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On fractional order differential equations model for nonlocal epidemics

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Abstract

A fractional order model for nonlocal epidemics is given. Stability of fractional order equations is studied. The results are expected to be relevant to foot-and-mouth disease, SARS and avian flu.

Keywords: Fractional order differential equations; Stability; Applications to nonlocal epidemic models

1. Introduction

Recently three major epidemic diseases have occurred namely foot-and-mouth disease, severe acute respiratory syndrome (SARS) and avian (bird's) flu. Hopefully this will increase the awareness of modeling infectious diseases spreading that is an important topic in mathematical biology [1]. There are different approaches to this topic, e.g. ordinary differential equations, difference equations, partial differential equations and coupled map lattice. Here we use fractional order differential equations (FOD). The reason is that FOD are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. Consider the following evolution equation [2]:

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = -\lambda^2 \int_0^t k(t-t')f(t')\,\mathrm{d}t'.$$

If the system has no memory then $k(t-t') = \delta(t-t')$, and one gets $f(t) = f_0 \exp(-\lambda^2 t)$. If the system has an ideal memory, then

$$k(t-t') = \begin{cases} 1 & if \quad t \geq t' \\ 0 & if \quad t < t' \end{cases},$$

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hence $f \approx f_0 \cos \lambda t$. Using Laplace transform $L[f] = \int_0^\infty f(t)\exp(-st)\,\mathrm{d}t$, one gets $L[f] = 1$ if there is no memory and $L[f] = 1/s$ if there is ideal memory hence the case of non-ideal memory is expected to be given by $L[f] = 1/s^\alpha$, $0<\alpha<1$. In this case the above equation becomes

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-t')^{\alpha-1} f(t')\,\mathrm{d}t',$$

where $\Gamma(\alpha)$ is the Gamma function. This system has the following solution:

$$f(t) = f_0 E_{\alpha+1}(-\lambda^2 t^{\alpha+1}),$$

where $E_\alpha(z)$ is the Mittag–Leffler function given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}.$$

It is direct to see that $E_1(z) = \exp(z)$, $E_2(z) = \cos z$.

Following a similar procedure to study a random process with memory, one obtains the following fractional evolution equation:

$$\frac{\partial^{\alpha+1}P(x,t)}{\partial t^{\alpha+1}} = \sum_n \frac{(-1)^n}{n!}\frac{\partial^n[K_n(x)P(x,t)]}{\partial x^n}, \quad 0<\alpha<1,$$

where $P(x,t)$ is a measure of the probability to find a particle at time t at position x. We expect that the above result will be relevant to many complex adaptive systems and to systems where fractal structures are relevant since it is argued that there is a strong relevance between fractals and fractional differentiation [3].

For the case of fractional diffusion equation the results are

$$\frac{\partial^{\alpha+1}P(x,t)}{\partial t^{\alpha+1}} = D\frac{\partial^2 P(x,t)}{\partial x^2}, \quad P(x,0) = \delta(x), \quad \frac{\partial P(x,0)}{\partial t} = 0,$$

then

$$P = \frac{1}{2\sqrt{D}t^\beta} M\left(\left(\frac{|x|}{\sqrt{D}t^\beta}\right);\beta\right), \quad \beta = \frac{\alpha+1}{2},$$

$$M(z;\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(-\beta n+1-\beta)}.$$

For the case of no memory $\alpha = 0 \Rightarrow M(z;\ 1/2) = \exp(-z^2/4)$.

The paper is organized as follows: in Section 2, we study the stability of FOD. The stability conditions are derived and several examples are given. The stability conditions for some fractional order differential coupled map lattices are concluded in Section 3. Applications to nonlocal epidemics is introduced in Section 4. The stability conditions for the disease-free state are discussed. Some conclusions are presented in Section 5.

2. Stability of fractional order differential equations [4–6]

Consider the following system:

$$D^\alpha x(t) = f(x,y), D^\alpha y(t) = g(x,y), \quad \alpha \in [0,1), \tag{1}$$

where the fractional derivative in Eq. (1) is in the sense of Caputo. The equilibrium solutions are defined by $f(x_{eq},\ y_{eq}) = 0$, $g(x_{eq},\ y_{eq}) = 0$ and it is locally asymptotically stable if all the eigenvalues λ of the Jacobian matrix $A = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$ evaluated at the equilibrium point satisfies the following condition [4,5]:

$$\left|\arg(\lambda)\right| > \frac{\alpha\pi}{2}. \tag{2}$$

The condition in Eq. (2) poses an interesting question namely:
What are the conditions that all the roots of the polynomial equation

$$P(\lambda) = 0, P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{3}$$

satisfy Eq. (2) where all the coefficients in Eq. (3) are real?
For $\alpha = 1$, the solution is the Routh–Hurwitz conditions [7]

$$a_1 > 0, \quad \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \ldots. \tag{4}$$

For $\alpha \in [0, 1)$, these conditions are sufficient but not necessary. Since most biologically interesting systems are 1,2 and 3-dimensions, we will study the problem (3) for $n = 1, 2, 3$.

Definition 1. The discriminant $D(f)$ of a polynomial

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

is defined by $D(f) = (-1)^{n(n-1)/2}R(f, f')$, where f' is the derivative of f, if $g(x) = x^n + b_1x^{l-1} + b_2x^{l-2} + \cdots + b_l$, $R(f, g)$ is the determinant of the corresponding Sylvester $(n+l)\otimes(n+l)$ matrix. The Sylvester matrix is formed by filling the matrix beginning with the upper left corner with the coefficients of $f(x)$, then shifting down one row and one column to the right and filling in the coefficients starting there until they hit the right side. The process is then repeated for the coefficients of $g(x)$.

Using the results of Ref. [3], if $D(f) > 0 (< 0)$ then there is an even (odd) number of pairs of complex roots for the equation $f(x) = 0$. For $n = 3$, this implies that $D(f) > 0$, and all the roots are real while $D(f) < 0$ implies that there is only one real root and one complex root and its complex conjugate. For $n = 3$, we have

$$D(f) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2.$$

Proposition 1.

(i) For $n = 1$, the condition for (3) is $a_1 > 0$.
(ii) For $n = 2$, the conditions for (3) are either Routh–Hurwitz conditions or

$$a_1 < 0,\ 4a_2 > (a_1)^2,\ \left|\tan^{-1}\left(\frac{\sqrt{4a_2 - (a_1)^2}}{a_1}\right)\right| > \frac{\alpha\pi}{2}. \tag{5}$$

(iii) For $n = 3$, if the discriminant of $P(\lambda)$, $D(P)$ is positive, then Routh–Hurwitz conditions are the necessary and sufficient conditions for (3) i.e.

$$a_1 > 0,\ a_3 > 0,\ a_1a_2 > a_3 \quad \text{if} \quad D(P) > 0. \tag{6}$$

(iv) If $D(P) < 0$, $a_1 \geqslant 0$, $a_2 \geqslant 0$, $a_3 > 0$, $\alpha < 2/3$, then condition (3) is satisfied. Also if $D(p) < 0$, $a_1 < 0$, $a_2 < 0$, $\alpha > 2/3$, then all roots of $P(\lambda) = 0$ satisfies $|\arg(\lambda)| < \alpha\pi/2$.
(v) If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 = a_3$ then condition (3) is satisfied for all $\alpha \in [0,1)$.
(vi) For general n, $a_n > 0$ is a necessary condition for condition (3) to be satisfied.
(vii) If $\forall\lambda$, $P(\lambda) = P(-\lambda)$ then define $x = \lambda^2$ and Routh–Hurwitz conditions for the resulting polynomial in x are necessary conditions for (3) for all $\alpha \in [0,1)$.

(viii) For $n>1$, the necessary and sufficient condition for (3) is $\int_{-\infty}^{0} dz/P(z)|_{C1} + \int_{0}^{\infty} dz/P(z)|_{C2} = 0$, where C_1 is the curve $z = x(1-i\tan\alpha\pi/2)$ and C_2 is the curve $z = x(1+i\tan\alpha\pi/2)$, $i=\sqrt{-1}$.

Proof. Case (i) is obvious. For (ii), the two roots are $\lambda_{\pm} = \left[-a_1 \pm \sqrt{(a_1^2 - 4a_2)}\right]/2$. If both roots are real or complex conjugates with negative real parts then condition (3) is equivalent to the Routh–Hurwitz case. If the two roots are complex conjugate with positive real parts, then the two roots become $\lambda_{\pm} = \left[-a_1 \pm i\sqrt{4a_2 - (a_1)^2}\right]/2$ and one gets Eq. (5).

To prove (iii) not that if $n=3$, $D\ (p)>0$, then all the roots of $p(\lambda)=0$ are real hence Routh–Hurwitz conditions are both necessary and sufficient for (3).

To prove (iv) not that if $n=3$, $D\ (P)>0$, then the roots of $P(\lambda)=0$ are one real and a complex conjugate pair thus

$$p(\lambda) = (\lambda + b)(\lambda - \beta - i\gamma)(\lambda - \beta + i\gamma) \Rightarrow$$
$$a_1 = b - 2\beta,\ a_2 = \beta^2 + \gamma^2 - 2b\beta,\ a_3 = b(\beta^2 + \gamma^2),\ b \geqslant 0 \tag{7}$$

and $a_1>0 \Rightarrow b>2\beta$, $a_2>0 \Rightarrow \beta^2\sec^2\theta > 2b\beta > 4\beta^2 \Rightarrow \theta>\pi/3$, where $\theta = |\arg(\lambda)|$. The second part is proved similarly.

To prove (v) if $n=3$, $D(P)>0$, then Eq. (7) is valid. Now $a_1a_2 = a_3 \Rightarrow b^2\beta + \beta(\beta^2+\gamma^2) = 2b\beta^2 \Rightarrow \beta = 0$ or $\beta^2+\gamma^2+b^2 = 2b\beta$

The last equality is not valid if both $a_1>0$, $a_2>0$.

To prove (vi) use the fact that for general n

$$P(\lambda) = \left[\prod_j (\lambda + b_j)\right]\left[\prod_k (\lambda - \beta_k - i\gamma_k)(\lambda - \beta_k + i\gamma_k)\right] \Rightarrow$$
$$a_n = \left[\prod_j b_j\right]\left[\prod_k (\beta_k^2 + \gamma_k^2)\right]. \tag{8}$$

To prove (vii) use Eq. (8) and $P(\lambda) = P(-\lambda)\forall\lambda \Rightarrow P(\lambda)$ contains only even power of

$$\lambda \Rightarrow b_j = 0 \quad \forall j, \quad \beta_k = 0 \forall k \Rightarrow$$
$$P(\lambda) = \prod_k (\lambda^2 + \gamma_k^2) = \prod_k (x + \gamma_k^2).$$

And all the roots x should be negative.

To prove (viii) not that if $P(z)$ has no roots in the region $|\arg(\lambda)| < \alpha\pi/2$, hence the function $1/P(z)$ will be analytic in this region. Using Cauchy theorem $\oint_C f(z)\,dz = 0$ for all $f(z)$ analytic within and on the curve C, and that $P(z)$ is polynomial of degree >1 this completes the proof. □

One can give explicit examples where the Routh–Hurwitz conditions are not valid yet Eq. (3) is satisfied for explicit α, e.g. $a = \frac{1}{2}$. The first example is

$$\lambda^3 - \frac{\lambda^2}{2} + \frac{\lambda}{2} + \frac{1}{2} = 0,$$
$$\lambda^3 - \left(1 - \frac{1}{\sqrt{2}}\right)\lambda^2 + \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right)\lambda + \frac{1}{2\sqrt{2}} = 0.$$

For $n=3$, one can solve the polynomial equation explicitly and apply Eq. (3) to the solutions. The solution method is:

Define $y = \lambda - a_1/2$, then the polynomial becomes

$$y^3 + py + q = 0, \quad p = -a_1^2/3 + a_2, \ \ q = 2a_1^3/27 - a_1a_2/3 + a_3,$$

$$y=\rho+\sigma,\ \rho=\sqrt[3]{-(q/2)+\sqrt{(q/2)^2+(p/3)^3}},\ \ \sigma=\sqrt[3]{-(q/2)-\sqrt{(q/2)^2+(p/3)^3}},$$

choose the roots such that $\rho\sigma=-p/3$.

Conjecture 1. For all $n>3$, if $\Delta_1,\Delta_2,\ldots,\Delta_n$ are Routh–Hurwitz determinants

$$\Delta_1=a_1,\ \Delta_2=\begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix},\ \Delta_3=\begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix},\ \ldots,$$

then the conditions

$$\Delta_i>0,\ i=1,2,\ldots,n-2,\quad a_n>0,\quad \Delta_{n-1}=0, \tag{9}$$

are sufficient conditions that Eq. (3) is valid for all $\alpha\in[0,\ 1)$.

This conjecture can be proved for $n=4$. The case $n=3$ is proved in Proposition 1.

Now we apply Proposition 1 to derive the value of α at which chaos or instability disappears at some models. Fractional order Lotka-Volterra (FLV) predator–rey model is given by

$$D^\alpha x=\beta x-xy,\ D^\alpha y=-\gamma y+xy,\ \alpha\in[0,1],$$

where β, γ are positive constants. The equilibrium solutions of FLV are (0, 0) and $(\gamma,\ \beta)$. The eigenvalues corresponding to (0, 0) are β, $-\gamma$ respectively hence (0, 0) is unstable for all $\alpha\in[0,\ 1]$. The eigenvalues corresponding to $(\gamma,\ \beta)$ are $\pm i\sqrt{\beta\gamma}$ hence $|\arg(\lambda)|=\pi/2$, then it is locally asymptotically stable for all $\alpha\in[0,\ 1)$. This agrees with the numerical results of Ref. [5].

Fractional order Chen (FOC) model is given by [7]

$$D^\alpha x=a(y-x),\ D^\alpha y=(c-a)x-xz+cy,\ D^\alpha z=xy-bz,$$

where $a=35$, $b=3$, $c=28$. The equilibrium solutions for FOC are (0, 0, 0) and $(x^*,x^*,x^{*2}/b)$, where $x^*=\pm\sqrt{b(2c-a)}$. The eigenvalues of (0, 0, 0) are $\lambda=-b$, $\lambda^2+(a-c)\lambda-\mathrm{a}(2c-a)=0$. Using Proposition 1 part (vi), then (0, 0, 0) is unstable for all $\alpha\in[0,\ 1]$. The eigenvalues of the internal equilibrium solution is $\lambda^3+(a-c+b)\lambda^2+bc\lambda+2ab(2c-a)=0$. Since all the coefficients of this polynomial are positive and it is easy to check that $D(P)<0$, using Proposition 1 part (iv) then the internal solution of FOC will be stable for $a<\frac{2}{3}$ which is in excellent agreement with the numerical result obtained by and Chen [8].

A similar result is obtained for fractional order Lorenz system

$$D^\alpha x=c(y-x),\ D^\alpha y=rx-xz-y,\ D^\alpha z=xy-bz,$$

where r, b, c are positive constants and $r>1$.

The fractional order Rossler system (FOR) is defined by [9]

$$D^\alpha x=-(y+z),\ D^\alpha y=x+ay,\ D^\alpha z=0.2+xz-10z,\ a=0.6$$

There are one equilibrium $z=-y$, $x=-ay$, $y=[-10+\sqrt{100-0.8a}]/2a$, and the characteristic polynomial is

$$\lambda^3+(ay-a+10)\lambda^2+(1-ya^2-10a-y)\lambda+10+2ay=0.$$

It is direct to see that $a_2<0$, $a_1>0$ hence by solving the characteristic equation directly one gets the following roots:

$$\lambda=-9.986,\ \lambda=0.3-0.954i,\ \lambda=0.3+0.954i.$$

This implies that the equilibrium solution is locally asymptotically stable if $\alpha<0.8$ which is in excellent agreement with the result obtained numerically in Ref. [9]: $\alpha\in(0.7,\ 0.8)$.

Considering the fractional order Chua model

$$D^{\alpha}x = a(y-x) - g(x),\ D^{\alpha}y = s[-a(y-x)+z],$$
$$D^{\alpha}z = -c(y+rz),$$

$$g(x) = \begin{cases} -x & \text{if } |x|<1, \\ [-1+b(|x|-1)]\mathrm{sgn}(x) & \text{if } 1<|x|<10, \\ [10(|x|-10)+9b-1]\mathrm{sgn}(x) & \text{if } |x|>10, \end{cases}$$

where $a = 0.923$ or 1, $b = 0.636$, $r = 0.071$ or 0, $s = 0.066$. The equilibrium solution is $x \approx \pm(1-b)/(a-b)$, $y \approx 0$, $z \approx -ax$. Linearizing about the quilibrium solution one gets

$$\begin{bmatrix} -a+b & a & 0 \\ sa & -sa & s \\ 0 & -c & -rc \end{bmatrix},$$

whose eigenvalues for $a = 1$, $b = 0.636$, $c = 0.779$, $s = 0.066$, $r = 0.071$ are $\lambda = -0.486$, $\lambda = 5\times 10^{-4} \pm 0.184i$ which implies that this equilibrium is locally asymptotically stable for $\alpha \in [0,1)$. The eigenvalues for $a = 0.923$, $b = 0.636$, $c = 0.779$, $s = 0.066$, $r = 0$ are $\lambda = -0.406$, $\lambda = 0.029 \pm .188i$ which implies that this equilibrium is locally asymptotically stable for $\alpha < 0.9$.

3. Stability conditions for some fractional order differential coupled map lattices

Spatial effects are important in many biological systems. Thus generalizing fractional order systems (FOS) to include them is important. The standard approach is fractional order partial differential equations. However since most biologically interesting systems are nonlinear [10], one gets fractional order nonlinear partial differential equations whose existence and uniqueness has not been established yet. Therefore, we use coupled map lattices (CML) [11] as an alternative approach to include spatial effects in FOS. Consider the 1-system CML

$$D^{\alpha}u_i(t) = (1-D)f(u_i) + \frac{D}{2}[f(u_{i+1}) + f(u_{i-1})],$$
$$i = 1, 2, \ldots, n, \tag{10}$$

where D is a positive constant. The homogeneous equilibrium solution of Eq. (10) satisfies $f(u_{eq}) = 0$ and it is stable if all the eigenvalues of the circulant matrix [12]

$$\begin{bmatrix} (1-D)f' & (D/2)f' & 0 & \ldots & 0 & (D/2)f' \\ (D/2)f & (1-D)f' & (D/2)f & 0 & \ldots & 0 \\ 0 & (D/2)f' & (1-D)f' & (D/2)f' & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & (D/2)f' & (1-D)f' & (D/2)f' \end{bmatrix}$$

satisfies Eq. (2). Since the eigenvalues of the circulant matrices [12] are known, the local asymptotic stability conditions become

$$\left|\arg\left[(1-D)f'(u_{eq}) + Df'(u_{eq})\cos\frac{2\pi k}{n}\right]\right| > \frac{\alpha\pi}{2},$$
$$k = 0, 1, \ldots, n-1 \tag{11}$$

Generalizing to 2-system CML

$$\begin{aligned} D^\alpha x_i &= (1-D)f(x_j, y_j) \\ &\quad + \frac{D}{2}\left[f(x_{j+1}, y_{j+1}) + f(x_{j-1}, y_{j-1})\right], \\ D^\alpha y_i &= (1-D)g(x_j, y_j) \\ &\quad + \frac{D}{2}\left[g(x_{j+1}, y_{j+1}) + g(x_{j-1}, y_{j-1})\right], \end{aligned} \tag{12}$$

$j = 1, 2, \ldots, n$. The homogeneous equilibrium solutions are given by $f(x_{eq}, y_{eq}) = 0$, $g(x_{eq}, y_{eq}) = 0$ and is locally asymptotically stable if all the eigenvalues of the following matrix B satisfies Eq. (2):

$$B = \left[(1-D) + \frac{D}{2}\cos\left(\frac{2\pi k}{n}\right)\right]\begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}, \tag{13}$$

where the Jacobian matrix is evaluated at the equilibrium point.

4. Application to a nonlocal epidemic model

Now we present a nonlocal interacting epidemic model. The nonlocal interactions in the epidemic spreading is widely observed in many outbreaks, especially the recent ones of FMD, SARS and bird's flu. The Lajmanovich-Yorke model [13,14] is a special case of coupled map lattices. It is a globally interacting model that emphasizes the diffusion of a disease from other infected farms beside its spread within the same farm. We have generalized it to the inhomogeneous case [15] and introduced it as an approximation for the epidemic spreading on small-world networks [15]. Consider n patches, where each one contains a certain number of individuals (say animals). In general these patches are not identical. Infection spreads from infected animals within the patch and due to those diffusing from other patches. The fractional order form of the model is given by

$$D^\alpha y_i = \lambda_i y_i(1-y_i) + \mu_i(1-y_i)\sum_{j\neq i} y_j - \gamma_i y_i, \; i = 1, 2, \ldots, n, \tag{14}$$

where y_i is the number of infected individuals in the ith patch. The first term represents the infection within the patch with a rate λ_i. The second term represents the effect of other patches both nearby and far away at a rate μ_i. The recovery rate is represented by γ_i. Since the effect of other patches (second term in Eq. (14)) is significantly smaller than the first one, we expect that $\mu_i \ll \lambda_i$. The disease is eradicated if $y_i = 0$, $\forall\, i = 1, 2, \ldots, n$. So the stability of this solution is studied. Using the same procedures in the previous section, the stability of the zero solution (disease free state) is that all the eigenvalues x of the matrix A satisfy $|\arg(x)| > \alpha\pi/2$ where

$$A = \begin{bmatrix} \lambda_1 - \gamma_1 & \mu_1 & \mu_1 & .. & \mu_1 \\ \mu_2 & \lambda_2 - \gamma_2 & \mu_2 & .. & \mu_2 \\ \mu_3 & \mu_3 & \lambda_3 - \gamma_3 & \mu_3 & \mu_3 \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ \mu_n & \mu_n & \mu_n & \ldots & \lambda_n - \gamma_n \end{bmatrix}.$$

For $n = 2$, the solutions $y_i = 0$, $i = 1, 2$ are locally asymptotically stable if

$$\left|\tan^{-1}\left(\frac{\sqrt{(\lambda_1 - \gamma_1 + \lambda_2 - \gamma_2)^2 - 4[(\lambda_1 - \gamma_1)(\lambda_1 - \gamma_1) - \mu_1\mu_2]}}{\lambda_1 - \gamma_1 + \lambda_2 - \gamma_2}\right)\right| > \frac{\alpha\pi}{2}.$$

For $n = 3$, the characteristic polynomial of A is

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

where

$$a_1 = -[\lambda_1 - \gamma_1 + \lambda_2 - \gamma_2 + \lambda_3 - \gamma_3],$$

$$a_2 = (\lambda_1 - \gamma_1)(\lambda_2 - \gamma_2) - \mu_1\mu_2 + (\lambda_1 - \gamma_1)(\lambda_3 - \gamma_3) - \mu_1\mu_3 + (\lambda_3 - \gamma_3)(\lambda_2 - \gamma_2) - \mu_3\mu_2$$

and

$$a_3 = -[(\lambda_1 - \gamma_1)(\lambda_2 - \gamma_2)(\lambda_3 - \gamma_3) + 2\mu_1\mu_2\mu_3 - (\lambda_1 - \gamma_1)\mu_2\mu_3 - (\lambda_3 - \gamma_3)\mu_2\mu_1 - (\lambda_2 - \gamma_2)\mu_1\mu_3].$$

Using part (vi) in Proposition 1, the disease free state is unstable if $a_3 < 0$.

Finally vaccination may affect the parameters λ_i but not μ_i. Also it is important to know whether the vaccine works once administered or it takes time till it becomes effective.

5. Conclusions

A fractional order evolution equation describing a random process with memory is concluded. This will be relevant to many complex adaptive systems. The stability conditions for both fractional order differential equations and fractional order differential coupled map lattices are derived. A fractional order form of Lajmanovich-Yorke model is introduced. This model emphasis the nonlocal interaction beside the local interaction in the epidemic spreading. Both types of interaction are widely observed in the recent FMD, SARS and bird's flu outbreaks.

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