Dynamics of a novel nonlinear SIR model with double epidemic hypothesis and impulsive effects

Xinzhu Meng · Zhenqing Li · Xiaoling Wang

Abstract In this paper, the propagation of a nonlinear delay SIR epidemic using the double epidemic hypothesis is modeled. In the model, a system of impulsive functional differential equations is studied and the sufficient conditions for the global attractivity of the semi-trivial periodic solution are drawn. By use of new computational techniques for impulsive differential equations with delay, we prove that the system is permanent under appropriate conditions. The results show that time delay, pulse vaccination, and nonlinear incidence have significant effects on the dynamics behaviors of the model. The conditions for the control of the infection caused by viruses A and B are given.

X. Meng · Z. Li (✉)
State Key Laboratory of Vegetation and Environmental Change, Institute of Botany, Chinese Academy of Sciences, Beijing 100093, People's Republic of China
e-mail: lizq@ibcas.ac.cn

X. Meng
College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao 266510, People's Republic of China
e-mail: mxz721106@sdust.edu.cn

X. Wang
College of Agronomy, Henan University of Science and Technology, Luoyang 471003, People's Republic of China

1 Introduction

In real world, there are two epidemics, one epidemic caused by virus A and another epidemic caused by virus B. The most likely origin of virus A is a mutation or recombination event from virus B [1]. Also, it has been observed by scientists about possibilities where viruses A and B are of different origins but would cause an overlapping immune response of the host. Both epidemics spread in parallel, and the epidemic caused by virus B, which is rather innocuous, protects against epidemic A. The SIR infections disease model is an important biologic model and has been studied by many authors [2–9]. It is well known that one of the strategies to control infectious diseases is vaccination. Then a number of epidemic models in ecology can be formulated as dynamical systems of differential equations with vaccination [10–17]. Systems with sudden perturbations lead to impulsive differential equations, which have been studied intensively and systematically in [18–23]. It is very important that one investigates under what conditions a given agent can invade partially vaccinated population, i.e., how large a fraction of the population do we have to keep vaccinated in order to prevent the agent from establishing.

Pulse vaccination seems more reasonable than traditional continuous constant vaccination in real world. Pulse vaccination strategy (PVS) [10–14] consists of periodical repetitions of impulsive vaccinations in a population, on all the age cohorts, which is different from the traditional constant vaccination.

A model for the spread of an infectious disease (involving only susceptibles and infective individuals) transmitted by a vector (e.g., mosquitoes) after an incubation time was proposed by Cooke [24]. This is called the phenomena of "time delay." Many authors have directly incorporated time delays in modeling equations, and, as a result, the models take the form of delay differential equations [2–9, 25–30].

In recent years, the research on delay SIR epidemic models with impulsive perturbations is a relevant, but not totally developed, subject in mathematical biology. See [10, 30] and the references therein. However, this is an interesting problem in mathematical biology. Since an adopted incidence form like the $\beta e^{-\mu\omega} S^q(t) I(t - \omega)$ term with time delay in this paper is different from the incidence forms in [10, 30] by use of new computational techniques for impulsive differential equations with delay such as inequality techniques, the construction of an appropriate Lyapunov function, and the classification analysis method on the discussion of permanence of system, which is very different from [10, 30].

2 SIR model and preliminary information

Cooke [24] formulated an SIR model with time delay effect by assuming that the force of infection at time t is given by

$$\beta S(t) I(t - \omega),$$

where β is the average number of contacts per infective per day, and $\omega > 0$ is a fixed time during which the infectious agents develop in the vector and it is only after that time that the infected vector can infect a susceptible human. Recently, Beretta, Takeuchi and Ma [2–5] considered the SIR model with vital dynamics

$$\begin{cases} S'(t) = -\beta S(t) I(t - \omega) - \mu S(t) + \mu, \\ I'(t) = \beta S(t) I(t - \omega) - \mu I(t) - r I(t), \\ R'(t) = r I(t) - \mu R(t), \end{cases} \quad (2.1)$$

which represents an SIR model with epidemics spread via a vector with an incubation time ω. Here, μ is birth and death rate, and r is a daily recovery rate. Of course, $\beta, \mu, r \in R_+$.

Levin et al. have adopted a nonlinear incidence rate form like

$$\beta S^q(t) I^p(t),$$

which depends on different infective diseases and environments [31, 32]. Of course, $p, q \in R_+$. In this paper, we consider the case where $p = 1$ and $q \in N$ is positive integer, i.e.,

$$\beta e^{-\mu\omega} S^q(t) I(t - \omega). \quad (2.2)$$

Let I_A be the total population of infectives with virus A at time t, and I_B be the total population of infectives with virus B at time t. Both epidemics spread in parallel, and the epidemic caused by virus B, which is rather innocuous, protects against epidemic caused by virus A. When pulse and the force of infection (2.2) are introduced in (2.1), we have

$$\left. \begin{aligned} S'(t) &= \mu - \beta_1 e^{-\mu\omega_1} S^{q_1}(t) I_A(t - \omega_1) \\ &\quad - \beta_2 e^{-\mu\omega_2} S^{q_2}(t) I_B(t - \omega_2) \\ &\quad - \mu S(t), \\ I_A'(t) &= \beta_1 e^{-\mu\omega_1} S^{q_1}(t) I_A(t - \omega_1) \\ &\quad - \mu I_A(t) - r_1 I_A(t), \\ I_B'(t) &= \beta_2 e^{-\mu\omega_2} S^{q_2}(t) I_B(t - \omega_2) \\ &\quad - \mu I_B(t) - r_2 I_B(t), \\ R'(t) &= r_1 I_A(t) + r_2 I_B(t) - \mu R(t), \end{aligned} \right\} \quad t \neq n\tau,$$

$$\left. \begin{aligned} S(t^+) &= (1 - \delta) S(t), \\ I_A(t^+) &= I_A(t), \\ I_B(t^+) &= I_B(t), \\ R(t^+) &= R(t) + \delta S(t), \end{aligned} \right\} \quad t = n\tau, \quad (2.3)$$

where $\beta_i, r_i, \omega_i \in R_+$ ($i = 1, 2$), $q_i \in N$ ($i = 1, 2$) is positive integer, and δ ($0 \le \delta < 1$) is the proportion of those vaccinated successfully to all of the susceptible. The $\beta_i e^{-\mu\omega_i} S^{q_i}(t) I(t - \omega_i)$ ($i = 1, 2$) term exhibits more clearly the death of the exposed population within finite incubation times (with ω_i) than the $\beta_i S^{q_i}(t) I(t - \omega_i)$ ($i = 1, 2$) term. For both systems (2.1) and (2.3), the total population size $N(t) = S(t) + I_A(t) + I_B(t) + R(t)$ satisfies $N'(t) = \mu(1 - N(t))$, and $N(t) \to 1$ as $t \to \infty$. System (2.3)

can be regarded as a model with constant total population. Hence it is sufficient to consider the first three equations in (2.3) with respect to $\Omega = \{(S, I_A, I_B) \in R_+^3 | S + I_A + I_B \leq 1\}$.

The initial condition of (2.3) is given as

$$S(\theta) = \phi_1(\theta), \qquad I_A(\theta) = \phi_2(\theta), \qquad I_B(\theta) = \phi_3(\theta),$$

$$R(\theta) = \phi_4(\theta) \quad (-\omega \leq \theta \leq 0),$$

where $\omega = \max\{\omega_1, \omega_2\}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C$ are such that $\phi_i(\theta) \geq 0 (-\omega \leq \theta \leq 0, i = 1, 2, 3, 4)$. C denotes the Banach space $C([-\omega, 0], R^4)$ of continuous functions mapping the interval $[-\omega, 0]$ into R^4. For a biological meaning, we further assume that $\phi_i(0) > 0$ for $i = 1, 2, 3, 4$.

Note that the variable R does not appear in the first three equations of system (2.3); hence we only need to consider the following subsystem of (2.3):

$$\left.\begin{aligned}
S'(t) &= \mu - \beta_1 e^{-\mu\omega_1} S^{q_1}(t) I_A(t - \omega_1) \\
&\quad - \beta_2 e^{-\mu\omega_2} S^{q_2}(t) I_B(t - \omega_2) \\
&\quad - \mu S(t), \\
I_A'(t) &= \beta_1 e^{-\mu\omega_1} S^{q_1}(t) I_A(t - \omega_1) \\
&\quad - \mu I_A(t) - r_1 I_A(t), \\
I_B'(t) &= \beta_2 e^{-\mu\omega_2} S^{q_2}(t) I_B(t - \omega_2) \\
&\quad - \mu I_B(t) - r_2 I_B(t),
\end{aligned}\right\}, \quad t \neq n\tau,$$

$$\left.\begin{aligned}
S(t^+) &= (1 - \delta)S(t), \\
I_A(t^+) &= I_A(t), \\
I_B(t^+) &= I_B(t),
\end{aligned}\right\}. \quad t = n\tau. \tag{2.4}$$

Before starting our theorem, we give the following lemma.

Lemma 2.1 (*See* [33]) *Consider the following impulse differential inequalities*:

$$w'(t) \leq (\geq) p(t)w(t) + q(t), \quad t \neq t_k,$$
$$w(t_k^+) \leq (\geq) d_k w(t_k) + b_k, \quad t = t_k, \ k \in N,$$

where $p(t), q(t) \in C[R_+, R]$, *and* $d_k \geq 0$ *and* b_k *are constants. Assume that*:

(A_0) *the sequence* $\{t_k\}$ *satisfies* $0 \leq t_0 < t_1 < t_2 < \cdots$ *with* $\lim_{t \to \infty} t_k = \infty$;

(A_1) $w \in PC'[R_+, R]$, *and* $w(t)$ *is left continuous at* $t_k, \ k \in N$.

Then

$$w(t) \leq (\geq) w(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s)ds\right)$$

$$+ \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^t p(s)ds\right)\right)b_k$$

$$+ \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(\theta)d\theta\right)q(s)ds,$$

$t \geq t_0.$

Lemma 2.2 (*Kuang* [34]) *Consider the delay differential equation*

$$\frac{dx(t)}{dt} = ax(t - \omega) - bx(t),$$

where a, b, ω *are all positive constants, and* $x(t) > 0$ *for* $t \in [-\omega, 0]$:

(i) *If* $a < b$, *then* $\lim_{t \to \infty} x(t) = 0$.
(ii) *If* $a > b$, *then* $\lim_{t \to \infty} x(t) = +\infty$.

Let $R_+ = [0, +\infty)$, $R_+^3 = \{X \in R^3 : X \geq 0, \ X = (S, I_A, I_B)\}$, $\Omega = \text{int} R_+^3$, and N be the set of nonnegative integers. Denote by $f = (f_1, f_2, f_3)^T$ the map defined by the right-hand side of the anterior three equations of system (2.4). Let $V : R_+ \times R_+^3 \to R_+$. Then V is said to belong to class V_0 if:

(i) V is continuous in $(n\tau, (n+1)\tau] \times R_+^3$, and for all $X \in R_+^3$ and $n \in N$, $\lim_{(t,y) \to ((n\tau)^+, X)} V(t, y) = V((n\tau)^+, X)$ exists.
(ii) V is locally Lipschitzian in X.

Lemma 2.3 [33] *Let* $V : R_+ \times R_+^3 \to R_+$ *and* $V \in V_0$. *Assume that*

$$D^+ V(t, z(t)) \leq (\geq) g(t, V(t, z(t))), \quad t \neq n\tau,$$
$$V(t, z(t)^+) \leq (\geq) \Psi_n(V(t, z(t))), \qquad t = n\tau,$$

where $g : R_+ \times R_+ \to R$ *is continuous in* $(n\tau, (n+1)\tau] \times R_+$; *for all* $x \in R_+$ *and* $n \in N$, $\lim_{(t,y) \to ((n\tau)^+, x)} g(t, y) = g((n\tau)^+, x)$ *exists; and* $\Psi_n : R_+ \to R_+$ *is nondecreasing. Let* $r(t) = r(t, 0, u_0)$ *be the maximal (minimal) solution of the scalar impulsive differential equation*

$$u' = g(t, u), \qquad t \neq n\tau,$$
$$u(t^+) = \Psi_n(u(t)), \quad t = n\tau,$$

existing on $[0, \infty)$. Then $V(0^+, z_0) \leq (\geq) u_0$ implies that $V(t, z(t)) \leq (\geq) r(t)$, $t \geq 0$, where $z(t) = z(t, 0, z_0)$ is any solution of (2.4) existing on $[0, \infty)$.

3 Main results

Definition 3.1 System (2.4) is said to be permanent if there exists a compact region $\Omega \subset \text{int} R_+^3$ such that every solution of system (2.4) with initial conditions ϕ eventually enters and remains in region Ω.

We begin the analysis of (2.4) by first demonstrating the existence of an infection-free solution in which infectious individuals are entirely absent from the population permanently, i.e.,

$$I_A(t) = 0, \qquad I_B(t) = 0, \quad t \geq 0. \tag{3.1}$$

Assuming (3.1), we know that the growth of the susceptible in the time interval $n\tau < t \leq (n+1)\tau$ and give some basic properties of the subsystem of (2.4)

$$\begin{cases} S'(t) = -\mu S(t) + \mu, & t \neq n\tau, \ n \in N, \\ S(t^+) = (1-\delta)S(t), & t = n\tau, \ n \in N. \end{cases} \tag{3.2}$$

Solving (3.2) between pulses and using the discrete dynamical system determined by a fixed-point theory in Poincaré map yields

$$\begin{cases} \widetilde{S(t)} = 1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}}e^{-\mu(t-n\tau)}, \\ \qquad t \in (n\tau, (n+1)\tau], \ n \in N, \\ \widetilde{S(0^+)} = \widetilde{S(n\tau^+)} = 1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}}, \end{cases} \tag{3.3}$$

which is a unique globally asymptotically stable positive periodic solution of system (3.2).

Since the solution of (3.2) is

$$\begin{cases} S(t) = \left(S(0^+) - \left(1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}}\right)\right)e^{-\mu t} \\ \qquad + \widetilde{S(t)}, \ t \in (nT, (n+1)T], \\ \widetilde{S(0^+)} = 1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}}, \end{cases} \tag{3.4}$$

we have the following Lemma 3.1.

Lemma 3.1 *System* (3.2) *has a unique positive periodic solution* $\widetilde{S(t)}$, *that is, system* (2.4) *has an infection-free periodic solution* $(\widetilde{S(t)}, 0, 0)$ *for* $t \in (n\tau, (n+1)\tau]$, $n \in N$, *and for any solution* $(S(t),$

$I_A(t), I_B(t))$ *of* (2.4) *with positive initial conditions, we have* $S(t) \to \widetilde{S(t)}$ *as* $t \to \infty$. *Denote*

$$\begin{aligned} R_1 &= \frac{\beta_1 e^{-\mu\omega_1}}{r_1 + \mu}\left(\frac{e^{\mu\tau} - 1}{e^{\mu\tau} - 1 + \delta}\right)^{q_1}, \\ R_2 &= \frac{\beta_2 e^{-\mu\omega_2}}{r_2 + \mu}\left(\frac{e^{\mu\tau} - 1}{e^{\mu\tau} - 1 + \delta}\right)^{q_2}. \end{aligned} \tag{3.5}$$

Theorem 3.1 *If* $\mathcal{R}_1 = \max\{R_1, R_2\} < 1$, *then the infection-free periodic solution* $(\widetilde{S(t)}, 0, 0)$ *of system* (2.4) *is globally attractive.*

Proof Let $(S(t), I_A(t), I_B(t))$ be any solution of system (2.4). Since $\mathcal{R}_1 < 1$, we can easily see that

$$\begin{cases} \beta_1 e^{-\mu\omega_1}\left(\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta}\right)^{q_1} < r_1 + \mu, \\ \beta_2 e^{-\mu\omega_2}\left(\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta}\right)^{q_2} < r_2 + \mu. \end{cases}$$

Then we can choose an $\varepsilon > 0$ small enough such that

$$\begin{cases} \beta_1 e^{-\mu\omega_1}\left(\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta} + \varepsilon\right)^{q_1} < r_1 + \mu, \\ \beta_2 e^{-\mu\omega_2}\left(\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta} + \varepsilon\right)^{q_2} < r_2 + \mu. \end{cases} \tag{3.6}$$

Note that $S'(t) \leq \mu - \mu S(t)$. Then we consider the impulse differential inequalities

$$\begin{cases} S'(t) \leq \mu - \mu S(t), & t \neq n\tau, \ n \in N \\ S(t^+) = (1-\delta)S(t), & t = n\tau, \ n \in N. \end{cases}$$

Using Lemma 2.1, we have

$$\begin{aligned} S(t) &\leq S(n_1\tau^+) \prod_{n_1\tau < n\tau < t}(1-\delta)\exp\left(\int_{n_1\tau}^t -\mu\, ds\right) \\ &\quad + \mu\int_{n_1\tau}^t \prod_{s < n\tau < t}(1-\delta)\exp\left(\int_s^t -\mu d\theta\right)ds \\ &= \Delta_1 + \Delta_2, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= S(n_1\tau^+)\left(\prod_{n_1 < n < t/\tau}(1-\delta)\right. \\ &\quad \left.\times \exp\left(\int_{n_1}^{t/\tau} -\mu\, d\tau\xi\right)\right) \\ &= S(n_1\tau^+)(1-\delta)^{[t/\tau]}e^{-\mu(t-n_1\tau)}, \end{aligned}$$

$$\Delta_2 = e^{-\mu t} \int_{n_1\tau}^{t} \prod_{s<n\tau<t} (1-\delta)e^{\mu s}\, d\mu s$$

$$= e^{-\mu t} \int_{n_1}^{t/\tau} \prod_{\xi<n/\tau<t/\tau} (1-\delta)e^{\mu\tau\xi}\, d\mu\tau\xi$$

$$= e^{-\mu t}\Bigg[\int_{n_1}^{n_1+1} \prod_{\xi<n/\tau<t/\tau} (1-\delta)e^{\mu\tau\xi}\, d\mu\tau\xi$$

$$+ \int_{n_1+1}^{n_1+2} \prod_{\xi<n/\tau<t/\tau} (1-\delta)e^{\mu\tau\xi}\, d\mu\tau\xi + \cdots$$

$$+ \int_{[t/\tau]-1}^{[t/\tau]} \prod_{\xi<n/\tau<t/\tau} (1-\delta)e^{\mu\tau\xi}\, d\mu\tau\xi$$

$$+ \int_{[t/\tau]}^{t/\tau} \prod_{\xi<n/\tau<t/\tau} (1-\delta)e^{\mu\tau\xi}\, d\mu\tau\xi \Bigg]$$

$$= e^{-\mu t}\big[(1-\delta)^{[t/\tau]-n_1}(e^{\mu\tau}-1)e^{n_1\mu\tau}$$

$$+ (1-\delta)^{[t/\tau]-(n_1+1)}(e^{\mu\tau}-1)e^{(n_1+1)\mu\tau} + \cdots$$

$$+ (1-\delta)(e^{\mu\tau}-1)e^{([t/\tau]-1)\mu\tau}$$

$$+ e^{\mu t} - e^{\mu\tau[t/\tau]}\big]$$

$$= e^{-\mu t}$$

$$\times \Bigg[\frac{(1-\delta)^{[t/\tau]-n_1+1}(e^{\mu\tau}-1)e^{n_1\mu\tau}-(1-\delta)(e^{\mu\tau}-1)e^{\mu\tau[t/\tau]}}{1-\delta-e^{\mu\tau}}$$

$$+ e^{\mu t} - e^{\mu\tau[t/\tau]} \Bigg].$$

Thus,

$$S(t) \leq \Delta_1 + \Delta_2 = \gamma(t) + 1 - \frac{\delta e^{\mu\tau([t/\tau]+1-t/\tau)}}{e^{\mu\tau}-1+\delta},$$

where

$$\gamma(t) = e^{-\mu t}\Bigg[S(n_1\tau^+)(1-\delta)^{[t/\tau]}e^{n_1\tau\mu}$$

$$- \frac{(1-\delta)^{[t/\tau]-n_1+1}(e^{\mu\tau}-1)e^{n_1\mu\tau}}{e^{\mu\tau}-1+\delta} \Bigg]$$

$$\leq e^{-\mu t} S(n_1\tau^+)(1-\delta)^{[t/\tau]}e^{n_1\tau\mu}$$

$$= e^{-\mu t} S(n_1\tau)(1-\delta)^{[t/\tau]+1}e^{n_1\tau\mu}$$

$$< S(n_1\tau)e^{n_1\tau\mu}e^{-\mu t}.$$

Then

$$S(t) \leq S(n_1\tau)e^{n_1\tau\mu}e^{-\mu t} + 1 - \frac{\delta}{e^{\mu\tau}-1+\delta},$$

which implies

$$\limsup_{t\to\infty} S(t) \leq 1 - \frac{\delta}{e^{\mu\tau}-1+\delta},$$

so, there exist a positive integer n_1 and an arbitrarily small positive constant ε such that for all $t \geq n_1\tau$,

$$S(t) \leq 1 - \frac{\delta}{e^{\mu\tau}-1+\delta} + \varepsilon \equiv S^{\Delta}. \tag{3.7}$$

From (3.7) and from the second and third equations of (2.4), we get that

$$\begin{cases} I_A'(t) \leq \beta_1 e^{-\mu\omega_1}(S^{\Delta})^{q_1} I_A(t-\omega_1) \\ \qquad - (r_1+\mu)I_A(t), \quad t > n_1\tau+\omega_1, \\ I_B'(t) \leq \beta_2 e^{-\mu\omega_2}(S^{\Delta})^{q_2} I_B(t-\omega_2) \\ \qquad - (r_2+\mu)I_B(t), \quad t > n_1\tau+\omega_2. \end{cases} \tag{3.8}$$

Consider the following comparison equation:

$$\begin{cases} z_A'(t) = \beta_1 e^{-\mu\omega_1}(S^{\Delta})^{q_1} z_A(t-\omega_1) \\ \qquad - (r_1+\mu)z_A(t), \quad t > n_1\tau+\omega_1, \\ z_B'(t) = \beta_2 e^{-\mu\omega_2}(S^{\Delta})^{q_2} z_B(t-\omega_2) \\ \qquad - (r_2+\mu)z_B(t), \quad t > n_1\tau+\omega_2. \end{cases} \tag{3.9}$$

From (3.6) we have that $\beta_i e^{-\mu\omega_i}(S^{\Delta})^{q_i} < r_i + \mu$, $i = 1, 2$. By Lemma 2.2 we obtain that $\lim_{t\to\infty} z_A(t) = 0$ and $\lim_{t\to\infty} z_B(t) = 0$. Since $I_A(s) = z_A(s) = \phi_2(s) > 0$ and $I_B(s) = z_B(s) = \phi_3(s) > 0$ for all $s \in [-\omega, 0]$, by the comparison theorem for differential equations and the nonnegativity of solution (with $I_A(t) \geq 0$ and $I_B(t) \geq 0$), we have that $I_A(t) \to 0$ and $I_B(t) \to 0$ as $t \to \infty$. Without loss of generality, for any sufficiently small $0 < \varepsilon_i < \mu e^{\mu\omega_i}/\beta_i$, $i = 1, 2$, we may assume that $0 < I_A(t) < \varepsilon_1$ and $0 < I_B(t) < \varepsilon_2$ for all $t \geq 0$. From the first equation of system (2.4) we have

$$S'(t) \geq \mu - \big(\mu + \beta_1 e^{-\mu\omega_1}\varepsilon_1 + \beta_2 e^{-\mu\omega_2}\varepsilon_2\big)S(t).$$

Then we have $z_1(t) \leq S(t)$, where $z_1(t)$ is the solution of the following system (3.10) with initial value $z_1(0^+) = S(0^+)$, and $\widetilde{z_1(t)}$ is the unique positive periodic solution of

$$\begin{cases} z_1'(t) = \mu - (\mu + \beta_1 e^{-\mu\omega_1}\varepsilon_1 \\ \qquad + \beta_2 e^{-\mu\omega_2}\varepsilon_2)z_1(t), \quad t \neq n\tau, \ n \in N, \\ z_1(t^+) = (1-\delta)z_1(t), \quad t = n\tau, \ n \in N, \\ z_1(0^+) = S(0^+). \end{cases} \tag{3.10}$$

From (3.10) we have that, for $n\tau < t \le (n+1)\tau$,

$$\widetilde{z_1(t)} = \frac{\mu}{\mu + \beta_1 e^{-\mu\omega_1}\varepsilon_1 + \beta_2 e^{-\mu\omega_2}\varepsilon_2}$$

$$\times \left[1 - \frac{\delta e^{-(\mu+\beta_1 e^{-\mu\omega_1}\varepsilon_1+\beta_2 e^{-\mu\omega_2}\varepsilon_2)(t-n\tau)}}{1-(1-\delta)e^{-(\mu+\beta_1 e^{-\mu\omega_1}\varepsilon_1+\beta_2 e^{-\mu\omega_2}\varepsilon_2)\tau}}\right].$$

Hence, for any $\varepsilon_3 > 0$, there exists an integer $n_2 > n_1$ such that

$$S(t) > \widetilde{z_1(t)} - \varepsilon_3 \quad \text{for all } t > n_2\tau. \tag{3.11}$$

On the other hand, from the first equation of (2.4) it follows that $S'(t) \le \mu - \mu S(t)$. By Lemma 2.3, we have $S(t) \le \widetilde{S(t)}$. Then, for any above $\varepsilon_3 > 0$, there exists an integer $n_3 > n_2$ such that

$$S(t) < \widetilde{S(t)} + \varepsilon_3 \quad \text{for } t > n_3\tau. \tag{3.12}$$

Let $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Then from (3.11) and (3.12) it follows that

$$\widetilde{S(t)} - \varepsilon_3 < S(t) < \widetilde{S(t)} + \varepsilon_3$$

for t large enough, which implies that $S(t) \to \widetilde{S(t)}$ as $t \to \infty$. The proof is completed. $\square$

Corollary 3.1 (i) *If $\beta_i e^{-\mu\omega_i} \le r_i + \mu$, $i = 1, 2$, then the infection-free periodic solution $(\widetilde{S(t)}, 0, 0)$ is globally attractive.*

(ii) *If $\beta_i e^{-\mu\omega_i} > r_i + \mu$ ($i = 1, 2$) and $\delta > \delta^*$, or $\tau < \tau_*$, or $\omega > \omega^*$, or $q > q^*$, then the infection-free periodic solution $(\widetilde{S(t)}, 0, 0)$ is globally attractive, where the critical values δ^*, τ_*, ω^*, and q^* are listed as follows:*

$$\delta^* = \max\left\{(e^{\mu\tau}-1)\left(\sqrt[q_1]{\frac{\beta_1 e^{-\mu\omega_1}}{r_1+\mu}}-1\right),\right.$$

$$\left.(e^{\mu\tau}-1)\left(\sqrt[q_2]{\frac{\beta_2 e^{-\mu\omega_2}}{r_2+\mu}}-1\right)\right\},$$

$$\tau_* = \min\left\{\frac{1}{\mu}\ln\left(1 + \frac{\delta}{\sqrt[q_1]{\frac{\beta_1 e^{-\mu\omega_1}}{r_1+\mu}}-1}\right),\right.$$

$$\left.\frac{1}{\mu}\ln\left(1 + \frac{\delta}{\sqrt[q_2]{\frac{\beta_2 e^{-\mu\omega_2}}{r_2+\mu}}-1}\right)\right\},$$

$$\omega^* = \max\left\{\frac{1}{\mu}\ln\left[\frac{\beta_1}{r_1+\mu}\cdot\left(\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta}\right)^{q_1}\right],\right.$$

$$\left.\frac{1}{\mu}\ln\left[\frac{\beta_2}{r_2+\mu}\cdot\left(\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta}\right)^{q_2}\right]\right\},$$

$$q^* = \max\left\{\log_{\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta}}\frac{r_1+\mu}{\beta_1 e^{-\mu\omega_1}},\right.$$

$$\left.\log_{\frac{e^{\mu\tau}-1}{e^{\mu\tau}-1+\delta}}\frac{r_2+\mu}{\beta_2 e^{-\mu\omega_2}}\right\}.$$

From the proof of this section, we can easily obtain the following Corollary 3.2.

Corollary 3.2

(i) *If $R_1 < 1$ and $R_2 > 1$, then the epidemic I_A dies out, and the epidemic I_B is permanent.*

(ii) *If $R_1 > 1$ and $R_2 < 1$, then the epidemic I_A is permanent, and the epidemic I_B dies out.*

Denote

$$R_3 = \frac{\beta_1 e^{-\mu\omega_1}}{r_1+\mu}$$

$$\times \left(\frac{\mu-\beta_2 e^{-\mu\omega_2}}{\mu}\cdot\frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta}\right)^{q_1},$$

$$m_1^* = \frac{(\mu-\beta_2 e^{-\mu\omega_2})(\sqrt[q_1]{R_3}-1)}{\beta_1\mu\sqrt[q_1]{R_1}e^{-\mu\omega_1}},$$

$$R_4 = \frac{\beta_2 e^{-\mu\omega_2}}{r_2+\mu}$$

$$\times \left(\frac{\mu-\beta_1 e^{-\mu\omega_1}}{\mu}\cdot\frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta}\right)^{q_2},$$

$$m_2^* = \frac{(\mu-\beta_1 e^{-\mu\omega_1})(\sqrt[q_2]{R_4}-1)}{\beta_2\mu\sqrt[q_2]{R_2}e^{-\mu\omega_2}},$$

$$S_1^* = \sqrt[q_1]{\frac{r_1+\mu}{\beta_1 e^{-\mu\omega_1}}}, \qquad S_2^* = \sqrt[q_2]{\frac{r_2+\mu}{\beta_2 e^{-\mu\omega_2}}}.$$

Theorem 3.2 *If $\mathcal{R}_2 = \min\{R_3, R_4\} > 1$, then there exist constants $\gamma_i : 0 < \gamma_i < 1$, $i = 1, 2$, such that*

$$\liminf_{t\to\infty} I_A(t) \ge \min\left\{\frac{\gamma_1 m_1^*}{2}, \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}\right\} \equiv m_1,$$

$$\liminf_{t\to\infty} I_B(t) \ge \min\left\{\frac{\gamma_2 m_2^*}{2}, \gamma_2 m_2^* e^{-(r_2+\mu)\omega_2}\right\} \equiv m_2.$$

Proof Suppose that $(S(t), I_A(t), I_B(t))$ is any positive solution of system (2.4). Since $\mathcal{R}_2 = \min\{R_3, R_4\} > 1$,

we have $m_1^* > 0$ and $m_2^* > 0$. In the following, we claim that for any $\gamma_i : 0 < \gamma_i < 1$, $i = 1, 2$, we have $I_A(t) > \gamma_1 m_1^*$ and $I_B(t) > \gamma_2 m_2^*$ for t large enough. For convenience, we will show this in the following two steps.

Step I. We claim that there exist $t_1, t_2 \in (0, \infty)$ such that $I_A(t_1) \geq \gamma_1 m_1^* > 0$ and $I_B(t_2) \geq \gamma_2 m_2^* > 0$. Otherwise, there will be three cases:

(i) There exists $t_2 > 0$ such that $I_B(t_2) \geq \gamma_2 m_2^*$, but $I_A(t) < \gamma_1 m_1^*$ for all $t > 0$;
(ii) There exists $t_1 > 0$ such that $I_A(t_1) \geq \gamma_1 m_1^*$, but $I_B(t) < \gamma_2 m_2^*$ for all $t > 0$;
(iii) $I_A(t) < \gamma_1 m_1^*$ and $I_B(t) < \gamma_2 m_2^*$ for all $t > 0$.

We first consider case (i). According to the above assumption, we get

$$\begin{cases} S'(t) \geq \mu - \beta_1 \gamma_1 m_1^* e^{-\mu\omega_1} - \beta_2 e^{-\mu\omega_2} - \mu S(t), \\ \qquad\qquad\qquad\qquad\qquad t \neq n\tau, \ n \in N, \\ S(t^+) = (1 - \delta)S(t), \quad t = n\tau, \ n \in N. \end{cases}$$

Let $\eta = \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu\omega_1} - \beta_2 e^{-\mu\omega_2}}{\mu} \frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta} > 0$. By Lemma 2.1, there exists $\varepsilon > 0$ small enough such that $S(t) > \eta - \varepsilon$. Then

$$I_A'(t) \geq \beta_1 e^{-\mu\omega_1}(\eta - \varepsilon)^{q_1} I_A(t - \omega_1) \\ \qquad\qquad - (\mu + r_1)I_A(t). \qquad (3.13)$$

Since $R_1 > 1$, it is clear that

$$\beta_1 e^{-\mu\omega_1}(\eta - \varepsilon)^{q_1} > \mu + r_1. \qquad (3.14)$$

Using Lemma 2.2 along with (3.13) and (3.14), we have $I_A(t) \to \infty$ as $t \to \infty$, which is a contradiction.

Similarly, we can prove that $I_B(t) \to \infty$ as $t \to \infty$ in case (ii), which also is a contradiction.

Last, we consider case (iii). For $t \geq 0$, we define the differentiable function $V(t)$ by

$$V(t) = I_A(t) + I_B(t) + \beta_1 e^{-\mu\omega_1}(S_1^*)^{q_1} \int_{t-\omega_1}^t I_A(\theta) \, d\theta \\ \qquad + \beta_2 e^{-\mu\omega_2}(S_2^*)^{q_2} \int_{t-\omega_2}^t I_B(\theta) \, d\theta.$$

Then, the derivative of $V(t)$ satisfies

$$\dot{V}(t) = \dot{I}_A(t) + \beta_1 e^{-\mu\omega_1}(S_1^*)^{q_1} I_A(t) \\ \qquad - \beta_1 e^{-\mu\omega_1}(S_1^*)^{q_1} I_A(t - \omega_1)$$

$$\begin{aligned} &\quad + \dot{I}_B(t) + \beta_2 e^{-\mu\omega_2}(S_2^*)^{q_2} I_B(t) \\ &\quad - \beta_2 e^{-\mu\omega_2}(S_2^*)^{q_2} I_B(t - \omega_2) \\ &= \beta_1 e^{-\mu\omega_1}(S^{q_1}(t) - (S_1^*)^{q_1}) I_A(t - \omega_1) \\ &\quad + [\beta_1 e^{-\mu\omega_1}(S_1^*)^{q_1} - (r_1 + \mu)]I_A(t) \\ &\quad + \beta_2 e^{-\mu\omega_2}(S^{q_2}(t) - (S_2^*)^{q_2}) I_B(t - \omega_2) \\ &\quad + [\beta_2 e^{-\mu\omega_2}(S_2^*)^{q_2} - (r_2 + \mu)]I_B(t) \\ &= \beta_1 e^{-\mu\omega_1}(S^{q_1}(t) - (S_1^*)^{q_1}) I_A(t - \omega_1) \\ &\quad + \beta_2 e^{-\mu\omega_2}(S^{q_2}(t) - (S_2^*)^{q_2}) I_B(t - \omega_2) \\ &= \beta_1 e^{-\mu\omega_1}(S^{q_1-1}(t) + S^{q_1-2}(t)S_1^* \\ &\quad + \cdots + S(t)(S_1^*)^{q_1-2} \\ &\quad + (S_1^*)^{q_1-1})(S(t) - S_1^*)I_A(t - \omega_1) \\ &\quad + \beta_2 e^{-\mu\omega_2}(S^{q_2-1}(t) + S^{q_2-2}(t)S_2^* \\ &\quad + \cdots + S(t)(S_2^*)^{q_2-2} \\ &\quad + (S_2^*)^{q_2-1})(S(t) - S_2^*)I_B(t - \omega_2). \quad (3.15) \end{aligned}$$

Since $\mathcal{R}_2 > 1$, we have $m_1^* > 0$ and $m_2^* > 0$. For any $\gamma_i : 0 < \gamma_i < 1$, $i = 1, 2$, we have that

$$S_1^* < \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu\omega_1} - \beta_2 e^{-\mu\omega_2}}{\mu} \\ \qquad \times \frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta},$$

$$S_2^* < \frac{\mu - \beta_2 \gamma_2 m_2^* e^{-\mu\omega_2} - \beta_1 e^{-\mu\omega_1}}{\mu} \\ \qquad \times \frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta}.$$

Then there exist two positive constants ε_1 and ε_2 small enough such that

$$S_1^* < \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu\omega_1} - \beta_2 e^{-\mu\omega_2}}{\mu} \\ \qquad \times \frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta} - \varepsilon_1 \triangleq S_{\Delta_1},$$

$$S_2^* < \frac{\mu - \beta_2 \gamma_2 m_2^* e^{-\mu\omega_2} - \beta_1 e^{-\mu\omega_1}}{\mu} \\ \qquad \times \frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta} - \varepsilon_2 \triangleq S_{\Delta_2}.$$

$$(3.16)$$

Under the assumption of case (iii), from the first and fourth equations of system (2.4) we have

$$
\begin{cases}
S'(t) \geq \mu - \beta_1 \gamma_1 m_1^* e^{-\mu\omega_1} - \beta_2 e^{-\mu\omega_2} - \mu S(t), \\
\qquad\qquad\qquad\qquad\qquad t \neq n\tau, \ n \in N, \\
S(t^+) = (1-\delta)S(t), \quad t = n\tau, \ n \in N,
\end{cases}
$$

and

$$
\begin{cases}
S'(t) \geq \mu - \beta_2 \gamma_2 m_2^* e^{-\mu\omega_2} - \beta_1 e^{-\mu\omega_1} - \mu S(t), \\
\qquad\qquad\qquad\qquad\qquad t \neq n\tau, \ n \in N, \\
S(t^+) = (1-\delta)S(t), \quad t = n\tau, \ n \in N.
\end{cases}
$$

Using Lemma 2.1, we know that there exists $T_1 \geq t_0 + \omega$ such that, for $t \geq T_1$,

$$
S(t) > \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu\omega_1} - \beta_2 e^{-\mu\omega_2}}{\mu}
$$
$$
\times \frac{(1-\delta)(e^{\mu\tau} - 1)}{e^{\mu\tau} - 1 + \delta} - \varepsilon_1 = S_{\Delta_1}, \qquad (3.17)
$$
$$
S(t) > \frac{\mu - \beta_2 \gamma_2 m_2^* e^{-\mu\omega_2} - \beta_1 e^{-\mu\omega_1}}{\mu}
$$
$$
\times \frac{(1-\delta)(e^{\mu\tau} - 1)}{e^{\mu\tau} - 1 + \delta} - \varepsilon_2 = S_{\Delta_2}. \qquad (3.18)
$$

Therefore, for $t \geq T_1$, inserting (3.17) and (3.18) into (3.15), we have

$$
\dot{V}(t) = \beta_1 e^{-\mu\omega_1} \big(S^{q_1-1}(t) + S^{q_1-2}(t)S_1^* + \cdots
$$
$$
+ S(t)(S_1^*)^{q_1-2} + (S_1^*)^{q_1-1}\big)\big(S(t) - S_1^*\big)
$$
$$
\times I_A(t-\omega_1)
$$
$$
+ \beta_2 e^{-\mu\omega_2} \big(S^{q_2-1}(t) + S^{q_2-2}(t)S_2^* + \cdots
$$
$$
+ S(t)(S_2^*)^{q_2-2} + (S_2^*)^{q_2-1}\big)\big(S(t) - S_2^*\big)
$$
$$
\times I_B(t-\omega_2)
$$
$$
> q_1 \beta_1 e^{-\mu\omega_1}(S_1^*)^{q_1-1}\big(S_{\Delta_1} - S_1^*\big)I_A(t-\omega_1)
$$
$$
+ q_2 \beta_2 e^{-\mu\omega_2}(S_2^*)^{q_2-1}\big(S_{\Delta_2} - S_2^*\big)I_B(t-\omega_2).
$$

Let

$$
I_A^l = \min_{t \in [T_1, T_1+\omega]} I_A(t), \qquad I_B^l = \min_{t \in [T_1, T_1+\omega]} I_B(t).
$$

We show that $I_A(t) \geq I_A^l$ and $I_B(t) \geq I_B^l$ for all $t \geq T_1$. Otherwise, there exists a nonnegative constant T_2 such that $I_A(t) \geq I_A^l$ and $I_B(t) \geq I_B^l$ for $t \in [T_1, T_1+\omega+T_2]$, $I_A(T_1+\omega+T_2) = I_A^l$, and $I_B(T_1+\omega+T_2) =$

I_B^l, $\dot{I}_A(T_1+\omega+T_2) \leq 0$ or $\dot{I}_B(T_1+\omega+T_2) \leq 0$. Thus, inserting (3.18) into the second equation of (2.4), we have

$$
\dot{I}_A(T_1+\omega+T_2) \geq (\beta_1 e^{-\mu\omega_1} S^{q_1}(t) - (r_1+\mu))I_A^l
$$
$$
= (r_1+\mu)\left(\frac{\beta_1 e^{-\mu\omega_1}}{r_1+\mu} S^{q_1}(t) - 1\right)I_A^l
$$
$$
> (r_1+\mu)\left(\left(\frac{S_{\Delta_1}}{S_1^*}\right)^{q_1} - 1\right)I_A^l
$$
$$
> 0,
$$

which is a contradiction to $\dot{I}_A(T_1+\omega+T_2) \leq 0$. Similarly, we can obtain that it also is a contradiction to $\dot{I}_B(T_1+\omega+T_2) \leq 0$. Hence, we get that $I_A(t) \geq I_A^l > 0$ and $I_B(t) \geq I_B^l > 0$ for all $t \geq T_1$. Therefore, for all $t \geq T_1 + \omega$,

$$
\dot{V}(t) > q_1 \beta_1 e^{-\mu\omega_1}(S_1^*)^{q_1-1}\big(S_{\Delta_1} - S_1^*\big)I_A^l
$$
$$
+ q_2 \beta_2 e^{-\mu\omega_2}(S_2^*)^{q_2-1}\big(S_{\Delta_2} - S_2^*\big)I_B^l > 0,
$$

which implies $V(t) \to +\infty$ as $t \to +\infty$. This is a contradiction to $V(t) \leq 2 + \beta_1 \omega_1 e^{-\mu\omega_1}(S_1^*)^{q_1} + \beta_2 \omega_2 e^{-\mu\omega_2}(S_2^*)^{q_2}$ for t large enough. Hence, the claim is proved.

From the above three cases we conclude that there exist t_1, t_2 such that $I_A(t_1) \geq \gamma_1 m_1^* > 0$ and $I_B(t_2) \geq \gamma_2 m_2^* > 0$.

Step II. In the rest, we are left to consider two cases:

(i) $I_A(t) \geq \gamma_1 m_1^*$ and $I_B(t) \geq \gamma_2 m_2^*$ for all t large enough;
(ii) $I_A(t)$ oscillates about $\gamma_1 m_1^*$, or $I_B(t)$ oscillates about $\gamma_2 m_2^*$ for all large t.

Case (i). Our aim is obtained. Clearly, we only need to consider Case (ii).

Case (ii). If $I_A(t)$ oscillates about $\gamma_1 m_1^*$, then we will show that $I_A(t) \geq m_1$ for all large t, where

$$
m_1 = \min\left\{\frac{\gamma_1 m_1^*}{2}, \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}\right\}.
$$

There exist two positive constants $\bar{t}, \psi$ such that

$$
I_A(\bar{t}) = I_A(\bar{t}+\psi) = \gamma_1 m_1^*
$$

and

$$
I_A(t) < \gamma_1 m_1^* \quad \text{for } \bar{t} < t < \bar{t}+\psi.
$$

Since $I_A(t)$ is continuous and ultimately bounded and is not affected by impulses, we conclude that $I_A(t)$ is uniformly continuous. Hence there exists a constant T_3 (with $0 < T_3 < \omega_1$) independent of the choice of $\bar{t}$ such that $I_A(t) > \frac{m_1^*}{2}$ for all $\bar{t} \le t \le \bar{t} + T_3$. If $\psi \le T_3$, our aim is obtained. If $T_3 < \psi \le \omega_1$, from the second equation of (2.4) we have that $\dot{I}_A(t) \ge -(r_1 + \mu)I_A(t)$ for $\bar{t} < t \le \bar{t} + \psi$. Then we have $I_A(t) \ge \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}$ for $\bar{t} < t \le \bar{t} + \psi \le \bar{t} + \omega_1$ since $I_A(\bar{t}) = \gamma_1 m_1^*$. If $\psi > \omega_1$, then we have that $I_A(t) \ge \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}$ for $\bar{t} < t \le \bar{t} + \omega_1$. Then, proceeding exactly as in the proof of the above claim, we can show that $I_A(t) \ge \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}$ for $\bar{t} + \omega_1 \le t \le \bar{t} + \psi$. In fact, if not, there exists $T_4 > 0$ such that $I(t) \ge \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}$ for $\bar{t} \le t \le \bar{t} + \omega_1 + T_4$, $I(\bar{t} + \omega_1 + T_4) = \gamma_1 m_1^* e^{-(r_1+\mu)\omega_1}$, and $\dot{I}_A(\bar{t} + \omega_1 + T_4) \le 0$. When $\bar{t}$ is large enough, the inequality $S(t) > S_{\Delta_1}$ holds for $\bar{t} < t < \bar{t} + \psi$. On the other hand, we have from the second equation of (2.4) that

$$
\begin{aligned}
&\dot{I}_A(\bar{t} + \omega_1 + T_4) \\
&\ge \big(\beta_1 e^{-\mu\omega_1} S^{q_1}(t) - (r_1 + \mu)\big)\gamma_1 m_1^* e^{-(r_1+\mu)\omega_1} \\
&= (r_1 + \mu)\bigg(\frac{\beta_1 e^{-\mu\omega_1}}{r_1 + \mu} S^{q_1}(t) - 1\bigg)\gamma_1 m_1^* e^{-(r_1+\mu)\omega_1} \\
&> (r_1 + \mu)\bigg(\bigg(\frac{S_{\Delta_1}}{S_1^*}\bigg)^{q_1} - 1\bigg)\gamma_1 m_1^* e^{-(r_1+\mu)\omega_1} \\
&> 0.
\end{aligned}
$$

This is a contradiction to $\dot{I}_A(\bar{t} + \omega_1 + T_4) \le 0$. Therefore, $I_A(t) \ge m_1$ for $t \in [\bar{t}, \bar{t} + \psi]$. Since this kind of interval $[\bar{t}, \bar{t} + \psi]$ is arbitrarily chosen, we get that $I_A(t) \ge m_1$ for t large enough in Case (ii). In view of our arguments above, the choice of m_1 is independent of the positive solution of (2.4), which satisfies that $I_A(t) \ge m_1$ for sufficiently large t.

Similarly, if $I_B(t)$ oscillates about $\gamma_2 m_2^*$, then we can prove that $I_B(t) \ge m_2$ for all large t, where

$$
m_2 = \min\bigg\{\frac{\gamma_2 m_2^*}{2}, \gamma_2 m_2^* e^{-(r_2+\mu)\omega_2}\bigg\}.
$$

This completes the proof of Theorem 3.2. $\qquad\square$

Theorem 3.3 *If $\mathcal{R}_2 > 1$, then system (2.4) is permanent.*

Proof Suppose that $(S(t), I_A(t), I_B(t))$ is any positive solution of system (2.4). From the first and fourth equations of system (2.4) it is easy to see that

$$
\begin{cases}
S'(t) \ge \mu - (\mu + \beta_1 e^{-\mu\omega_1} + \beta_2 e^{-\mu\omega_2})S(t), \\
\qquad\qquad\qquad\qquad t \ne n\tau, \ n \in N, \\
S(t^+) = (1-\delta)S(t), \quad t = n\tau, n \in N.
\end{cases}
\tag{3.19}
$$

Let

$$
\begin{aligned}
m &= \frac{\mu}{\mu + \beta_1 e^{-\mu\omega_1} + \beta_2 e^{-\mu\omega_2}} \\
&\quad \times \frac{(1-\delta)(1 - e^{-(\mu+\beta_1 e^{-\mu\omega_1}+\beta_2 e^{-\mu\omega_2})\tau})}{1 - (1-\delta)e^{-(\mu+\beta_1 e^{-\mu\omega_1}+\beta_2 e^{-\mu\omega_2})\tau}} - \varepsilon \\
&> 0,
\end{aligned}
$$

where ε is a sufficiently small positive constant. Similarly, using Lemma 2.1 along with (3.19), we have $S(t) > m$ for t large enough.

Set

$$
\begin{aligned}
\Omega = \big\{&(S, I_A, I_B) \in R_+^3 \mid m \le S(t) \le 1, \\
&m_1 \le I_A(t) \le 1, \ m_2 \le I_B(t) \le 1\big\}.
\end{aligned}
$$

Then Ω is a bounded compact region which has positive distance from coordinate planes. By Theorem 3.2, one obtains that every solution of system (2.4) with initial condition ϕ eventually enters and remains in the region Ω. This completes the proof. $\qquad\square$

Corollary 3.3 *If $\delta < \delta_*$, or $\tau > \tau^*$, or $r < r_*$, or $q < q_*$, then system (2.4) is permanent, that is, the disease can generate an endemic, where the critical values δ_*, τ^*, ω_*, and q_* are listed as follows:*

$$
\delta_* = \min_{i,j=1,2,i\ne j}\bigg\{1 - \frac{\mu e^{\mu\tau}}{\mu + (e^{\mu\tau}-1)(\mu - \beta_j e^{-\mu\omega_j})\sqrt[q_i]{\frac{\beta_i e^{-\mu\omega_i}}{r_i+\mu}}}\bigg\},
$$

$$
\tau^* = \max_{i,j=1,2,i\ne j}\bigg\{\frac{1}{\mu}\ln\bigg(1 + \frac{\delta\mu}{(1-\delta)(\mu - \beta_j e^{-\mu\omega_j})\sqrt[q_i]{\frac{\beta_i e^{-\mu\omega_i}}{r_i+\mu}} - \mu}\bigg)\bigg\},
$$

$$r_* = \min_{i,j=1,2,i\neq j}\left\{\beta_i e^{-\mu\omega_i}\cdot\left(\frac{\mu-\beta_j e^{-\mu\omega_j}}{\mu}\cdot\frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta}\right)^{q_i}-\mu\right\},$$

$$q_* = \min_{i,j=1,2,i\neq j}\left\{\log_{\frac{\mu-\beta_j e^{-\mu\omega_j}}{\mu}\cdot\frac{(1-\delta)(e^{\mu\tau}-1)}{e^{\mu\tau}-1+\delta}}\frac{r_i+\mu}{\beta_i e^{-\mu\omega_i}}\right\}.$$

Note that $\mathcal{R}_1 \to 0 < 1$ and $\mathcal{R}_2 \to 0 < 1$ as $q \to \infty$, which implies that the epidemic disease will die out eventually when nonlinear incidence q is gradually increasing. This is very interesting since nonlinear incidence has a significant effect on the dynamics of epidemic model.

4 Conclusions

As an example, let $\delta = 0$ and $I_B = 0$: then (2.3) becomes the following system without pulse:

$$\begin{cases} S'(t) = -\beta e^{-\mu\omega}S^q(t)I(t-\omega) - \mu S(t) + \mu, \\ I'(t) = \beta e^{-\mu\omega}S^q(t)I(t-\omega) - \mu I(t) - rI(t), \\ R'(t) = rI(t) - \mu R(t). \end{cases}$$

(4.1)

According to Theorems 3.1 and 3.2, we can deduce the same results for system (4.1) as Ma and Song in [5, 6].

In this paper, we introduce time delay (with ω_i, $i = 1, 2$), pulse vaccination (with δ and τ), and nonlinear incidence (with q_i, $i = 1, 2$) into SIR model and obtain that the latent period of disease, pulse vaccination, and nonlinear incidence can bring effects on infection-eradication and the permanence of epidemic disease. The main results show that a short period of pulsing (with $\tau < \tau_*$), or a large pulse vaccination rate (with $\delta > \delta^*$), or a long latent period of the disease (with $\omega > \omega^*$), or a large nonlinear incidence (with $q > q^*$) is a sufficient condition for the global attractivity of infection-eradication periodic solution $(\widetilde{S(t)}, 0, 0)$; if not, the system becomes permanent. Therefore, we can choose the vaccination period (with τ) and increase the proportion (with δ) of those vaccinated successfully to all of the susceptible such that $\mathcal{R}_1 < 1$ in order to prevent the epidemic disease from generating endemic.

We find that infection caused by viruses A and B can be controlled when $R_1 < 1$ and $R_2 < 1$. Since I_B competes I_A, the milder infection caused by virus B acts like a vaccine against the virus A. Hence, with the help of this study, there is a possibility in the future to develop a vaccination strategy to fight the epidemic I_A.

Note that $\mathcal{R}_1 > \mathcal{R}_2$, and we obtain the results for $\mathcal{R}_1 < 1$ or $\mathcal{R}_2 > 1$. However, for the closed interval $[\mathcal{R}_2, \mathcal{R}_1]$, the dynamical behavior of model (2.4) has not been studied, and the threshold parameter for the reproducing number (or the pulse vaccination rate) between the extinction of the disease and the permanence of the disease has not been obtained. It is worthwhile for us to study the case for $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 < 1$ in the future work. Finding the threshold value $\mathcal{R} = \mathcal{R}_1 = \mathcal{R}_2$ is left for our future consideration.

References

1. Enjuanes, L., Sanchez, C., Gebauer, F., Mendez, A., Dopazo, J., Ballesteros, M.L.: Evolution and tropism of transmissible gastroenteritis coronavirus. Adv. Exp. Med. Biol. **342**, 35–42 (1993)
2. Beretta, E., Takeuchi, Y.: Global stability of an SIR epidemic model with time delays. J. Math. Biol. **33**(3), 250–260 (1995)
3. Beretta, E., Hara, T., Ma, W.B., Takenchi, Y.: Global asymptotic stability of an SIR epidemic model with distributed time delay. Nonlinear Anal. **47**, 4107–4115 (2001)
4. Takeuchi, Y., Ma, W.B., Beretta, E.: Global asymptotic properties of a delay SIR epidemic model with finite incubation times. Nonlinear Anal. **42**, 931–947 (2000)
5. Ma, W.B., Song, M., Takeuchi, Y.: Global stability of an SIR epidemic model with time delay. Appl. Math. Lett. **17**, 1141–1145 (2004)
6. Song, M., Ma, W.B., Takeuchi, Y.: Permanence of a delayed SIR epidemic model with density dependent birth rate. J. Comput. Appl. Math. **201**(2), 389–394 (2007)
7. Ma, W.B., Takeuchi, Y., Hara, T., Beretta, E.: Permanence of an SIR epidemic model with distributed time delays. Tohoku Math. J. **54**, 581–591 (2002)
8. Zhang, T.L., Teng, Z.D.: Permanence and extinction for a nonautonomous SIRS epidemic model with time delay. Appl. Math. Model. **33**(2), 1058–1071 (2009)
9. Zhang, T.L., Teng, Z.D.: Global behavior and permanence of SIRS epidemic model with time delay. Nonlinear Anal.: Real World Appl. **9**(4), 1409–1424 (2008)
10. Meng, X.Z., Chen, L.S.: The dynamics of a new SIR epidemic model concerning pulse vaccination strategy. Appl. Math. Comput. **197**(2), 582–597 (2008)

11. Shulgin, B., Stone, L., Agur, Z.: Pulse vaccination strategy in the SIR epidemic model. Bull. Math. Biol. **60**, 1–26 (1998)
12. Lu, Z.H., Chi, X.B., Chen, L.S.: The effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission. Math. Comput. Model. **36**, 1039–1057 (2002)
13. D'Onofrio, A.: On pulse vaccination strategy in the SIR epidemic model with vertical transmission. Appl. Math. Lett. **18**, 729–732 (2005)
14. Hui, J., Chen, L.S.: Impulsive vaccination of SIR epidemic models with nonlinear incidence rates. Discrete Continuous Dyn. Syst. Ser. B **4**, 595–605 (2004)
15. DeQuadros, C.A., Andrus, J.K., Olive, J.M.: Eradication of the poliomyelitis, progress. Am. Pediatr. Infect. Dis. J. **10**(3), 222–229 (1991)
16. Ramsay, M., Gay, N., Miller, E.: The epidemiology of measles in England and Wales: Rationale for 1994 nation vaccination campaign. Commun. Dis. Rep. **4**(12), 141–146 (1994)
17. Sabin, A.B.: Measles, killer of millions in developing countries: Strategies of elimination and continuation control. Eur. J. Epidemiology **7**, 1–22 (1991)
18. Zhang, H., Georgescu, P., Chen, L.S.: An impulsive predator–prey system with Beddington–Deangelis functional response and time delay. Int. J. Biomath. **1**(1), 1–17 (2008)
19. Sun, S.L., Chen, L.S.: Permanence and complexity of the eco-epidemiological model with impulsive perturbation. Int. J. Biomath. **1**(2), 121–132 (2008)
20. Liu, B., Teng, Z.D., Liu, W.B.: Dynamic behaviors of the periodic Lotka–Volterra competing system with impulsive perturbations. Chaos Solitons Fractals **31**(2), 356–370 (2007)
21. Li, Z.X., Chen, L.S.: Periodic solution of a turbidostat model with impulsive state feedback control. Nonlinear Dyn. (in press). doi10.1007/s11071-009-9498-8
22. Wei, C.J., Chen, L.S.: Dynamic analysis of mathematical model of ethanol fermentation with gas stripping. Nonlinear Dyn. **57**(1–2), 13–23 (2009)
23. Shi, R.Q., Chen, L.S.: The study of a ratio-dependent predator–prey model with stage structure in the prey. Nonlinear Dyn. (in press). doi10.1007/s11071-009-9491-2
24. Cooke, K.L.: Stability analysis for a vector disease model. Rocky Mt. J. Math. **9**, 31–42 (1979)
25. Wei, H., Li, X., Martcheva, M.: An epidemic model of a vector-borne disease with direct transmission and time delay. J. Math. Anal. Appl. **342**, 895–908 (2008)
26. Mukandavire, Z., Garira, W., Chiyaka, C.: Asymptotic properties of an HIV/AIDS model with a time delay. J. Math. Anal. Appl. **330**, 916–933 (2007)
27. McCluskey, C.: Global stability for a class of mass action systems allowing for latency in tuberculosis. J. Math. Anal. Appl. **338**, 518–535 (2008)
28. Beretta, E., Luang, Y.: Modeling and analysis of a marine bacteriophage infection with latency period. Nonlinear Anal.: Real World Appl. **2**, 35–74 (2001)
29. Ghosh, S., Bhattacharyya, S., Bhattacharya, D.K.: Role of latency period in viral infection: A pest control model. Math. Biosci. **210**, 619–646 (2007)
30. Meng, X., Chen, L., Cheng, H.: Two profitless delays for the SEIRS epidemic disease model with nonlinear incidence and pulse vaccination. Appl. Math. Comput. **186**, 516–529 (2007)
31. Wei-min, Levin, S.A., Lwasa, Y.: Influence of nonlinear incidence rates upon the behavior of SIRS Epidemiological models. J. Math. Biol. **25**, 359–380 (1987)
32. Wei-min, Hethcote, H.W., Levin, S.A.: Dynamical behavior of epidemiological models with nonlinear incidence rates. J. Math. Biol. **23**, 187–240 (1986)
33. Lakshmikantham, V., Bainov, D., Simeonov, P.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
34. Kuang, Y.: Delay Differential Equations with Applications in Population Dynamics. Academic Press, San Diego (1993)