\xi. \tag{2.3}$$

Here, ξ is any 2-by-l vector, $k_{\min}$ and $k_{\max}$ are two positive constants.

Denoting Γ_d and Γ_t as the Dirichlet. boundary and the traction boundary for the elastic variables, denoting Γ_p and Γ_f as the pressure Dirichlet boundary and the fluid normal flux boundary, we assume that $\partial\Omega = \Gamma_d \cup \Gamma_t$ and $\partial\Omega = \Gamma_p \cup \Gamma_f$. The boundary conditions and initial conditions for the above Biot. system read as:

$$\mathbf{u} = 0, \quad \text{on } \Gamma_d \times (0, T], \tag{2.4a}$$

$$\sigma\mathbf{n} = \mathbf{0}, \quad \text{on } \Gamma_t \times (0, T], \tag{2.4b}$$

$$p = 0, \quad \text{on } \Gamma_p \times (0, T], \tag{2.4c}$$

$$\mathbf{q} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_f \times (0, T], \tag{2.4d}$$

$$p(\cdot, 0) = p_0, \quad \text{in} \quad \Omega, \tag{2.4e}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad \text{in} \quad \Omega. \tag{2.4f}$$

Here, **n** denotes the unit outward normal vector.

Let us introduce some notations. As usual, $H^s(\mathscr{D})$ denotes the standard Sobolev space of functions with regularity exponent $s \geq 0$. The associated norm and the semi-norm are denoted as $\|\cdot\|_{s,\mathscr{D}}$ and $|\cdot|_{s,\mathscr{D}}$. When $s = 0$, $H^0(\mathscr{D})$ is $L^2(\mathscr{D})$. For simplicity, when $\mathscr{D} = \Omega$, the norm $\|\cdot\|_{s,\Omega}$ is written as $\|\cdot\|_s$. For the space $(H^s(\mathscr{D}))^2$, its norm is still denoted by $\|\cdot\|_{s,\mathscr{D}}$. A subspace of $H^1(\Omega)$ with vanishing trace on Γ_d is denoted by $H^1_{0,\Gamma_d}(\Omega) = \left\{v \in H^1(\Omega): v|_{\Gamma_d} = 0\right\}$. Furthermore, we define $\mathrm{H}(\mathrm{div};\Omega) = \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^2: \nabla \cdot \mathbf{v} \in L^2(\Omega)\right\}$ with its graph norm $\|\mathbf{v}\|_{\mathrm{div}} = \left(\|\mathbf{v}\|_0^2 + \|\nabla \cdot \mathbf{v}\|_0^2\right)^{1/2}$. Two subspaces of H(div; Ω) are $\mathrm{H}_{0,\Gamma_f}(\mathrm{div};\Omega) = \left\{\mathbf{v} \in \mathrm{H}(\mathrm{div};\Omega): \mathbf{v} \cdot \mathbf{n}|_{\Gamma_f} = 0\right\}$ and $\mathrm{H}_{0,\Gamma_d}(\mathrm{div};\Omega) = \left\{\mathbf{v} \in H(\mathrm{div};\Omega): \mathbf{v} \cdot \mathbf{n}|_{\Gamma_d} = 0\right\}$. For the ease of notations, we set $\mathscr{P} = L^2(\Omega), \mathscr{Q} = \mathrm{H}_{0,\Gamma_f}(\mathrm{div};\Omega)$, and $\mathscr{V} = \left(H^1_{0,\Gamma_d}(\Omega)\right)^2$.

Multiplying by test functions and integrating by parts, the standard mixed weak formulation of (2.1) reads as: find $(p, \mathbf{q}, \mathbf{u}) \in \mathscr{P} \times \mathscr{Q} \times \mathrm{V}$ such that, $t \in (0, T]$,

$$c_0\left((p)_t, w\right) + \alpha\left(\nabla \cdot (\mathbf{u})_t, w\right) + (\nabla \cdot \mathbf{q}, w) = (\psi, w), \quad \forall w \in \mathscr{P}, \tag{2.5a}$$

$$\left(\mathbf{K}^{-1}\mathbf{q}, \mathbf{z}\right) - (p, \nabla \cdot \mathbf{z}) = 0, \quad \forall \mathbf{z} \in \mathscr{Q}, \tag{2.5b}$$

$$a(\mathbf{u}, \mathbf{v}) - \alpha(p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathscr{V}. \tag{2.5c}$$

Here and hereafter, $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$, $a(\mathbf{u}, \mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})) + \lambda(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$ with $(\boldsymbol{\sigma} : \boldsymbol{\tau}) = \sum_{i=1}^2 \sum_{j=1}^2 \boldsymbol{\sigma}_{ij}\boldsymbol{\tau}_{ij}$ being the product of tensors.

In the sequel, we shall deal with functions of time and space. To this end, we introduce the standard Bochner space $L^p(0, T; H^s(\Omega))$, which consists of all functions $u: [0, T] \to H^s(\Omega)$ with norm

$$\|u\|_{L^p\left(0, T; H^s(\Omega)\right)} = \left(\int_0^T \|u(t)\|_s^p dt\right)^{1/p}$$

for $1 \le p < \infty$. When $p = \infty$, the space $L^{\infty}\left(0, T; H^{s}(\Omega)\right)$ is endowed with the norm

$$\|u\|_{L^{\infty}\left(0, T; H^{s}(\Omega)\right)} = \sup_{0 \le t \le T} \|u(t)\|_{s}.$$

3. A semi-discrete scheme

In this section, we discuss on how to conduct the spatial discretization and present the corresponding semi-discrete numerical scheme. Let $\mathcal{T}_h = \{K\}$ be a shape-regular triangulation of Ω. We denote h_K as the diameter of K and $h = \max_{K \in \mathcal{T}_h} h_K$. Moreover, we denote $\mathcal{E}_h^0$ as the set of interior edges of elements in $\mathcal{T}_h$, $\mathcal{E}_h^d$ as the set of boundary edges on Γ_d and $\mathcal{E}_h^t$ as the set of boundary edges on Γ_t. Set $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^d \cup \mathcal{E}_h^t$. The length of an edge $e \in \mathcal{E}_h$ is denoted by h_e. Moreover, we introduce the set $\mathcal{E}_h^K = \left\{e \in \mathcal{E}_h \mid e \subset \partial K\right\}$. The shape-regularity of the mesh implies that there exits an integer $N_\partial > 0$, independent of h, such that

$$\max_{K \in \mathcal{T}_h} \operatorname{card}\left(\mathcal{E}_h^K\right) \le N_\partial. \tag{3.1}$$

This means that the maximum number of edges that are related to K is uniformly bounded (see Lemma 1.41 in [13]). We associate each edge $e \in \mathcal{E}_h$ with a fixed unit normal **n** and ensure that the unit normal for each edge on the boundary ∂Ω is exactly the exterior unit normal **n**. Let $e \in \mathcal{E}_h^0$ be an interior edge, shared by two elements K_1 and K_2. For a scalar piecewise smooth function φ with $\varphi^i = \varphi|_{K_i}$, we define the average and jump by

$$\{\varphi\} = \frac{1}{2}\left(\varphi^1 + \varphi^2\right), \quad [\varphi] = \varphi^1 - \varphi^2, \quad \text{on } e \in \mathcal{E}_h^0.$$

On a boundary edge $e \in \mathcal{E}_h^d \cup \mathcal{E}_h^t$,

$$\{\varphi\} = \varphi, \quad [\varphi] = \varphi.$$

Define

$$\mathcal{Q}_h = \left\{\mathbf{q} \in \mathrm{H}_{0, \Gamma_f}(\mathrm{div}; \Omega) : \mathbf{q}|_K \in \mathrm{BDM}_k(K)\right\}, \tag{3.2}$$

$$\mathcal{V}_h = \left\{\mathbf{v} \in \mathrm{H}_{0, \Gamma_d}(\mathrm{div}; \Omega) : \mathbf{v}|_K \in \mathrm{BDM}_k(K)\right\}, \tag{3.3}$$

and

$$\mathscr{P}_h = \left\{ w \in L^2(\Omega) : w|_K \in P_{k-1}(K) \right\}. \tag{3.4}$$

Here, $\mathrm{BDM}_k(k \geq 1)$ is the H(div)-conforming space introduced by Brezzi, Douglas and Marini [6], and $P_k(K)$ denotes the space of polynomials of degree less than or equal to k on K. Let $\Pi_h : \mathscr{Q} \to \mathscr{Q}_h$ be the BDM_k interpolation [6], and P_h be the L^2– projection from $L^2(\Omega)$ onto $\mathscr{P}_h$. It is well known that the following properties hold true [6]:

$$\left(z - \mathscr{P}_h z, w\right) = 0, \quad \forall w \in \mathscr{P}_h, \tag{3.5a}$$

$$\left|z - \mathscr{P}_h z\right|_{0,K} \leq Ch^l \left|z\right|_{l,K}, \quad \forall K \in \mathscr{T}_h, \quad 0 \leq l \leq k, \tag{3.5b}$$

$$\left(\nabla \cdot \left(\mathbf{v} - \Pi_h \mathbf{v}\right), w\right) = 0, \quad \forall w \in \mathscr{P}_h, \tag{3.5c}$$

$$\left|\mathbf{v} - \Pi_h \mathbf{v}\right|_{s,K} \leq Ch^{l-s} \left|\mathbf{v}\right|_{l,K}, \quad \forall K \in \mathscr{T}_h, \quad s = 0, 1, \quad 1 \leq l \leq k+1, \tag{3.5d}$$

$$\left|\nabla \cdot \left(\mathbf{v} - \Pi_h \mathbf{v}\right)\right|_{s,K} \leq Ch^{l-s} \left|\nabla \cdot \mathbf{v}\right|_{l,K}, \quad \forall K \in \mathscr{T}_h, \quad s = 0, 1, \quad 0 \leq l \leq k. \tag{3.5e}$$

Here and in the following, we use C to denote a positive generic constant (may take different values at different occurrences), which is independent of h, $\triangle t$, and Lamé constants μ and λ.

3.1. An H(div)-conforming element method

Multiplying the equation (2.1c) by any $\mathbf{v} \in \mathscr{V}_h$, integrating by parts on every element K, and then summing over all elements in $\mathscr{T}_h$, we obtain

$$\begin{aligned} &2\mu \sum_{K \in \mathscr{T}_h} \int_K \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) dx - 2\mu \sum_{e \in \varepsilon_h^0 \cup \varepsilon_h^d} \int_e [(\epsilon(\mathbf{u})\mathbf{n}) \cdot \nabla \ ds] + \lambda \int_\Omega \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v} dx \\ &-\alpha \int_\Omega p \nabla \cdot \mathbf{v} dx - \sum_{e \in \mathscr{E}_h^t} \int_e (\boldsymbol{\sigma} n) \cdot \mathbf{v} ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathscr{V}_h. \end{aligned} \tag{3.6}$$

Note that in the above equality we have used the fact that $\mathbf{v} \cdot \mathbf{n}$ is continuous across each interior edge. For an edge e, if $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit normal and tangential vectors which form a right-handed coordinate system, there holds the following decomposition,

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} + (\mathbf{v} \cdot \tau)\tau.$$

Applying the above decomposition yields

$$(\epsilon(\mathbf{u})\mathbf{n})\cdot\mathbf{v} = (((\epsilon(\mathbf{u})\mathbf{n})\cdot\mathbf{n})\mathbf{n} + ((\epsilon(\mathbf{u})\mathbf{n})\cdot\boldsymbol{\tau})\boldsymbol{\tau})\cdot((\mathbf{v}\cdot\mathbf{n})\mathbf{n} + (\mathbf{v}\cdot\boldsymbol{\tau})\boldsymbol{\tau})$$
$$= ((\epsilon(\mathbf{u})\mathbf{n})\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}) + (\epsilon(\mathbf{u})\mathbf{n})\cdot\boldsymbol{\tau})(\mathbf{v}\cdot\boldsymbol{\tau}).$$

Noting from the above decomposition, the equality $[ab] = [a]\{b\} + \{a\}[b]$, the regularity of the exact solution, and the fact $\mathbf{v}\cdot\mathbf{n}$ is continuous across each interior edge, one can derive that

$$2\mu\sum_{e\in\mathscr{E}_h^0\cup\mathscr{E}_h^d}\int_e[(\epsilon(\mathbf{u})\mathbf{n})\cdot\mathbf{v}]ds = 2\mu\sum_{e\in\mathscr{E}_h^0\cup\mathscr{E}_h^d}\int_e\{(\epsilon(\mathbf{u})\mathbf{n})\cdot\tau\}[\mathbf{v}\cdot\tau]ds.$$

Thus (3.6) is reduced to

$$\begin{aligned}&2\mu\sum_{K\in\mathscr{T}_h}\int_K\epsilon(\mathbf{u}):\epsilon(\mathbf{v})dx - 2\mu\sum_{e\in\mathscr{E}_h^0\cup\mathscr{E}_h^d}\int_e\{(\epsilon(\mathbf{u})\mathbf{n})\cdot\tau\}[\mathbf{v}\cdot\boldsymbol{\tau}]ds + \lambda\int_\Omega\nabla\cdot\mathbf{u}\nabla\cdot\mathbf{v}dx\\&-\alpha\int_\Omega p\nabla\cdot\mathbf{v}dx - \sum_{e\in\mathscr{E}_h^t}\int_e(\sigma\mathbf{n})\cdot\mathbf{v}ds = \int_\Omega\mathbf{f}\cdot\mathbf{v}ds,\quad\forall\mathbf{v}\in\mathscr{V}_h.\end{aligned}\tag{3.7}$$

As with the usual interior penalty DG methods [1], adding some stabilized terms in the above equation, and noting that $\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}$ on Γ_t, our DG approximation of (2.1c) is

$$a_h(\mathbf{u},\mathbf{v}) - \alpha\int_\Omega p\nabla\cdot\mathbf{v}dx = \int_\Omega\mathbf{f}\cdot\mathbf{v}dx,\tag{3.8}$$

where

$$\begin{aligned}a_h(\mathbf{u},\mathbf{v}) &= 2\mu\sum_{K\in\mathscr{T}_h}\int_K\epsilon(\mathbf{u}):\epsilon(\mathbf{v})dx - 2\mu\sum_{e\in\mathscr{E}_h^0\cup\mathscr{E}_h^d}\int_e\{(\epsilon(\mathbf{u})\mathbf{n})\cdot\tau\}[\mathbf{v}\cdot\tau]ds\\&-2\mu\sum_{e\in\mathscr{E}_h^0\cup\mathscr{E}_h^d}\int_e\{(\epsilon(\mathbf{v})\mathbf{n})\cdot\tau\}[\mathbf{u}\cdot\tau]ds + \frac{2\mu\gamma}{h_e}\sum_{e\in\mathscr{E}_h^0\cup\mathscr{E}_h^d}\int_e[\mathbf{u}\cdot\tau][\mathbf{v}\cdot\tau]ds + \lambda\int_\Omega\nabla\cdot\mathbf{u}\nabla\cdot\mathbf{v}dx.\end{aligned}\tag{3.9}$$

From the definitions of functional spaces and a_h, we note that the exact solutions of (2.1a), (2.1b) and (2.1c) satisfy

$$c_0\big((p)_t, w\big) + \alpha\big(\nabla\cdot(\mathbf{u})_t, w\big) + (\nabla\cdot\mathbf{q}, w) = (\psi, w),\quad\forall w\in\mathscr{P}_h,\tag{3.10a}$$

$$\big(\mathbf{K}^{-1}\mathbf{q},\mathbf{z}\big) - (p,\nabla\cdot\mathbf{z}) = 0,\quad\forall\,\mathbf{z}\in\mathscr{Q}_h,\tag{3.10b}$$

$$a_h(\mathbf{u},\mathbf{v}) - \alpha(p,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}),\quad\forall\,\mathbf{v}\in\mathscr{V}_h.\tag{3.10c}$$

Naturally, the corresponding H(div)-conforming finite element method for (2.1a), (2.1b) and (2.1c) reads as: given the initial conditions $\mathscr{P}_h(0) = \mathscr{P}_h p_0$ and $\mathbf{u}_h(0) = \Pi_h \mathbf{u}_0$, find $(\mathscr{P}_h, \mathbf{q}_h, \mathbf{u}_h) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$ such that

$$c_0\left((\mathscr{P}_h)_t, w\right) + \alpha\left(\nabla \cdot (\mathbf{u}_h)_t, w\right) + \left(\nabla \cdot \mathbf{q}_h, w\right) = (\psi, w), \quad \forall w \in \mathscr{P}_h, \tag{3.11a}$$

$$\left(\mathbf{K}^{-1}\mathbf{q}_h, \mathbf{z}\right) - \left(\mathscr{P}_h, \nabla \cdot \mathbf{z}\right) = 0, \quad \forall \mathbf{z} \in \mathscr{Q}_h, \tag{3.11b}$$

$$a_h\left(\mathbf{u}_h, \mathbf{v}\right) - \alpha\left(\mathscr{P}_h, \nabla \cdot \mathbf{v}\right) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathscr{V}_h. \tag{3.11c}$$

3.2. The existence and uniqueness

In order to prove the existence and uniqueness of the solutions of (3.11), we will use the theory of differential-algebraic equations (DAEs) developed in [36].

By introducing the corresponding finite element basis functions, one can represent the solutions $\mathbf{q}_h(x,t)$, $p_h(x,t)$ and $\mathbf{u}_h(x,t)$ as

$$\mathbf{q}_h(x,t) = \sum_j^{n_\mathbf{q}} \mathbf{q}_j(t)\varphi_{\mathbf{q},j} = \overline{\mathbf{q}}_h(t)\varphi_\mathbf{q},$$

$$\mathscr{P}_h(x,t) = \sum_j^{n_p} p_j(t)\varphi_{p,j} = \overline{p}_h(t)\varphi_p,$$

$$\mathbf{u}_h(x,t) = \sum_j^{n_\mathbf{u}} \mathbf{u}_j(t)\varphi_{\mathbf{u},j} = \overline{\mathbf{u}}_h(t)\varphi_\mathbf{u},$$

Here, $\overline{\mathbf{q}}_h(t) = \left[\mathbf{q}_1(t), \ldots, \mathbf{q}_{n_q}(t)\right]$, $\varphi_\mathbf{q} = \left[\varphi_{\mathbf{q},1}, \ldots, \varphi_{\mathbf{q},n_q}\right]^T$; $\overline{p}_h(t) = \left[p_1(t), \ldots, p_{n_p}(t)\right.$, $\varphi_p = \left[\varphi_{p,1}, \ldots, \varphi_{p,n_p}\right]^T$, $\overline{\mathbf{u}}_h(t) = \left[\mathbf{u}_1(t), \ldots, \mathbf{u}_{n_u}(t)\right]$ and $\varphi_\mathbf{u} = \left[\varphi_{\mathbf{u},1}, \ldots, \varphi_{\mathbf{u},n_u}\right]^T$. Similarly, we define row vectors $\overline{\mathbf{f}}_h(t)$ and $\overline{\psi}_h(t)$ according to the right hand side. Rearranging the above equations, one can rewrite (3.11) as an equivalent system of DAEs:

$$\mathbf{M}\mathbf{x}'(t) + \mathbf{N}x(t) = L(t). \tag{3.12}$$

Here, $\mathbf{x}(t) = \left[\overline{\mathbf{u}}_h(t), \overline{\mathbf{q}}_h(t), \overline{p}_h(t))\right]^T$, $L(t) = \left[\overline{\mathbf{f}}_h(t), 0, -\overline{\psi}_h(t)\right]^T$, and

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{\mathbf{u}p} & 0 & -a_{pp} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} a_{\mathbf{u}u} & 0 & a_{\mathbf{u}p}^T \\ 0 & a_{\mathbf{q}q} & a_{\mathbf{q}p}^T \\ 0 & a_{\mathbf{q}p} & 0 \end{pmatrix}, \tag{3.13}$$

where $a_{\mathbf{uu}}$, $a_{\mathbf{qq}}$, a_{pp}, $a_{\mathbf{u}p}$ and, $a_{\mathbf{q}p}$ denote the matrices corresponding to the bilinear forms $a_h(\mathbf{u}_h, \mathbf{v})$, $(\mathbf{K}^{-1}\mathbf{q}_h, \mathbf{z})$, $c_0(p, w)$, $\alpha(\nabla \cdot \mathbf{u}_h, w)$ and $(\nabla \cdot \mathbf{q}_h, w)$ in (3.11), respectively. According to the theory of DAEs, as pointed out in [36], it is sufficient to prove the existence and uniqueness of (3.12) by verifying the existence and uniqueness of the following saddle point problem: find $(\mathscr{P}_h, \mathbf{q}_h, \mathbf{u}_h) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$

$$A\big((\mathbf{u}_h, \mathbf{q}_h), (\mathbf{v}, \mathbf{z})\big) + B\big((\mathbf{v}, \mathbf{z}), \mathscr{P}_h\big) = (\mathbf{f}, \mathbf{v}), \quad \forall (\mathbf{v}, \mathbf{z}) \in \mathscr{V}_h \times \mathscr{Q}_h, \tag{3.14a}$$

$$B\big((\mathbf{u}_h, \mathbf{q}_h), w\big) - C\big(\mathscr{P}_h, w\big) = -(\psi, w), \quad \forall w \in \mathscr{P}_h, \tag{3.14b}$$

where

$$A((\mathbf{u}, \mathbf{q}), (\mathbf{v}, \mathbf{z})) = a_h(\mathbf{u}, \mathbf{v}) + \left(\mathbf{K}^{-1}\mathbf{q}, \mathbf{z}\right),$$

$$B((\mathbf{v}, \mathbf{z}), p) = -\alpha(\nabla \cdot \mathbf{v}, p) - (\nabla \cdot \mathbf{z}, p),$$

$$C(p, w) = c_0(p, w).$$

To prove the existence and uniqueness of problem (3.14), by using the theory of saddle point problems [7], it is enough to prove that the above bilinear forms satisfy certain LBB conditions. For the subsequent analysis, we define two mesh-dependent, norm $\|\cdot\|h$ and $\|\cdot\|h$ by

$$\|\mathbf{v}\|_h = \left(\sum_{K \in \mathscr{T}_h} 2\mu \|\epsilon(\mathbf{v})\|_{0,K}^2 + \sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} 2\mu h_e^{-1} \|[\mathbf{v} \cdot \tau]\|_{0,e}^2 + \lambda \|\nabla \cdot \mathbf{v}\|_{0,\Omega}^2 \right)^{1/2}$$

and

$$\|\mathbf{v}\|_h = \left(\|\mathbf{v}\|_h^2 + \sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} 2\mu h_e \left\| \left\{ (\epsilon(\mathbf{v})\mathbf{n}) \cdot \tau \right\} \right\|_{0,e}^2 \right)^{1/2}.$$

Actually, one can define another norm by

$$\|\mathbf{v}\|_{d,h} = \left(\sum_{K \in \mathcal{T}_h} 2\mu \|\nabla \mathbf{v}\|_{0,K}^2 + \lambda \|\nabla \cdot \mathbf{v}\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_h^0 \cup \mathcal{E}_h^d} 2\mu h_e^{-1} \|[\mathbf{v} \cdot \boldsymbol{\tau}]\|_{0,e}^2 \right)^{1/2}.$$

Using the discrete version of the Korn's inequality [4], it can be proved that $|||\cdot|||h$, $\|\cdot\|h$ and $|||\cdot|||_{d,h}$ are equivalent on $\mathcal{V}_h$. The details of the proof can be found in [2,15].

Let K be an element with e as an edge. For all $w \in H^1(K)$, it is well known [4] that there exists a constant $C > 0$ such that

$$\|w\|_{0,e}^2 \leq C\left(h_K^{-1}\|w\|_{0,K}^2 + h_K\|\nabla w\|_{0,K}^2\right). \tag{3.15}$$

Then, by the shape-regularity of the mesh, there holds [4, 24]

$$h_e\|\{(\epsilon(\mathbf{w})\mathbf{n}) \cdot \tau\}\|_{0,e}^2 \leq C\left(\|\epsilon(\mathbf{w})\|_{0,K}^2 + h_K^2\|\epsilon(\mathbf{w})\|_{1,K}^2\right). \tag{3.16}$$

Applying the standard inverse inequality to the last term of the above inequality, we see that

$$h_e\|\{(\epsilon(\mathbf{w})\mathbf{n}) \cdot \tau\}\|_{0,e}^2 \leq C_{tr}\|\epsilon(\mathbf{w})\|_{0,K}^2, \quad \forall \mathbf{w} \in \mathcal{V}_h, \tag{3.17}$$

where C_{tr} depends only on the polynomial degree k and the shape-regularity of the mesh. Thus, there exists a constant $C_0 > 0$ such that

$$\|\mathbf{v}\|_h^2 \leq C_0\|\mathbf{v}\|_h^2, \quad \forall \mathbf{v} \in \mathcal{V}_h \tag{3.18}$$

with $C_0=1+C_{tr}$.

Setting $\mathcal{V}(h) = \mathcal{V} + \mathcal{V}_h$, then we have the following lemma.

Lemma 3.1.— *There exists a constant $C_{cont} > 0$, independent of μ and λ, such that*

$$a_h(\mathbf{w}, \mathbf{v}) \leq C_{\text{cont}}\|\mathbf{w}\|_h\|\mathbf{v}\|_h, \quad \forall \mathbf{w}, \mathbf{v} \in \mathcal{V}(h). \tag{3.19}$$

Furthermore, if the penalty parameter γ is sufficiently large, then there exists a constant $C_{coer} > 0$ such that.

$$a_h(\mathbf{v}, \mathbf{v}) \geq C_{\text{coer}}\|\mathbf{v}\|_h^2, \quad \forall \mathbf{v} \in \mathcal{V}_h. \tag{3.20}$$

Here, C_{coer} does not. depend on the Lamé constants μ and λ.

Proof.: The inequality of (3.19) can be easily derived from the Cauchy-Schwarz inequality. It leaves us to prove (3.20). Using Young's inequality and a trace inequality (3.17), we have, for any $\varepsilon > 0$,

$$\sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} \int_e \left\{ (\epsilon(\mathbf{v})\mathbf{n}) \cdot \tau \right\} [\mathbf{v} \cdot \tau] ds$$

$$\leq \sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} h_e^{1/2} \| \left\{ (\epsilon(\mathbf{v})\mathbf{n}) \cdot \tau \right\} \|_{0,e} h_e^{-1/2} \| [\mathbf{v} \cdot \tau] \|_{0,e}$$

$$\leq \left(\sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} h_e \| \left\{ (\epsilon(\mathbf{v})\mathbf{n}) \cdot \tau \right\} \|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} h_e^{-1} \| [\mathbf{v} \cdot \tau] \|_{0,e}^2 \right)^{1/2}$$

$$\leq \left(\sum_{K \in \mathscr{T}_h} N_\partial C_{tr} \| \epsilon(\mathbf{v}) \|_{0,K}^2 \right)^{1/2} \left(\sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} h_e^{-1} \| [\mathbf{v} \cdot \tau] \|_{0,e}^2 \right)^{1/2}$$

$$\leq \frac{N_\partial C_{tr}}{2\varepsilon} \sum_{K \in \mathscr{T}_h} \| \epsilon(\mathbf{v}) \|_{0,K}^2 + \frac{\varepsilon}{2} \sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} h_e^{-1} \| [\mathbf{v} \cdot \boldsymbol{\tau}] \|_{0,e}^2 . \tag{3.21}$$

Here N_∂ and C_{tr} axe defined in (3.1) and (3.17), respectively. Substituting the above inequality into (3.9) yields

$$a_h(\mathbf{v}, \mathbf{v}) \geq 2\mu \sum_{K \in \mathscr{T}_h} \| \epsilon(\mathbf{v}) \|_{0,K}^2 + 2\mu\gamma \sum_{e \in \mathscr{E}_b^0 \cup \mathscr{E}_h^d} h_e^{-1} \| [\mathbf{v} \cdot \tau] \|_{0,e}^2 + \lambda \| \nabla \cdot \mathbf{v} \|_{0,\Omega}^2$$

$$- \frac{2\mu N_\partial C_{tr}}{\varepsilon} \sum_{K \in \mathscr{T}_h} \| \epsilon(\mathbf{v}) \|_{0,K}^2 - 2\mu\varepsilon \sum_{e \in \mathscr{E}_h^0 \cup \mathscr{E}_h^d} h_e^{-1} \| [\mathbf{v} \cdot \tau] \|_{0,e}^2$$

$$\geq \left(2\mu - \frac{2\mu N_{\partial} C_{tr}}{\varepsilon}\right) \sum_{K \in \mathcal{T}_h} \|\epsilon(\mathbf{v})\|_{0,K}^2 + (2\mu\gamma - 2\mu\varepsilon) \sum_{e \in \mathcal{E}_h^0 \cup \mathcal{E}_h^d} h_e^{-1} \|[\mathbf{v} \cdot \tau]\|_{0,e}^2 + \lambda \|\nabla \cdot \mathbf{v}\|_{0,\Omega}^2. \tag{3.22}$$

Setting $\varepsilon = 2N_{\partial}C_{tr}$ in the above inequality and choosing a sufficiently large penalty parameter γ to ensure $2\mu\gamma - 2\mu\varepsilon = 2\mu\gamma - 4\mu N_{\partial} C_{tr} > 0$, we have

$$a_h(\mathbf{v}, \mathbf{v}) \geq C_1 |||\mathbf{v}|||_h^2, \quad \forall \mathbf{v} \in \mathcal{V}_h. \tag{3.23}$$

Here, $0 < C_1 = \min\left\{1/2, \gamma - 2N_{\partial}C_{tr}\right\} < 1/2$. Combining (3.23) with (3.18), we have

$$a_h(\mathbf{v}, \mathbf{v}) \geq \frac{C_1}{C_0} \|\mathbf{v}\|_h^2, \quad \forall \mathbf{v} \in \mathcal{V}_h \tag{3.24}$$

The inequality (3.20) follows by setting $C_{\text{coer}} = \frac{C_1}{C_0}$. Since C_{tr} depends only on the polynomial degree k and the shape-regularity of the mesh and $C_0 = 1 + C_{tr}$, we see that $C_{\text{coer}} = \frac{C_1}{C_0}$ depend on the Lamé constants μ and λ.

Remark 3.1.—*In general, as in other interior penalty DG methods, one can choose* $\gamma > \gamma_{\min}$ $\gamma_{\min} = N_{\partial}C_{tr}$ *to obtain* (3.20). *In fact, in* (3.22), *setting* $2\mu - \frac{2\mu N_{\partial}C_{tr}}{\varepsilon} > 0$ *and* $2\mu\gamma - 2\mu\varepsilon > 0$, *i.e.,* $\gamma > \varepsilon > N_{\partial}C_{tr}$, *we have*

$$a_h(\mathbf{v}, \mathbf{v}) \geq C_1 |||\mathbf{v}|||_h^2, \quad \forall \mathbf{v} \in \mathcal{V}_h \tag{3.25}$$

where $0 < C_1 = \min\left\{1 - \frac{N_{\partial}C_{tr}}{\varepsilon}, \gamma - \varepsilon\right\} < 1$. *This, together with* (3.18), *gives* (3.20). *We further comment here that the constant* $\gamma_{\min} = N_{\partial}C_{tr}$ *depends on the polynomial degree k. For two dimensional triangle elements,* C_{tr} *scales as* $k(k+2)$. *More comments on* C_{tr} *can be found in* [14, 34] *and Remark 1.48 in* [13]. *In actual computation, one can choose* $\gamma = 10k^2$. *More discussions on choosing* γ *can be found in Remark 2.1 in* [16].

For the space $\mathcal{V}_h \times \mathcal{Q}_h$, we equip it with a discrete norm

$$\|(\mathbf{v}, \mathbf{z})\|_{1,h} = \left(\|\mathbf{v}\|_h^2 + \|\mathbf{z}\|_{\text{div}}^2\right)^{1/2}.$$

Lemma 3.2.—*If the penalty parameter γ is sufficiently large, then there exists a constant C > 0 such that*

$$A((\mathbf{v},\mathbf{z}),(\mathbf{v},\mathbf{z})) \geq C\|(\mathbf{v},\mathbf{z})\|_{1,h}, \quad \forall(\mathbf{v},\mathbf{z}) \in \mathscr{V}_h \times \mathscr{Q}_h. \tag{3.26}$$

Proof.: The lemma follows from the definition of $\|\cdot\|_{1,h}$ (3.20) and (2.3).

Lemma 3.3.—*There exists a positive constant $\beta > 0$ such that*

$$\sup_{(\mathbf{v},\mathbf{z}) \in \mathscr{V}_h \times \mathscr{Q}_h} \frac{B((\mathbf{v},\mathbf{z}),w)}{\|(\mathbf{v},\mathbf{z})\|_{1,h}} \geq \beta\|w\|_0, \quad \forall w \in \mathscr{P}_h. \tag{3.27}$$

Proof.: For any $w \in \mathscr{P}_h$, there exists a $z \in H^1(\Omega)^2 \cap \mathscr{V}$ such that (cf. Lemma 11.2.3 in [5])

$$\nabla \cdot \mathbf{z} = -w, \quad \text{and} \quad \|\mathbf{z}\|_1 \leq C_1\|w\|_0. \tag{3.28}$$

From (3.5d), we note that

$$\|\Pi_h\mathbf{z}\|_1 \leq C_2\|\mathbf{z}\|_1, \quad \forall \mathbf{z} \in \left(H^1(\Omega)\right)^2. \tag{3.29}$$

Setting $\boldsymbol{v} = \mathbf{0}$, by using (3.28) and (3.29), we see that

$$\frac{B\left(\left(\mathbf{0},\Pi_h\mathbf{z}\right),w\right)}{\left\|\left(\mathbf{0},\Pi_h\mathbf{z}\right)\right\|_{1,h}} = \frac{\|w\|_0^2}{\|\Pi_h\mathbf{z}\|_{\text{div}}} \geq \frac{\|w\|_0^2}{\|\Pi_h\mathbf{z}\|_1} \geq \frac{1}{C_2}\frac{\|w\|_0^2}{\|\mathbf{z}\|_1} \geq \frac{1}{C_1C_2}\|w\|_0.$$

The lemma follows by setting $\beta = \frac{1}{C_1C_2}$.

In Lemmas 3.2 and 3.3, we have proved the LBB condition of the saddle point problems (3.14). Noting that the bilinear form $C(\cdot,\cdot)$ is symmetric positive semidefinite, we then obtain the following main result of this subsection.

Theorem 3.1.—*The semidiscrete scheme* (3.11) *has a unique solution.*

Remark 3.2.—*The author of a recent work* [38] *has pointed out that if* $\ker\left(a_{\mathbf{u}p}^T\right) = 0$, *one can remove spurious pressure oscillations which arise when* $c_0 = 0$ *and* $K \to 0$. *Since we use standard mixed finite element spaces* $\mathscr{V}_h = \left\{\mathbf{v} \in \mathrm{H}_{0,\Gamma_d}(\text{div};\Omega) : \mathbf{v}|_K \in \mathrm{BDM}_k(K)\right\}$ *and* $\mathscr{P}_h = \left\{w \in L^2(\Omega) : w|_K \in P_{k-1}(K)\right\}$ *for the displacement and pressure variables, there naturally holds* $\left(a_{\mathbf{u}p}^T\right) = 0$. *Therefore, there will be no spurious pressure oscillation by using our method.*

3.3. Error estimates for the semi-discrete scheme

3.3.1. Error estimates for the case $c_0 \geq \beta_0 > 0$

Theorem 3.2.: *Let* $(\mathscr{P}, \mathbf{q}, \mathbf{u}) \in \times \mathscr{Q} \times \mathscr{V}$ *and* $(\mathscr{P}_h, \mathbf{q}_h, \mathbf{u}_h) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$ *be the solutions of* (2.5) *and* (3.11), *respectively. Moreover, we assume that*

$$\mathbf{u} \in L^{\infty}\left(0, T; H^{k+1}(\Omega)\right), \ \ \mathbf{u}_t \in L^2\left(0, T; H^{k+1}(\Omega)\right), \ \ q \in L^2\left(0, T; H^k(\Omega)\right).$$

Then, provided that the penalty parameter γ *is sufficiently large, the following finite element error estimate holds.*

$$\|\mathbf{u} - \mathbf{u}_h\|^2_{L^{\infty}\left(0, T; E_h\right)} + \|p - \mathscr{P}_h\|^2_{L^{\infty}\left(0, T; L^2(\Omega)\right)} + \|\mathbf{q} - \mathbf{q}_h\|^2_{L^2\left(0, T; L^2(\Omega)\right)} \leq Ch^{2k}, \tag{3.30}$$

where $\|\mathbf{u}\|_{L^{\infty}\left(0, T; E_h\right)} = \sup_{0 \leq s \leq T} \|\mathbf{u}(s)\|_h.$

Proof.: Subtracting (3.10a), (3.10b) and (3.10c) from (3.11a), (3.11b) and (3.11c), respectively, we have

$$c_0\left(\left(p - p_h\right)_t, w\right) + \alpha\left(\nabla \cdot \left(\mathbf{u} - \mathbf{u}_h\right)_t, w\right) + \left(\nabla \cdot \left(\mathbf{q} - \mathbf{q}_h\right), w\right) = 0, \quad \forall w \in p_h, \tag{3.31a}$$

$$\left(\mathbf{K}^{-1}\left(\mathbf{q} - \mathbf{q}_h\right), \mathbf{z}\right) - \left(p - p_h, \nabla \cdot \mathbf{z}\right) = 0, \quad \forall \mathbf{z} \in \mathscr{Q}_h, \tag{3.31b}$$

$$a_h\left(\mathbf{u} - \mathbf{u}_h, \mathbf{v}\right) - \alpha\left(p - p_h, \nabla \cdot \mathbf{v}\right) = 0, \quad \forall \mathbf{v} \in \mathscr{V}_h. \tag{3.31c}$$

We then split the error p–p_h as $p - p_h = \xi_p + \theta_p$ with $\xi_p = p - p_h p$ and $\theta_p = p_h p - p_h$. Similarly,$\mathbf{q} - \mathbf{q}_h = \xi_{\mathbf{q}} + \theta_{\mathbf{q}}$ with $\xi_{\mathbf{q}} = \mathbf{q} - \Pi_h \mathbf{q}$ and $\theta_{\mathbf{q}} = \Pi_h \mathbf{q} - \mathbf{q}_h \cdot \mathbf{u} - \mathbf{u}_h = \xi_{\mathbf{u}} + \theta_{\mathbf{u}}$ with $\xi_{\mathbf{u}} = \mathbf{u} - \Pi_h \mathbf{u}$ and $\theta_{\mathbf{u}} = \Pi_h \mathbf{u} - \mathbf{u}_h$. Since the estimates for ξ_p, $\xi_{\mathbf{q}}$ and $\xi_{\mathbf{u}}$ can be derived by the interpolation error bounds in (3.5b) and (3.5d), it leaves us to estimate θ_p, $\theta_{\mathbf{q}}$ and $\theta_{\mathbf{u}}$. To this end, using (3.5c) and (3.5e), we can rewrite (3.31a), (3.31b) and (3.31c) by

$$c_0\left(\left(\theta_p\right)_t, w\right) + \alpha\left(\nabla \cdot \left(\theta_{\mathbf{u}}\right)_t, w\right) + \left(\nabla \cdot \theta_{\mathbf{q}}, w\right) = 0, \quad \forall w \in p_h, \tag{3.32a}$$

$$\left(\mathbf{K}^{-1}\left(\theta_{\mathbf{q}}\right), \mathbf{z}\right) - \left(\theta_p, \nabla \cdot \mathbf{z}\right) = -\left(\mathbf{K}^{-1}\left(\xi_{\mathbf{q}}\right), \mathbf{z}\right), \forall \mathbf{z} \in \mathscr{Q}_h, \tag{3.32b}$$

$$a_h\left(\theta_{\mathbf{u}}, \mathbf{v}\right) - \alpha\left(\theta_p, \nabla \cdot \mathbf{v}\right) = -a_h\left(\xi_{\mathbf{u}}, \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathscr{V}_h. \tag{3.32c}$$

Setting $w = \theta_p$, $\mathbf{Z} = \theta_{\mathbf{q}}$ and $\mathbf{v} = (\theta_{\mathbf{u}})_t$ in the above equations and using the chain rule in time and the symmetry of $a_h(\cdot,\cdot)$, we obtain

$$\frac{1}{2}c_0\frac{\partial}{\partial t}\left(\theta_p,\theta_p\right) + \alpha\left(\nabla\cdot\left(\theta_{\mathbf{u}}\right)_t,\theta_p\right) + \left(\nabla\cdot\theta_{\mathbf{q}},\theta_p\right) = 0, \tag{3.33}$$

$$\left(\mathbf{K}^{-1}\left(\theta_{\mathbf{q}}\right),\theta_{\mathbf{q}}\right) - \left(\theta_p,\nabla\cdot\ \theta_{\mathbf{q}}\right) = -\left(\mathbf{K}^{-1}\left(\xi_{\mathbf{q}}\right),\theta_{\mathbf{q}}\right), \tag{3.34}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(a_h\left(\theta_{\mathbf{u}},\theta_{\mathbf{u}}\right)\right) - \alpha\left(\theta_p,\nabla\cdot\ \left(\theta_{\mathbf{u}}\right)_t\right) = -a_h\left(\xi_{\mathbf{u}},\left(\theta_{\mathbf{u}}\right)_t\right). \tag{3.35}$$

The initial conditions $\mathscr{P}_h(0) = \mathscr{P}_h p_0$ and $\mathbf{u}_h(0) = \Pi_h\mathbf{u}_0$ imply that $\theta_p(0) = 0$ and $\theta_{\mathbf{u}}(0) = 0$. Using this fact, summing equations (3.33)–(3.35), integrating in time from 0 to $t(\le T)$, we obtain

$$\frac{1}{2}a_h\left(\theta_{\mathbf{u}}(t),\theta_{\mathbf{u}}(t)\right) + \frac{1}{2}c_0\|\theta_p(t)\|_0^2 + \int_0^t \|\mathbf{K}^{-\frac{1}{2}}\theta_{\mathbf{q}}(s)\|_0^2 ds = B_1 + B_2. \tag{3.36}$$

Here,

$$B_1 = -\int_0^t\left(\mathbf{K}^{-1}\xi_{\mathbf{q}}(s),\theta_{\mathbf{q}}(s)\right)ds \text{ and } B_2 = -\int_0^t a_h\left(\xi_{\mathbf{u}}(s),\left(\theta_{\mathbf{u}}\right)_t(s)\right)ds.$$

For B_1, we can bound it as follows:

$$\begin{aligned} B_1 &\le \int_0^t \|\mathbf{K}^{-\frac{1}{2}}\xi_{\mathbf{q}}(s)\|_0\|\mathbf{K}^{-\frac{1}{2}}\theta_{\mathbf{q}}(s)\|_0 ds \\ &\le \frac{1}{2}\int_0^t \|\mathbf{K}^{-\frac{1}{2}}\xi_{\mathbf{q}}(s)\|_0^2 ds + \frac{1}{2}\int_0^t \|\mathbf{K}^{-\frac{1}{2}}\theta_{\mathbf{q}}(s)\|_0^2 ds. \end{aligned} \tag{3.37}$$

For B_2, integrating by parts, we firstly obtain

$$B_2 = \int_0^t a_h\left(\left(\xi_{\mathbf{u}}\right)_t(s),\theta_{\mathbf{u}}(s)\right)ds - a_h\left(\xi_{\mathbf{u}}(t),\theta_{\mathbf{u}}(t)\right). \tag{3.38}$$

Then noting that $\theta_{\mathbf{u}}(0) = 0$, using (3.19) and Young's inequality, we further have

$$B_2 \le C\left(\int_0^t\left(\|\left(\xi_{\mathbf{u}}\right)_t(s)\|_h^2 + \|\theta_{\mathbf{u}}(s)\|_h^2\right)ds + \|\xi_{\mathbf{u}}(t)\|_h^2\right) + \varepsilon\|\theta_{\mathbf{u}}(t)\|_h^2 \tag{3.39}$$

with ε being an arbitrarily small number.

Noting from the above bounds, using (2.3) and (3.20), we have

$$\left(\frac{C_{\text{coer}}}{2} - \varepsilon\right)\|\theta_{\mathbf{u}}(t)\|_h^2 + \frac{1}{2}c_0\|\theta_p(t)\|_0^2 + \frac{1}{2}\int_0^t \left\|\mathbf{K}^{-\frac{1}{2}}\theta_{\mathbf{q}}(s)\right\|_0^2 ds \\ \leq C\int_0^t \|\theta_{\mathbf{u}}(s)\|_h^2 ds + C\int_0^t \left(\|\xi_{\mathbf{q}}(s)\|_0^2 + \|(\xi_{\mathbf{u}})_t(s)\|_h^2\right)ds + \|\xi_{\mathbf{u}}(t)\|_h^2. \tag{3.40}$$

We can choose ε small enough to make $C_{\min} = \min\left\{\frac{C_{\text{coer}}}{2} - \varepsilon, \frac{1}{2}c_0, \frac{1}{2k_{\max}}\right\}$ be positive. We note that the above inequality still holds if one replaces the left-hand side of (3.40) by $C_{\min}\left(\|\theta_{\mathbf{u}}(t)\|_h^2 + \|\theta_p(t)\|_0^2 + \int_0^t\|\theta_{\mathbf{q}}(s)\|_0^2 ds\right)$. Therefore, dividing both sides of the above inequality by $C_{\min}$ and using Gronwall's lemma, we have

$$\|\theta_{\mathbf{u}}(t)\|_h^2 + \|\theta_p(t)\|_0^2 + \int_0^t \|\theta_{\mathbf{q}}(s)\|_0^2 ds \leq C \\ \left(\int_0^t \left(\|\xi_{\mathrm{q}}(s)\|_0^2 + \|(\xi_{\mathrm{u}})_t(s)\|_h^2\right)ds + \|\xi_{\mathrm{u}}(t)\|_h^2\right). \tag{3.41}$$

Noting that the above estimate holds for all $0 \leq t \leq T$, and using some appropriate approximation properties of P_h in (3.5b) and Π_h in (3.5d), we obtain

$$\sup_{0 \leq s \leq T}\|\theta_{\mathbf{u}}(s)\|_h^2 + \sup_{0 \leq s \leq T}\|\theta_p(s)\|_0^2 + \int_0^T \|\theta_{\mathbf{q}}(s)\|_0^2 ds \\ \leq C\left(h^{2k}\left(\int_0^T \|\mathbf{q}(s)\|_k^2 + \|\mathbf{u}_t(s)\|_k^2 ds\right) + h^{2k}\sup_{0 \leq s \leq T}\|\mathbf{u}(s)\|_k^2\right), \tag{3.42}$$

where, $\|\mathbf{u}\|_k^2 = \mu\|\mathbf{u}(s)\|_{k+1}^2 + \lambda\|\nabla\cdot u\|_k^2$. This estimate can be rewritten by the following equivalent formulation:

$$\|\theta_{\mathbf{u}}(s)\|_{L^\infty(0,T;E^h)}^2 + \|\theta_p(s)\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\theta_{\mathbf{q}}(s)\|_{L^2(0,T;L^2(\Omega))}^2 \\ \leq Ch^{2k}\left(\int_0^T \|\mathbf{q}\|_k^2 + \|\mathbf{u}_t\|_k^2 ds + \sup_{0 \leq s \leq T}\|\mathbf{u}(s)\|_k^2\right). \tag{3.43}$$

Combining the above estimate with the interpolation error estimates for ξ_p, $\xi_{\mathbf{q}}$ and $\xi_{\mathbf{u}}$, and using the triangle inequality, we obtain the assertion (3.30).

3.3.2. Error estimates for the case $c_0 \geq 0$—Note that the results in Theorem 3.2 in the previous subsection hold under the assumption that $c_0 > 0$. If $c_0 = 0$, the optimal error estimates are derived using the weaker $L^2(0, T; L^2(\Omega))$ norm. To this end, we need the following lemma [10].

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Lemma 3.4.—*Let* $(p, \mathbf{q}, \mathbf{u}) \in \mathscr{P} \times \mathscr{Q} \times \mathscr{V}$ and $(\mathscr{P}_h, \mathbf{q}_h, \mathbf{u}_h) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$ *be the solutions of* (2.5) *and* (3.11), *respectively. Then, there exists a constant* $c_0 > 0$ *such that*

$$\|\theta_p\|_0 \le C_p \|q - q_h\|.$$

By using the above result, we can obtain the following main result.

Theorem 3.3.—Under the same assumption as that in Theorem 3.2, the following error estimate holds.

$$\|\mathbf{u} - \mathbf{u}_h\|^2_{L^\infty\left(0, T; E_h\right)} + \|p - \mathscr{P}_h\|^2_{L^2\left(0, T; L^2(\Omega)\right)} + \|\mathbf{q} - \mathbf{q}_h\|^2_{L^2\left(0, T; L^2(\Omega)\right)} \le Ch^{2k}. \tag{3.45}$$

Proof.: Squaring both sides of (3.44) and then integrating them in time from 0 to T, we see that $\|\theta_p\|_{L^2\left(0, T; L^2(\Omega)\right)} \le C\|\mathbf{q} - \mathbf{q}_h\|_{L^2\left(0, T; L^2(\Omega)\right)}$. Then, the desired result follows from the error bound in (3.30), the interpolation estimates and the triangle inequality.

4. The fully discrete scheme

4.1. The fully discrete scheme

For simplicity, we apply the backward Euler method as the time discretization scheme. Let N be a positive integer and let $\Delta t = T/N$. Set $t^n = n\Delta t \quad (1 \le n \le N)$. The fully discrete approximation of (3.11) reads as: given the initial conditions $p_h^0 = \mathscr{P}_h p_0$ and $\mathbf{u}_h^0 = \Pi_h \mathbf{u}_0$, at each time $t = t^n$, find $\left(p_h^n, q_h^n, \mathbf{u}_h^n\right) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$ such that

$$c_0\left(\frac{p_h^n - p_h^{n-1}}{\Delta t}, w\right) + \alpha\left(\frac{\nabla \cdot \left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)}{\Delta t}, w\right) + \left(\nabla \cdot \mathbf{q}_h^n, w\right) = \left(\psi^n, w\right), \quad \forall w \in \mathscr{P}_h, \tag{4.1a}$$

$$\left(\mathbf{K}^{-1}\mathbf{q}_h^n, \mathbf{z}\right) - \left(p_h^n, \nabla \cdot \mathbf{z}\right) = 0, \quad \forall \mathbf{z} \in \mathscr{Q}_h, \tag{4.1b}$$

$$a_h\left(\mathbf{u}_h^n, \mathbf{v}\right) - \alpha\left(p_h^n, \nabla \cdot \mathbf{v}\right) = \left(\mathbf{f}^n, \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathscr{V}_h. \tag{4.1c}$$

4.2. The existence and uniqueness

In this subsection, we will show the existence and uniqueness of solutions of (4.1) for each time step $t = t^n$, $1 \le n \le N$. Firstly, the Eqs. (4.1a)–(4.1c) can be transformed into the following equivalent variational formulation.

$$A_h\left(\left(\mathbf{u}_h^n,\mathbf{q}_h^n\right),(\mathbf{v},\mathbf{z})\right)+B_h\left((\mathbf{v},\mathbf{z}),p_h^n\right)=\left(\mathbf{f}^n,\mathbf{v}\right), \tag{4.2a}$$

$$B_h\left(\left(\mathbf{u}_h^n,\mathbf{q}_h^n\right),w\right)-C_h\left(p_h^n,w\right)=-\Delta t\left(\psi^n,w\right)-\left(c_0p_h^{n-1}+\alpha\nabla\cdot\mathbf{u}_h^{n-1},w\right). \tag{4.2b}$$

Here, the bilinear forms are

$$A_h((\mathbf{u},\mathbf{q}),(\mathbf{v},\mathbf{z}))=a_h(\mathbf{u},\mathbf{v})+\Delta t\left(K^{-1}\mathbf{q},\mathbf{z}\right),B_h((\mathbf{v},\mathbf{z}),p)=-\alpha(\nabla\cdot\mathbf{v},p)-\Delta t(\nabla\cdot\mathbf{z},p),C_h(p,w)=c_0(p,w).$$

Similar to the semi-discrete case, to prove the existence and uniqueness of the saddle point problem (4.2), it is sufficient to verify that these bilinear forms satisfy LBB conditions [7]. To this end, we need to define a discrete time-dependent, norm for the space $\mathscr{V}_h\times\mathscr{Q}_h$, namely,

$$\|(\mathbf{v},\mathbf{z})\|_{1,h}=\left(\|\mathbf{v}\|_h^2+(\Delta t)^2\|\mathbf{z}\|_{\mathrm{div}}^2\right)^{1/2}.$$

Lemma 4.1.—*If the penalty parameter* γ *is sufficiently large, then there exists a constant* $C>0$ *such that*

$$A_h((\mathbf{v},\mathbf{z}),(\mathbf{v},\mathbf{z}))\geq C\|(\mathbf{v},\mathbf{z})\|_{1,h},\quad\forall(\mathbf{v},\mathbf{z})\in\mathscr{V}_h\times\mathscr{Q}_h. \tag{4.4}$$

Proof.: The assertion follows from the definition of $|||\cdot|||_{1,h}$ in (4.3), (3.20) and (2.3).

Lemma 4.2.—*There exists a positive constant* $\beta>0$ *such that*

$$\sup_{(\mathbf{v},\mathbf{z})\in\mathscr{V}_h\times\mathscr{Q}_h}\frac{B_h((\mathbf{v},\mathbf{z}),w)}{|||(\mathbf{v},\mathbf{z})|||_{1,h}}\geq\beta\|w\|_0,\quad\forall w\in\mathscr{P}_h. \tag{4.5}$$

Proof.: For any $w\in\mathscr{P}_h$, there exists a a $\mathrm{z}\in H^1(\Omega)^2\cap\mathscr{V}$ such that (cf. Lemma 11.2.3 in [5])

$$\nabla\cdot\mathbf{z}=-w,\quad\text{and}\quad\|\mathbf{z}\|_1\leq C_1\|w\|_0. \tag{4.6}$$

From (3.5d), we obtain

$$\|\Pi_h\mathbf{z}\|_1\leq C_2\|\mathbf{z}\|_1,\quad\forall\mathbf{z}\in\left(H^1(\Omega)\right)^2. \tag{4.7}$$

In view of (4.6) and (4.7), and setting $\boldsymbol{v}=\mathbf{0}$, we have

$$\frac{B\left((\mathbf{0}, \Pi_h \mathbf{z}/\Delta t), w\right)}{|||(\mathbf{0}, \Pi_h \mathbf{z}/\Delta t)|||_{1,h}} = \frac{\|w\|_0^2}{\|\Pi_h \mathbf{z}\|_{\mathrm{div}}} \geq \frac{\|w\|_0^2}{\|\Pi_h \mathbf{z}\|_1} \geq \frac{1}{C_2}\frac{\|w\|_0^2}{\|\mathbf{z}\|_1} \geq \frac{1}{C_1 C_2}\|w\|_0 .$$

The desired result follows by setting $\beta = \frac{1}{C_1 C_2}$.

In Lemmas 4.1 and 4.2, we have proved the LBB conditions of the saddle point problem (4.2). Noting that the bilinear form $C_h(\cdot,\cdot)$ is symmetric positive semidefinite, then we obtain the following main result.

Theorem 4.1.—*At each time $t = t^n$($1 \leq n \leq N$), the fully discrete numerical scheme* (4.1) *has a unique solution $\left(p_h^n, \mathbf{q}_h^n, \mathbf{u}_h^n\right) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$, if the penalty parameter γ is sufficiently large.*

4.3. Error estimates for the fully discrete scheme

4.3.1. Error estimates for the case $c_0 \geq \beta_0 > 0$—

For any function *g(t, x),* at each time $t^n = n\Delta t, n = 1, \ldots, N$, we denote $g^n = g\left(t^n, x\right), \forall x \in \Omega$. By Taylor's expansion, there hold

$$\frac{p^n - p^{n-1}}{\Delta t} = p_t^n + \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\left(t^{n-1} - s\right)p_{tt}(s)ds, \tag{4.8}$$

$$\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} = \mathbf{u}_t^n + \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\left(t^{n-1} - s\right)\mathbf{u}_{tt}(s)ds . \tag{4.9}$$

Theorem 4.2.—*Let* $(p, \mathbf{q}, \mathbf{u}) \in \ \times \mathscr{Q} \times \mathscr{V}$ and $\left(p_h^n, \mathbf{q}_h^n, \mathbf{u}_h^n\right) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$ *be the solutions of* (2.5) *and* (4.1), *respectively. Moreover, we assume that*

$$\mathbf{u} \in L^\infty\left(0, T; H^{k+1}(\Omega)\right),\ \mathbf{u}_t \in L^\infty\left(0, T; H^{k+1}(\Omega)\right),\ \mathbf{u}_{tt} \in L^2\left(0, T; H^{k+1}(\Omega)\right),$$
$$\nabla \cdot \mathbf{u}_{tt} \in L^2\left(0, T; L^2(\Omega)\right),\ p_{tt} \in L^2\left(0, T; L^2(\Omega)\right),\ q \in L^\infty\left(0, T; H^k(\Omega)\right),$$

and that the penalty parameter γ is sufficiently large. Then, the following error estimate holds.

$$\max_{1 \leq n \leq N}\|\mathbf{u}^n - \mathbf{u}_h^n\|_h^2 + \max_{1 \leq n \leq N}\|p^n - p_h^n\|_0^2 + \Delta t\sum_{n=1}^{N}\|\mathbf{q}^n - \mathbf{q}_h^n\|_0^2 \leq C\left(h^{2k} + (\Delta t)^2\right). \tag{4.10}$$

Proof.: We note that (3.10) holds for the exact solution at any time $t = t^n$. Using this fact, combining with (4.8) and (4.9), we see that

$$c_0\left(\frac{p^n - p^{n-1}}{\Delta t}, w\right) + \alpha\left(\nabla \cdot \left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}\right), w\right) + (\nabla \cdot \mathbf{q}^n, w) = (\psi^n, w) + \frac{c_0}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1} - s\right)p_{tt}(s)ds, w\right) + \frac{\alpha}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1} - s\right)\nabla \cdot \mathbf{u}_{tt}(s)ds, w\right), \tag{4.11a}$$

$$\left(\mathbf{K}^{-1}\mathbf{q}^n, \mathbf{z}\right) - (p, \nabla \cdot \mathbf{z}) = 0, \tag{4.11b}$$

$$a_h(\mathbf{u}^n, \mathbf{v}) - \alpha(p^n, \nabla \cdot \mathbf{v}) = (\mathbf{f}^n, \mathbf{v}), \tag{4.11c}$$

for any $(w, \mathbf{z}, \mathbf{v}) \in \mathscr{P}_h \times \mathscr{Q}_h \times \mathscr{V}_h$.

Subtracting (4.1a), (4.1b) and (4.1c) from (4.11a), (4.11b) and (4.11c), respectively, we obtain

$$c_0\left(\frac{\left(p^n - p_h^n\right) - \left(p^{n-1} - p_h^{n-1}\right)}{\Delta t}, w\right) + \alpha\left(\nabla \cdot \left(\frac{\left(\mathbf{u}^n - \mathbf{u}_h^n\right) - \left(\mathbf{u}^{n-1} - \mathbf{u}_h^{n-1}\right)}{\Delta t}\right), w\right) + \left(\nabla \cdot \left(\mathbf{q}^n - \mathbf{q}_h^n\right), w\right) = \frac{c_0}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1} - s\right)p_{tt}(s)ds, w\right) + \frac{\alpha}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1} - s\right)\nabla \cdot \mathbf{u}_{tt}(s)ds, w\right),$$

$$\left(\mathbf{K}^{-1}\left(\mathbf{q}^n - \mathbf{q}_h^n\right), \mathbf{z}\right) - \left(p^n - p_h^n, \nabla \cdot \mathbf{z}\right) = 0, \tag{4.12b}$$

$$a_h\left(\mathbf{u}^n - \mathbf{u}_h^n, \mathbf{v}\right) - \alpha\left(p^n - p_h^n, \nabla \cdot \mathbf{v}\right) = 0. \tag{4.12c}$$

We then split the error $p^n - p_h^n$ into $p^n - p_h^n = \xi_p^n + \theta_p^n$ with $\xi_p^n = p^n - \mathscr{P}_h p^n$ and $\theta_{\mathbf{p}}^n = \mathscr{P}_h p^n - p_h^n$. *Similarly,*
$\mathbf{q}^n - \mathbf{q}_h^n = \xi_{\mathbf{q}}^n + \theta_{\mathbf{q}}^n$ with $\xi_{\mathbf{q}}^n = \mathbf{q}^n - \Pi_h \mathbf{q}^n$ and $\theta_{\mathbf{q}}^n = \Pi_h \mathbf{q}^n - \mathbf{q}_h^n$. $\mathbf{u}^n - \mathbf{u}_h^n = \xi_{\mathbf{u}}^n + \theta_{\mathbf{u}}^n$ with $\xi_{\mathbf{u}}^n = \mathbf{u}^n$Since $- \Pi_h \mathbf{u}^n$ and $\theta_{\mathbf{u}}^n = \Pi_h \mathbf{u}^n - \mathbf{u}_h^n$.
the estimates for $\xi_p^n, \xi_{\mathbf{q}}^n$ and $\xi_{\mathbf{u}}^n$ can be derived by the interpolation error bounds, it leaves us to estimate $\theta_p^n, \theta_{\mathbf{q}}^n$ and $\theta_{\mathbf{u}}^n$. To this end, using (3.5c) and (3.5e), we can rewrite (4.12a), (4.12b) and (4.12c) by

$$c_0\left(\frac{\theta_{\mathrm{p}}^n-\theta_{\mathrm{p}}^{n-1})}{\Delta t},w\right)+\alpha\left(\nabla\cdot\left(\frac{\theta_{\mathrm{u}}^n-\theta_{\mathrm{u}}^{n-1})}{\Delta t}\right),w\right)+\left(\nabla\cdot\theta_{\mathrm{q}}^n,w\right)$$
$$=\frac{c_0}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1}-s\right)p_{tt}(s)ds,w\right)+\frac{\alpha}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1}-s\right)\nabla\cdot\mathbf{u}_{tt}(s)ds,w\right), \tag{4.13a}$$

$$\left(\mathbf{K}^{-1}\theta_{\mathbf{q}}^n,\mathbf{z}\right)-\left(\theta_p^n,\nabla\cdot\mathbf{z}\right)=-\left(\mathbf{K}^{-1}\xi_{\mathbf{q}}^n,\mathbf{z}\right), \tag{4.13b}$$

$$a_h\left(\theta_{\mathbf{u}}^n,\mathbf{v}\right)-\alpha\left(\theta_p^n,\nabla\cdot\mathbf{v}\right)=-a_h\left(\xi_{\mathbf{u}}^n,\mathbf{v}\right). \tag{4.13c}$$

Setting $w=\theta_p^n,\mathbf{z}=\theta_{\mathbf{q}}^n$ and $\mathbf{v}=\frac{\left(\theta_{\mathbf{u}}^n-\theta_{\mathbf{u}}^{n-1}\right)}{\Delta t}$ in the above equations and adding them together, we obtain

$$a_h\left(\theta_{\mathbf{u}}^n,\theta_{\mathbf{u}}^n\right)+c_0\|\theta_p^n\|_0^2+\Delta t\|\mathbf{K}^{-\frac{1}{2}}\theta_{\mathbf{q}}^n\|_0^2=a_h\left(\theta_{\mathbf{u}}^n,\theta_{\mathbf{u}}^{n-1}\right)+c_0\left(\theta_p^{n-1},\theta_p^n\right)$$
$$+c_0\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1}-s\right)p_{tt}(s)ds,\theta_p^n\right)+\alpha\left(\int_{t^{n-1}}^{t^n}\left(t^{n-1}-s\right)\nabla\cdot\mathbf{u}_{tt}(s)(s)ds,\theta_p^n\right) \tag{4.14}$$
$$-\Delta t\left(\mathbf{K}^{-1}\xi_{\mathbf{q}}^n,\theta_{\mathbf{q}}^{n-1}\right)-a_h\left(\xi_{\mathbf{u}}^n,\theta_{\mathbf{u}}^n-\theta_{\mathbf{u}}^{n-1}\right).$$

To estimate the bounds for the above error equation, we need the following inequalities.

$$a_h\left(\theta_{\mathbf{u}}^n,\theta_{\mathbf{u}}^{n-1}\right)\le\frac{1}{2}\left(a_h\left(\theta_{\mathbf{u}}^{n-1},\theta_{\mathbf{u}}^{n-1}\right)+a_h\left(\theta_{\mathbf{u}}^n,\theta_{\mathbf{u}}^n\right)\right) \tag{4.15}$$

and

$$c_0\left(\theta_p^{n-1},\theta_p^n\right)\le\frac{1}{2}c_0\left(\|\theta_p^{n-1}\|_0^2+\|\theta_p^n\|_0^2\right). \tag{4.16}$$

In view of the above inequalities, summing (4.14) from 1 to $m(\le N)$, and noting that $\theta_{\mathbf{u}}^0=0$ and $\theta_p^0=0$, we obtain

$$\frac{1}{2}\left(a_h\left(\theta_{\mathbf{u}}^m,\theta_{\mathbf{u}}^m\right)+c_0\|\theta_p^m\|_0^2\right)+\Delta t\sum_{n=1}^{m}\|\mathbf{K}^{-\frac{1}{2}}\theta_{\mathbf{q}}^n\|_0^2\le T_1+T_2+T_3+T_4, \tag{4.17}$$

where

$$T_1 = c_0 \sum_{n=1}^{m} \left(\int_{t^{n-1}}^{t^n} \left(t^{n-1} - s\right) p_{tt}(s) ds, \theta_p^n \right),$$
$$T_2 = \alpha \sum_{n=1}^{m} \left(\int_{t^{n-1}}^{t^n} \left(t^{n-1} - s\right) \nabla \cdot \mathbf{u}_{tt}(s) ds, \theta_p^n \right),$$
$$T_3 = - \sum_{n=1}^{m} \Delta t \left(\mathbf{K}^{-1} \xi_{\mathbf{q}}^n, \theta_{\mathbf{q}}^n \right),$$
$$T_4 = - \sum_{n=1}^{m} a_h \left(\xi_{\mathbf{u}}^n, \theta_{\mathbf{u}}^n - \theta_{\mathbf{u}}^{n-1} \right).$$

The first term T_1 can be bounded by

$$T_1 = c_0 \sum_{n=1}^{m} \left(\int_{t^{n-1}}^{t^n} \left(t^{n-1} - s\right) p_{tt}(s) ds, \theta_p^n \right) \le c_0 \sum_{n=1}^{m} \left\| \int_{t^{n-1}}^{t^n} \left(t^{n-1} - s\right) p_{tt}(s) ds \right\|_0 \|\theta_p^n\|_0 .$$

Since

$$\left\| \int_{t^{n-1}}^{t^n} \left(t^{n-1} - s\right) p_{tt}(s) ds \right\|_0 \le (\Delta t)^{3/2} \left(\int_{t^{n-1}}^{t^n} \|p_{tt}(s)\|_0^2 ds \right)^{1/2},$$

then we further have

$$T_1 \le C \left(\Delta t \sum_{n=0}^{m} \|\theta_p^n\|_0^2 + (\Delta t)^2 \int_0^{t^m} \|p_{tt}(s)\|_0^2 ds \right). \tag{4.18}$$

Similarly, the second term T_2 can be bounded by

$$T_2 \le C \left(\Delta t \sum_{n=0}^{m} \|\theta_p^n\|_0^2 + (\Delta t)^2 \int_0^{t^m} \| \nabla \cdot \mathbf{u}_{tt}(s)\|_0^2 ds \right). \tag{4.19}$$

For the third term T_3, it is easy to show that

$$T_3 = - \sum_{n=1}^{m} \Delta t \left(K^{-1} \xi_{\mathbf{q}}^n, \theta_{\mathbf{q}}^n \right) \le \frac{1}{2} \Delta t \sum_{n=0}^{m} \left\| K^{-\frac{1}{2}} \theta_{\mathbf{q}}^n \right\|_0^2 + C \Delta t \sum_{n=1}^{m} \|\xi_{\mathbf{q}}^n\|_0^2 .$$

To bound the last term T_4, we need the following equalities.

$$\sum_{n=1}^{m} \left(f^n - f^{n-1}\right) g^{n-1} = f^m g^m - f^0 g^0 - \sum_{n=1}^{m} f^n \left(g^n - g^{n-1}\right) \tag{4.21}$$

and

$$\xi_{\mathbf{u}}^{n}-\xi_{\mathbf{u}}^{n-1}=\xi_{\mathbf{u}_t}^{n}+\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n}}\left(t^{n-1}-s\right)\xi_{\mathbf{u}_{tt}}(s)ds. \tag{4.22}$$

Then, using (4.21), (4.22), (3.19) and Young's inequality, and noting that $\xi_{\mathbf{u}_t}^{0}=0$, we have

$$\begin{aligned}T_4&=-\sum_{n=1}^{m}a_h\left(\xi_{\mathbf{u}}^{n},\theta_{\mathbf{u}}^{n}-\theta_{\mathbf{u}}^{n-1}\right)=-a_h\left(\xi_{\mathbf{u}}^{m},\theta_{\mathbf{u}}^{m}\right)+\sum_{n=1}^{m}a_h\left(\xi_{\mathbf{u}}^{n}-\xi_{\mathbf{u}}^{n-1},\theta_{\mathbf{u}}^{n-1}\right)\\&\le\varepsilon\|\theta_{\mathbf{u}}^{m}\|_h^2+C\left(\|\xi_{\mathbf{u}}^{m}\|_h^2+(\Delta t)^2\int_0^{t^m}\|\xi_{\mathbf{u}_{tt}}(s)\|_h^2ds+\Delta t\sum_{n=0}^{m}\left(\|\xi_{\mathbf{u}_t}^{n}\|_h^2+\|\theta_{\mathbf{u}}^{n}\|_h^2\right)\right),\end{aligned} \tag{4.23}$$

where ε is an arbitrarily small number.

Combining the bounds above, and using (2.3) and (3.20), we have

$$\begin{aligned}&\left(\frac{C_{\text{coer}}}{2}-\varepsilon\right)\|\theta_{\mathbf{u}}^{m}\|_h^2+\frac{1}{2}c_0\|\theta_p^{m}\|_0^2+\frac{\Delta t}{2k_{\max}}\sum_{n=1}^{m}\|\theta_{\mathbf{q}}^{n}\|_0^2\\&\le C\left(\Delta t\sum_{n=0}^{m}\left(\|\theta_p^{n}\|_0^2+\|\theta_{\mathbf{u}}^{n}\|_h^2\right)+(\Delta t)^2\int_0^{t^m}\|p_{tt}(s)\|_0^2ds\right.\\&+(\Delta t)^2\int_0^{t^m}\|\nabla\cdot\mathbf{u}_{tt}(s)\|_0^2ds+\Delta t\sum_{n=1}^{m}\|\xi_{\mathbf{q}}^{n}\|_0^2+\|\xi_{\mathbf{u}}^{m}\|_h^2+(\Delta t)^2\int_0^{T}\|\xi_{\mathbf{u}_{tt}}(s)\|_h^2ds\\&\left.+\Delta t\sum_{n=0}^{m}\|\xi_{\mathbf{u}_t}^{n}\|_h^2\right).\end{aligned} \tag{4.24}$$

We can choose t being small enough to ensure $C_{\min}=\min\left\{\frac{C_{\text{coer}}}{2}-\varepsilon,\frac{1}{2}c_0,\frac{1}{2k_{\max}}\right\}$ is positive. Then, we note that the above inequality still holds if one replaces the left-hand side of (4.24) by $C_{\min}\left(\|\theta_{\mathbf{u}}^{m}\|_h^2+\|\theta_p^{m}\|_0^2+\Delta t\sum_{n=1}^{m}\|\theta_{\mathbf{q}}^{n}\|_0^2\right)$. Using the discrete Gronwall's inequality, some approximation properties, and noting that (4.24) holds for any 1 ≤m < N, we see that

$$\begin{aligned}&\max^{1\le n\le N}\|\theta_{\mathbf{u}}^{n}\|_h^2+\max^{1\le n\le N}\|\theta_p^{n}\|_0^2+\Delta t\sum_{n=1}^{N}\|\theta_{\mathbf{q}}^{n}\|_0^2\\&\le C\left((\Delta t)^2\int_0^T\|p_{tt}(s)\|_0^2ds+(\Delta t)^2\int_0^T\|\nabla\cdot\mathbf{u}_{tt}(s)\|_0^2ds+h^{2k}\max^{1\le n\le N}\|\mathbf{q}^n\|_k^2\right.\\&\left.+h^{2k}\max^{1\le n\le N}|||\mathbf{u}^n|||_k^2+h^{2k}(\Delta t)^2\int_0^T|||\mathbf{u}_{tt}(s)|||_k^2ds+h^{2k}\max^{1\le n\le N}|||\mathbf{u}_t^n|||_k^2\right).\end{aligned} \tag{4.25}$$

Combining the above estimate with the interpolation error estimates for $\xi_p^n,\xi_{\mathbf{q}}^n$ and $\xi_{\mathbf{u}}^n$, and using the triangle inequality, we obtain the assertion (4.26).

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4.3.2. Error estimates for the case $c_0 \geq 0$

—Similar to the semi-discrete scheme, if $c_0 = 0$, one can derive an optimal error bound for the pressure under a weaker norm. Specifically, we have

Theorem 4.3.: *Let* $(p, \mathbf{q}, \mathbf{u}) \in \times \mathcal{Q} \times \mathcal{V}$ *and* $\left(p_h^n, \mathbf{q}_h^n, \mathbf{u}_h^n\right) \in \mathcal{P}_h \times \mathcal{Q}_h \times \mathcal{V}_h$ *be the solutions of* (2.5) *and* (4.1), *respectively. Under the same assumption as that in Theorem 4.2, the following error estimate holds:*

$$\max_{1 \leq n \leq N} \|\mathbf{u}^n - \mathbf{u}_h^n\|_h^2 + \Delta t \sum_{n=1}^{N} \|p^n - p_h^n\|_0^2 + \Delta t \sum_{n=1}^{N} \|\mathbf{q}^n - \mathbf{q}_h^n\|_0^2 \leq C\left(h^{2k} + (\Delta t)^2\right). \tag{4.26}$$

5. Concluding remarks

In this work, we propose an H(div)-conforming Finite Element method for solving Biot's consolidation model. In our method, both the displacement and the fluid velocity are approximated by using BDM_k space. As we use H(div)-conforming elements, the normal components of displacement and fluid velocity are continuous across element interfaces. Therefore, our method is locally conservative. Moreover, there is no pressure oscillation of our method because the continuity of the tangential component of elasticity part are imposed by using an interior penalty Discontinuous Galerkin method. After introducing the spatial discretization, we present a semi-discrete scheme and a fully discrete scheme. The existence and uniqueness of solutions of the semi-discrete scheme and fully discrete scheme are proved by analyzing the corresponding differential algebraic equations (DAEs). Then, under some assumptions on the regularities of the solution, we derive the optimal error bound for each variable.

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